INVESTMENT-EQUIVALENT REINSURANCE PRICING

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Abstract

Reinsurance pricing is usually described as market-driven. In order to have a more theoretical (and practical) basis for pricing, some description of the economic origin of reinsurance risk load should be given. A special-case algorithm is presented here that allows any investment criteria concerning return and risk to be applied to a combination of reinsurance contract terms and financial techniques. The inputs are the investment criteria, the loss distributions, and a criterion describing a reinsurer's underwriting conservatism. The outputs are the risk load and the time-zero assets allocated to the contract when it is priced as a stand-alone deal. Since most reinsurers already have a book of business and hence contracts mutually support each other, the risk load here can be regarded as a reasonable maximum. The algorithm predicts the existence of minimum premiums for rare event contracts, and generally suggests a reduction in risk load for pooling across contracts and/or years. Three major applications are: (1) pricing individual contracts, (2) packaging a reinsurance contract with financial techniques to create an investment vehicle, and (3) providing a tool for whole book management using risk and return to relate investment capital, underwriting, and pricing.

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1. INTRODUCTION

There has been an evolution over the last few years toward looking at an insurance or reinsurance enterprise as a whole, rather than seeing underwriting, investments, dividend policy, and so forth as a set of disjoint pieces. Whereas in modern financial theory various approaches to the interaction of risk and reward are reasonably well developed, for insurance in general and reinsurance in particular the measurement of risk has been (and arguably still is) more of an art than a science. It is generally agreed that surplus creates capacity and writing business uses up surplus, but there is no agreement on how that happens.

This paper proposes a way of obtaining models for the special case where the contract is priced on a stand-alone basis; i.e., it is the reinsurer's only business. The risk loads (and hence pricing) derived here are maximal because reinsurers generally have an ongoing book of business. This book is mutually supporting in that usually it does not all go bad at the same time. Pricing on a stand-alone basis is equivalent to assuming that the whole book is fully correlated. Thus stand-alone pricing in general will result in larger risk loads than are actually needed.

Although the give and take of the market will ultimately determine what prices are actually charged for contracts, both insurers and reinsurers can use an economic pricing model in order to help decide whether to write a contract. For the insurer, the decision not to reinsure externally is a decision to self-reinsure. The intent of this paper is to present a paradigm that will allow the combination of a reinsurance arrangement and suitable financial techniques to be thought of as an investment alternative. This allows a firm's investment criteria to be applied to the decision.

What is actually done is to assume investment criteria in the form of a target mean return and risk measure thereon. From the paradigm, the necessary risk load and notionally allocated assets for the reinsurance arrangement are obtained.

The paradigm is as follows: When the reinsurer accepts a contract, it arranges to have available at every time of loss sufficient liquid assets to cover possible losses up to some safety level. These assets arise from premium and from surplus, both of which are invested in appropriate financial instruments. The reinsurer wishes to have at least as favorable return and risk over the period of the contract as it would target in other investments of the allocated assets.

Note that this is not, at least to the author's knowledge, how reinsurers currently do their pricing, nor is it advocated (except in special circumstances) as an operating procedure for reinsurers. It is a way of deriving risk loads by relating them to investment criteria. At the same time, it makes intuitive sense. Certainly reinsurers had better plan to have assets available to pay losses; otherwise they are planning for bankruptcy. This paradigm essentially looks at risk load as an opportunity cost and represents it as a (partially offset) cost of liquidity. This is not the only way to look at risk loads, but it is a simple and intuitive one.

The loss safety level is essentially a measure of reinsurer company conservatism. It is intuitive that some measure of company conservatism would be present in a risk load paradigm.[1] The more conservative the company, the higher the safety level and the less probable it is that the safety level will be exceeded. Higher safety levels will typically result in more expensive contracts.

A mundane example of a safety level occurs when a person decides to build a house in snow country. The question is, how strong to build the roof for snow load? If it is a cabin intended

[1] A financial economics point of view suggests that there is a market equilibrium price, and that it reflects a risk load independent of specific company attitudes. However, without wanting to get into a complex discussion, the author feels that many of the assumptions of an efficient market are not particularly well satisfied in the reinsurance arena. A reinsurer with a large portfolio of Florida Homeowners' policies may very well not take any more at all, much less at the price that some other reinsurer is willing to accept. What constitutes "large" depends upon the reinsurer management.

for use for only a few years, perhaps building to survive the ten year storm will be enough. If it is meant for the grandchildren, perhaps surviving the two hundred year storm is more appropriate. It is, of course, more expensive to build it stronger. In any case, some level is chosen depending on the builder's criteria.

The safety level used in the examples here will be the amount of loss associated with a previously chosen probability, such as the 99.9% level; i.e., the loss associated with a one thousand year return time. In some circumstances (see Section 2.3), the full amount of the contract may be the appropriate safety level. There are, of course, alternatives to a probability level. One would be to choose a loss safety level high enough so that the average value of the excess loss over that level is an acceptably small fraction of the mean loss. Another is when the average excess loss over the safety level times the probability of hitting the safety level is below some value. While it would be interesting to examine various choices in the context of different management styles, the essential point is that any quantifiable measure can be used.

Clearly, a risk load paradigm must involve the cost of capital and, more specifically, measures of investment return and risk for comparison to the capital markets. A *reductio ad absurdum* shows the argument: If capital were free and freely available, insurance, much less reinsurance, would be unnecessary. A firm in temporary trouble would simply borrow through difficulties.

The measure of investment risk used here will be the standard deviation (or variance) of the rate of return. Equally applicable would be one of the more sophisticated, strictly downside measures such as the semi-variance or the average value of the (negative) excess of return below some trigger point such as the risk-free rate. Especially where very large losses may generate negative results, such a downside risk measure may be desirable. These measures do not give pretty formulae but are easily used numerically. Again, any quantifiable measure is feasible.

There are two types of financial techniques that will be considered. Other techniques are possible; these are just two of the simplest. The first is where the reinsurer takes the capital that it would have put into the target investment (which could be, for example, corporate bonds) and puts it into a risk-free instrument such as government securities. This will be referred to as a ***switch***.[2] The cost associated with this is the loss of investment income, but there is also a gain in that investment default risk is reduced.

This technique results in simple formulae;[3] but it often results in a higher risk load and, hence, is more expensive (to the cedent) and therefore less competitive than the second technique: buying "put" options. These options are the right to sell the underlying target investment at a predetermined strike price at maturity. We only consider European options which can be exercised only at maturity. Here the strike price will be what an investment in risk-free securities would have brought, so that the reinsurer is buying the right to sell the target investment at a return not less than the risk-free rate.

The Black–Scholes[4] formula is used to price the option. The distribution of price for the reinsurer's target investment is assumed lognormal, so that this formula applies. The cost of these options will contribute to the risk load, but this is partly offset because the options both increase the return and decrease the variance of the target investment.

This treatment will not include the effects of reinsurer expenses, nor of taxes. However these could be included, especially in the simulation models in the latter part of the paper. For taxes, one would have to make some assumptions as to whether the

[2] To the author's knowledge, this is not a technical financial term. If it is, apologies are offered. The meaning intended is only what is stated here.

[3] For the variance measure of investment risk. As remarked earlier, other measures in general will not give simple formulae.

[4] See the discussion of Black–Scholes in the standard Part 5 reading [1, page 502 ff.]

contract would affect any possible Alternative Minimum Tax situation. Probably this could best be treated by looking at the reinsurer's whole underwriting book and investment structure with and without the contract of interest. This is beyond the scope of this paper.

In Section 2, the paper addresses the case of a single loss payment at the end of one year. In Section 2A, the switch is treated and in Section 2B, the option. These simple discussions will illustrate the general principles so that they will hopefully not be obfuscated by the details of the subsequent development. For readability, technical details are relegated to appendices. In Section 2C the limiting case of a high excess layer is presented, where it is shown that a minimum premium results. This is in accord with actual market behavior. In Section 3, the single payment case is extended to an arbitrary known time of loss. Section 3A is a numerical example, and Section 3B includes some general remarks on pooling and other subjects.

The multiple payment case is discussed in Section 4. In this case, there are no simple formulae available, and simulation modeling must be explicitly used. Section 5 contains concluding remarks on the whole paradigm.

2. SINGLE PAYMENT AT ONE YEAR

Let the principal determinants be:

S = the dollar safety level associated with the loss distribution;

L = the mean value of the loss;

σ_L = the standard deviation of the loss;

r_f = the risk-free rate;

y = the yield rate of the target investment;

σ_y = the standard deviation of the investment yield rate; and

P = the premium net to the reinsurer after expenses.

Quantities derived from the above are:

A = the assets allocated by the reinsurer;

F = the funds initially invested = premium and assets less option cost, if applicable; and

R = the risk load.

The premium in all cases is the risk load plus the expected loss discounted at the risk-free rate. Note that this premium does not include any reinsurer expenses. For a single payment at one year,

$$P = R + \frac{L}{1 + r_f}. \tag{2.1}$$

The constraints of the paradigm may now be stated as: (1) the investment result from F as input must be at least S, and (2) the standard deviation of the overall result must be no larger than σ_y.

Although the fundamental cash flow relations are stochastic, it is possible in this Section to get explicit formulae for the mean and variances involved, and hence get explicit forms for the risk load. In Section 4, the mean is easily obtained, but the variance of the final outcome of the stochastic cash flows has to be determined by simulation.

A. Switch Case

At time zero, the reinsurer has an inflow of P and an outflow of

$$F = (P + A). \tag{2.2}$$

Since the investment is in risk-free securities, at the end of the year the reinsurer has an inflow of $(1 + r_f)F$ and an outflow of the loss. The internal rate of return (IRR) on these cash flows is defined by the fundamental stochastic relation

$$(1 + \text{IRR})A = (1 + r_f)F - \text{loss}, \tag{2.3}$$

where both the loss and the IRR are stochastic variables. Taking the mean value of this equation and asking that the mean value of the IRR be the yield rate y gives

$$(1+y)A = (1+r_f)F - L, \tag{2.4}$$

which may be expressed as[5]

$$R = \frac{(y-r_f)}{(1+r_f)}A. \tag{2.5}$$

Another equation is needed to solve the system, and there are two constraints that must be satisfied—a loss safety constraint and an investment variance constraint. In general, it is clear that by making the asset base large enough, the fractional variability of results can be made as small as desired and the funds available as large as desired. Hence there is always a solution. Both constraints may be phrased as placing lower limits on the allocated assets, so satisfying the more restrictive will satisfy both.

For the safety constraint, requiring the funds available at the year end to be greater than or equal to the safety level gives

$$(1+r_f)F \geq S. \tag{2.6}$$

Combining Equations 2.4 and 2.6 to eliminate F,

$$A \geq \frac{(S-L)}{1+y}, \tag{2.7}$$

and consequently, from Equations 2.5 and 2.7, the risk load at the equality is

$$R = \frac{(y-r_f)(S-L)}{(1+r_f)(1+y)}. \tag{2.8}$$

Having the risk load, Equation 2.1 gives the premium before expenses.

This is the result for the safety constraint. For the variance constraint, since there is no variability in the investment return

[5]For readability, derivations of more than one line are done in Appendix C.

(because it is risk-free), the standard deviation of the IRR is given by Equation 2.3 as

$$A\sigma_{\text{IRR}} = \sigma_L. \tag{2.9}$$

The investment constraint is that the IRR should have a variance less than or equal to that of the target investment, which gives

$$A \geq \sigma_L/\sigma_y \tag{2.10}$$

and, using Equation 2.5 again,

$$R = \frac{(y - r_f)}{(1 + r_f)}(\sigma_L/\sigma_y). \tag{2.11}$$

Given typical values for the loss distribution and the target investment, the latter is likely to be the more stringent constraint. This will be true when

$$(S - L)/\sigma_L < (1 + y)/\sigma_y. \tag{2.12}$$

For a one-in-a-thousand safety level and a normal distribution, the number on the left is around 3. For more positively skewed distributions, it will be larger; but, in the experience of the author, it is seldom as large as five for typical reinsurance layers. However, in the example used later of an unlimited cover with a lognormal distribution with coefficient of variation two, the ratio on the left is over ten. The unlimited cover is a mathematical convenience for illustration rather than a realistic contract, at least since pollution losses began to develop extremely adversely. Plausible values for the ratio on the right are easily around twelve for bonds and higher than five for equities. For example, for a bond with an 8% yield and an 8% standard deviation, the ratio is $1.08/0.08 = 13.5$; for a stock with a 12% yield and a 20% standard deviation, the ratio is $1.12/0.20 = 5.6$.

B. Option Case

At time zero, the reinsurer will receive the premium but keep the initial assets invested in the target investment. It will also buy an option to sell the target investment at the end of the year for the value that a risk-free investment would have reached. By doing so it has an instrument that eliminates that portion of the investment return distribution that lies below the risk-free rate. This will have the effect both of increasing the mean return from the investment and decreasing its standard deviation.

The rate of the put option (cost per dollar of investment protected) is here denoted c, and depends upon the underlying investment parameter σ, which is determined by y and σ_y and defined in Appendix A. For small values of the ratio of σ_y to $(1+y)$, it is approximately true[6] that

$$\sigma = \sigma_y/(1+y), \qquad \text{and} \tag{2.13}$$

$$c = \frac{1}{\sqrt{2\pi}}\sigma(1-\sigma^2/24). \tag{2.14}$$

However, the examples below use the exact formula from Appendix A. At time zero, the reinsurer has an inflow of P and an outflow of $(P+A)$. The funds available for investment have decreased by the cost of the option. Specifically, Equation 2.2 becomes

$$F = P + A - cF, \tag{2.15}$$

$$\text{so} \qquad F = (P+A)/(1+c). \tag{2.16}$$

Since the investment is now in risky securities (hedged at the bottom end to stay above or equal to the risk-free rate), at the end of the year the reinsurer has an inflow of $(1+\text{invest})F$ and an outflow of the loss. The internal rate of return on these cash flows is defined by a fundamental stochastic relation similar to Equation 2.3:

$$(1+\text{IRR})A = (1+\text{invest})F - \text{loss}. \tag{2.17}$$

Again, requiring that the mean value of IRR be the target yield rate gives

$$(1+y)A = (1+i)F - L, \tag{2.18}$$

[6]See Appendix A.

where i is the mean investment return (determined in Appendix B). This does not simplify easily, but fundamentally we have two unknowns, R and A, and this is one equation relating them. The other equation will come from whichever is the more restrictive constraint, as before.

The loss safety constraint on the funds available is again Equation 2.6:

$$(1 + r_f)F \geq S. \tag{2.6}$$

It should be noted that the actual funds available are likely to be larger than this, since r_f represents the *minimum* value of the realizable investment return, thanks to the option. Combining Equations 2.6 and 2.18 to eliminate F, the allocated assets are

$$A \geq \frac{1}{1+y}\left\{\frac{1+i}{1+r_f}S - L\right\}. \tag{2.19}$$

This is larger than in the switch case since $i > y > r_f$. The expression for the risk load at equality is[7]

$$R = \frac{1}{(1+r_f)(1+y)}\{S[(1+y)(1+c) - (1+i)] - L[y - r_f]\}. \tag{2.20}$$

For $i = r_f$ and $r = 0$, the results of the previous section are, of course, obtained in the above two formulae.

In order to express the investment variance constraint, it is necessary to decide the correlation between the loss and the investment return. The linkage by inflation suggests that there may be a negative correlation. If inflation rises, typically claims costs rise and bond values fall. In the interest of simplicity the assumption will be made that the correlation is zero, although there is no essential complication introduced by taking a non-zero value. The standard deviation of the investment return is derived in Appendix B and written as σ_i. When the variance of the IRR is

[7] See Appendix C.

required to be that of the target investment, Equation 2.17 (with zero correlation) implies that

$$(A\sigma_y)^2 = (F\sigma_i)^2 + (\sigma_L)^2. \tag{2.21}$$

The value of the initial fund F from Equation 2.18 may be substituted into this, resulting[8] in a quadratic equation for A of the form

$$-aA^2 + 2bA + c = 0, \tag{2.22}$$

with

$$a = (\sigma_y)^2(1+i)^2 - (\sigma_i)^2(1+y)^2,$$
$$b = L(1+y)(\sigma_i)^2, \qquad \text{and}$$
$$c = L^2(\sigma_i)^2 + (\sigma_L)^2(1+i)^2.$$

All three coefficients are positive; the last two because of their explicit construction, and the first because the option both decreases the variance and increases the mean of the investment return compared to the target values.

The positive solution is

$$A = \frac{b + \sqrt{b^2 + ac}}{a} \tag{2.23}$$

and[9]

$$R = A\frac{(1+c)(1+y) - (1+i)}{1+i} + L\left[\frac{1+c}{1+i} - \frac{1}{1+r_f}\right]. \tag{2.24}$$

As $\sigma_i \to 0$, the solution for A goes back to the ratio of standard deviations. With $i = r_f$ and $c = 0$, the risk load returns to the earlier form found in the switch case, as it should.

[8] See Appendix C. The forms corresponding to a non-zero correlation are also given there.

[9] See Appendix C.

C. High Excess Layer and Minimum Premium

An interesting application of these formulae is in the case of a high excess layer or any similar finite rare event cover. A non-zero rate on line (ratio of premium to limit) is predicted even for cases where the loss probability goes to zero.

For simplicity, take the loss distribution to be binomial: there is a probability p of hitting the layer, and if it does get hit, it is a total loss. Note that the 99.9% level is not an appropriate way to get the safety level (especially for $p < 0.001$). There is still in fact an intuitive value: the safety level S is taken to be the limit (total amount payable) of the layer.

The mean loss L is pS and the variance of the loss is $p(1-p)S^2$. As the probability p gets smaller, corresponding to higher and higher layers, in both the switch and option cases the variance constraint forces A and R both to zero as $\sqrt{p}$. However, the safety constraint in both cases is linear in p with a non-zero intercept. In the option case, the rate on line (ROL) in the limit as p goes to zero is

$$\text{ROL} = \frac{(1+y)(1+c) - (1+i)}{(1+r_f)(1+y)}. \tag{2.25}$$

This is obtained by setting $L = 0$ in Equation 2.20 and recognizing ROL as the ratio of R to S. As usual, the switch version may be obtained by letting $c = 0$ and $i = r_f$, which results in

$$\text{ROL} = \frac{1}{1+r_f} - \frac{1}{1+y} = \frac{y - r_f}{(1+r_f)(1+y)}. \tag{2.26}$$

The latter form suggests that the minimum ROL is of the order of the real target return; i.e., the excess of the return over the risk-free rate. However, often the option form Equation 2.25 will produce a smaller number. For the investment values used below it is typically half as large. As the investment standard deviation gets small, the switch ROL stays the same (of course) and the option ROL gets small because the option cost gets small and the mean investment return approaches the target yield. It is important to remember that Equations 2.25 and 2.26 and this discussion are all at the limit where $p = 0$. For this value, the standard deviation of the loss distribution is zero, which implies that the variance constraint is always satisfied. However, for a small but fixed probability, say in the range from 2% to 0.1%, which is typical of catastrophe contracts, as the target standard deviation of investment is made small, the variance constraint will eventually become dominant.

In the market, a minimum ROL is generally justified by underwriters as a charge for using surplus. This approach is consistent with that view and also allows for quantification of the charge.

3. SINGLE PAYMENT AT VARIABLE TIME

If all the returns in the preceding are interpreted as total return up to time t, then the formulae hold without modification. When we wish to express the returns in terms of the equivalent annualized returns, the results hold after the following replacements are made:

$$(1+i) \to (1+i)^t,$$

$$(1+y) \to (1+y)^t, \qquad \text{and}$$

$$(1+r_f) \to (1+r_f)^t.$$

The forms for the option rate and the standard deviations given in Appendix B contain the time dependence.

A. Numerical Example

For any one-payment situation, the recommended procedure is as follows: (1) calculate the four risk loads and allocated assets under the safety and variance constraints for the option and switch cases; (2) find for each financial technique the constraint with the larger allocated assets—this is the dominant one;

(3) compare the dominant risk loads for different techniques and choose the smaller—this is the preferred[10] solution.

This calculation is easily put on a spreadsheet. For a specific example, the following annualized values are used:[11] yield rate $y = 5.3\%$; standard deviation of the yield rate $\sigma_y = 8.4\%$; risk-free rate $r_f = 3.6\%$. The loss distribution is assumed to be lognormal with mean of \$1M (million) and standard deviation of \$2M. The loss safety level is taken as the 99.9% level, \$22,548,702. For a one-year interval this makes the left-hand side of Equation 2.12 equal to 10.8, while the right-hand side is 12.5, suggesting that variance will be the dominant constraint for the switch. For a two-year interval, the right-hand side changes to 8.9 and safety is dominant in the switch. The large value of the left-hand side is due to the fact that this is an unlimited contract.

As an example of the recommended procedure, the following results can be derived from the formulae in the preceding sections for a time of two years.

	SWITCH		OPTION	
constraint	variance	safety	variance	safety
assets	\$15,963,111	\$19,434,097	\$23,024,033	\$20,737,421
risk load	\$528,184	\$643,031	\$316,332	\$283,248

The results are incorporated in Table 1.

For the switch, the safety constraint is dominant; for the option the variance constraint is dominant. Of the two, the option risk load is smaller, and hence preferred.

[10]Preferred from the point of view of the cedent, and preferred from the point of view of offering competitive advantage to the reinsurer—less charge for the same return and risk. On the other hand, the reinsurer may prefer to charge more if the market will bear it. Of course, a higher market rate can always be recast as a more profitable target investment return.

[11]These values are long-term rates from [1, Table 7-1, page 131] except for the standard deviation, which is a private estimate. The return rates are clearly too small to represent current (January 1997) conditions where returns are high and deviations apparently small. Anyone using the pricing technique will use current values appropriate to their own targets.

TABLE 1

VALUES FOR THE OPTION TECHNIQUE ON A SINGLE PAYMENT

Time (years)	1	2	3	4
Option rate	3.18%	4.49%	5.50%	6.35%
Risk load	$235,225	$316,332	$399,548	$502,444
Risk-loaded premium	1,200,476	1,248,042	1,298,882	1,370,526
Total premium	1,379,857	1,434,531	1,492,967	1,575,317
Allocated assets	32,522,839	23,024,033	20,095,065	19,446,192
Initial investment	32,685,050	23,228,830	20,278,801	19,574,132
Determining constraint	variance	variance	safety	safety
Safety value	3,087 years	1,309 years	1,000 years	1,000 years
Annualized Std/target std	100%	100%	97%	93%

In order to get, for example, the second column of Table 1, time is taken as two years. Following the formulae and notation of the appendices for the investment, $2\mu = 9.69\%$ and $\sigma\sqrt{2} = 11.26\%$ at two years. The target investment mean and standard deviation are 10.88% and 12.53% as calculated from the lognormal formulae. The option rate is 4.49%. The mean and standard deviation of the option-protected investment are 14.21% and 8.95%, respectively higher and lower than the target, as previously advertised. The investment minimum value is 7.33%, the risk-free cumulative return.

The calculated risk loads and asset values are given above for both the option and the switch, and the option-variance technique-constraint combination is chosen.

Please note again that any form of loss distribution could have been used, including underwriter's intuition or simulation result. All that is needed for this choice of risk load is the mean, standard deviation, and safety level. Reinsurer expenses, needed to calculate total premium from risk loaded premium, are taken as 13% of the total.

The table also lists the safety level implied by the chosen asset allocation, and the ratio of the standard deviations of the annualized yield to the target standard deviation. Whichever is not the determining constraint is, of course, more than satisfied. It is noteworthy that as the contract period becomes longer, the safety constraint becomes the more restrictive. In numerical explorations this often seems to be true.

B. Pooling and Other Remarks

It is an intuitive expectation that the total risk load may be reduced by pooling. Pooling over contracts will be considered here; pooling over years will be considered later. The one-year contract from Table 1 has a risk load of $235,225. If two contracts are combined into a single contract, then the safety level on the combined contract is generally less than the sum of the individual safety levels, unless the contracts are fully correlated.[12] Specifically, taking the approximation that the sum of two uncorrelated lognormals may, for these purposes, be represented by a lognormal, the safety level for the combined contract is $29,455,245, which is only 65.3% of the sum.[13]

The risk load for the combined contract over one year is $331,156, which is 70.4% of the sum of the individual risk loads. This risk load results from the option-variance technique and constraint. However, one may question whether some other investment risk measure might have given a different result. The author knows of no general theorem, but experimentation has given consistent reduction in risk load from pooling.

More intuitively, both the safety levels and investment risk measures will be primarily sensitive to the tail of the loss distribution. When two contracts are imperfectly correlated, the bulk

[12] Or effectively taken as such, as in the high excess example.

[13] The sum is represented by a lognormal with mean and variance equal to twice the mean and variance of the individual contract. Since the individual contract has mean $1,000,000 and standard deviation $2,000,000, the sum has mean is $2,000,000 and standard deviation is $2,828,427, which implies the 99.9% level mentioned. The 99.9% level on the individual contract is $22,548,701, so twice that level (which would correspond to perfect correlation) is $45,097,403. The ratio is 65.3%.

of the tail results from one or the other of the contracts going bad, not both. The effect generally is to shorten the tail relative to the mean, making measures that depend on extreme values take on less dangerous significance.

A glance at the values in Table 1 shows that it is possible that if the loss is very bad, say at the 0.001% level, then the ending value will be negative. That is, the reinsurer will lose all the premium and allocated assets, and still have to put in more money to fulfill the contract. At the very least, this result cannot be from a lognormal distribution, which never becomes negative.

Nevertheless, it is convenient to express the mean and standard deviation of the distribution of the ending values in terms of the annualized parameters of a geometric Brownian motion investment with the same mean and standard deviation at the time horizon. This allows a direct comparison with the original investment possibility. It is in this sense that the combination of reinsurance contract and switch/option can be thought of as an equivalent investment, if the return and standard deviation are the same as the target.

To the extent that the investment risk measure is valid for general distributions, a comparison can always be made.

Should a reinsurer actually follow through on the indicated financial technique for each contract? Almost surely not, unless the reinsurer is very conservative or this is the only contract. The latter could be the case for a specialty reinsurer set up for a single contract—for example for a large catastrophe contract. In general, a method relating investment criteria to reinsurance contracts could be useful when specifically engineered deals are made to connect reinsureds and investors looking for new opportunities. Considering the hunger of capital for uncorrelated risks, this kind of bundling would seem natural.

This procedure takes as input the financial targets and safety criterion and produces as output the risk load and the allocated assets. It is also possible to take the financial targets and allocated assets as input (more the financial point of view). The two constraints then become requirements on the loss distribution. The corresponding risk loads will emerge. Knowing the desired loss characteristics and the necessary risk loads, market knowledge can be used to do selective underwriting and keep the overall distribution within acceptable risk levels at the target rate of return. This point of view is really more applicable to the book as a whole and requires a treatment of multiple payments.

4. MULTIPLE PAYMENTS

When there are multiple loss payments possible, the same basic paradigm is used but needs a more complex formulation. In the single payment case, simultaneously enforcing the safety constraint and the rate of return through the mean value of the stochastic equation gave an easy solution for the risk load. The risk load appropriate to a particular safety constraint is almost as easy to find in the multiple payment case. However in contrast to the single payment case, there is no simple formula from the variance constraint, but the constraint can be evaluated by simulation for any given level of allocated assets.

The main complication lies in the construction of the safety constraint: in the definitions appropriate to safety levels at different times and in different circumstances. For example, if the first year has a very large loss, do the safety levels for subsequent years change? There are many different formulations possible, all of which will lead to risk loads. Competitive efficiency would suggest looking for a formulation with the smallest possible risk load. This usually will involve using the least possible capital for the shortest period of time, and may depend upon the specifics of the payout pattern expected.

The suggested general procedure is:

1. Express the fundamental stochastic process on a spreadsheet. It is now more complex than a simple equation be-

cause of the interaction of the fund, loss, and investment levels at different times but it is still easily expressed. The complications come from the fact that there are separate simulations of the loss and investment variables at each time. Further, there may have to be other cash flows in either direction between the reinsurer's general assets and the fund set up for this contract, depending upon how the safety levels are defined.

2. Define the safety levels. Since the whole point of this exercise is to use notionally allocated funds to obtain the opportunity cost of capital, the definitions need to be fixed at time zero so the pricing can be done. The simplest version would be to define a single safety level, say 99.9% on the distribution of ending cumulative loss values or the largest 99.9% loss level encountered at any time. The problem with this formulation is that much of the time there will be unnecessary liquidity available which will add to the risk load. A more sophisticated version would be to define levels dependent on the loss distributions at each time. The definitions need not necessarily even result in fixed amounts; the amounts could be conditional on how the losses manifest over time during each particular simulation. A general rule of thumb suggested for safety level definitions is that, if the losses are almost entirely at one time, then the outcome of whatever definitions are used should closely approximate the single payment case for that time.

3. Use the definitions of the safety levels to determine what funds need to be available at various points in time and what options need to be bought (notionally) to protect the values of those funds. Funds not totally consumed at a given time can be carried forward and should be option protected for the expected carryforward. For example, if it is decided to use the 99.9% level at year one as a safety level, in almost all simulations the loss in year one

will be considerably smaller than this safety level. If so, the net can be carried forward to form part of, all of, or more than the safety level for year two. In the latter case it may be that funds flow back to the reinsurer's general account. The option cost on the carryforward will depend on the actual amount of funds carried and the time period when they exist. The time zero present value of the projected average cost is probably not a bad prescription for the initial funds necessary for these options. The switch case is an easier problem because of the lack of option costs, but it does not give the advantage of reducing the standard deviation of the investment and increasing its average return. Hence it will usually give larger risk loads.

4. Find the risk load corresponding to the target return for the safety constraint chosen. This can be done by using a trial risk load and running simulations to ascertain the value of the average final cash result of all the transactions. If this value does not correspond to the desired average return, then try another risk load until the desired target is attained. A faster, simpler, and usually almost as accurate procedure begins by putting all the stochastic variables at their mean values. Then the value of the final cash result of all the transactions is deterministic and can be adjusted to the desired value by varying the risk load.[14] This latter procedure can also be used as a starting point for the former.

5. Simulate to see if the variance constraint is satisfied. If it is and the return is acceptable, stop. It is convenient to represent the variability of the final cash result of all the transactions in terms of the annualized standard deviation of a lognormal investment with the same return and variance at the horizon.

[14] Using Goal Seek in Excel, for example.

6. If the variance constraint is not satisfied, then add more initial capital and simulate again. Clearly, the addition of enough capital beyond that required by the safety constraint can reduce the variability to any desired point. If during the course of iterations the variance constraint is more than satisfied, then take away some capital.

7. Repeat step 6 until both constraints are satisfied. The whole process can be treated numerically like a root-finding procedure, but it is necessary to be careful of the simulation uncertainty in the mean and standard deviation in creating the estimates of the next value of initial capital to be tried.

If it is decided to work with cumulative loss values in establishing definitions of safety levels, the time value of money for the loss in year one must be accounted for with an appropriate rate in order that that loss be economically comparable to a loss in year two. Since the reinsurer can think of this as borrowing from itself, the rate used is the risk-free rate. In the switch case, this is obvious, since the securities held are risk-free. In the option case, this still seems appropriate, since the lower limit which will be realized is the risk-free rate.

5. CONCLUDING REMARKS

The usefulness of safety levels is that they make explicit the minimum funds to be allocated. Unless the safety level is 100%, there is always the possibility in a particular realization of loss and investment that some safety level(s) will be breached. In this case the general account of the reinsurer will have to contribute to the cedent. This does not affect the validity of the original pricing, but the reinsurer's attitude toward this possibility will influence how the safety level was set and hence the price.

For convenience the losses are assumed to occur at the end of each year, although there is no great difficulty in generalizing to arbitrary times. Since simulations are being used, any measures of risk and return that can be defined on individual results can be used. Also, in real-world scenarios the individual years of multi-year contracts may well have some correlation simply because they are from the same firm or exposures. In the simulation environment, the overall contract can still be evaluated if one is willing to quantify the correlation.

A simplification used here is to ignore the fact that the spot rates for risk-free investment depend upon the length of time invested, usually rising with time. For example, incremental losses could be discounted back to time zero using the different spot rates. Here only one single risk-free rate is assumed to apply, for all times of a contract. However, if desired the calculations can be reformulated to include the current spot rates and the view of what the future values of the spot rates are likely to be over the contract period.

In the single payment case, the IRR is used because it is unequivocally defined and provides a natural way of talking about returns. It is not actually necessary to look at the IRR, and only the end result need be considered. In the multiple payment case, the IRR may not even be definable as a real number. This is particularly obvious when the final value is negative because of large losses, but can also happen otherwise. In order to consider the end value (future value of the cash flows), it is necessary to set up some description of the investment policy on the released funds. The target investment is the obvious choice.

It is intuitive that there should be a reduction in risk load from pooling over years, even allowing for the increased cost of liquidity of the later contract. Numerical experimentation seems to indicate that the benefits of pooling over time are usually present for uncorrelated contracts.

The pricing here is extremal pricing, in that each contract is priced as a stand-alone entity. In reality, each contract written is supported by the whole surplus of the reinsurer. A more accurate treatment of the actual risk load needed to satisfy investment criteria would be to consider the whole book with and without the proposed contract. Perhaps a satisfactory compromise would be to scale the extremal risk load contemplated here by the ratio of the overall portfolio risk charge to the sum of the extremal risk loads.

If this paradigm is to be used in connection with a complete book of business, both the general unavailability of options for periods of more than one year and the changing nature of the ongoing book suggest something like looking at the distribution of the one-year forward value of the discounted payment streams, and re-evaluating the necessary risk load annually.

REFERENCES

[1] Brealey, Richard and Stewart Myers, *Principles of Corporate Finance*, 4th Edition, 1991, McGraw-Hill, Inc.

[2] Malliaris, A. G. and W. A. Brock, Stochastic Methods in Economics and Finance, North-Holland, 1982.

APPENDIX A

The form of the Black–Scholes formula for the price of a European call option on a security is[15]

$$\text{call price} = \Phi(\Delta_1)P_0 - \Phi(\Delta_2)PV(E),$$

where

$PV(E)$ = present value of the exercise price discounted at the risk-free rate,

P_0 = price of the security at time zero,

$$\Delta_1 = \frac{\ln\left(\frac{P_0}{PV(E)}\right)}{\sigma\sqrt{t}} + \sigma\sqrt{t}/2, \qquad \text{and}$$

$$\Delta_2 = \Delta_1 - \sigma\sqrt{t},$$

σ is a parameter of the distribution of the underlying security, and

$\Phi(x)$ is the cumulative distribution function for the normal distribution; that is

$$\Phi(x) = \int_{-\infty}^{x} \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} dz.$$

This function is available in at least one standard spreadsheet program.

A call option is the right to buy an underlying security at an exercise price at time t. The logarithm of the value of the security is assumed to follow a normal distribution with parameters μt and $\sigma\sqrt{t}$ for the mean and standard deviation, respectively.[16] Given

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[15] See Brealey and Myers [1, page 502].

[16] This is known as a geometric Wiener process or geometric Brownian motion process. See the development of Black–Scholes in [2] pages 220–223, and the discussion of the Brownian motion on pages 36-38, especially equation (7.13) and the development leading to it.

the expected annual yield rate y and its standard deviation σ_y, then

$$\sigma^2 = \ln\{1 + [\sigma_y/(1+y)]^2\},$$

and

$$\mu = \ln(1+y) - \sigma^2/2.$$

These equations are simply the inversion of the results for the mean and standard deviation of a log-normal distribution for $1 + y$. The approximation in Equation 2.13 comes from the first-order Taylor expansion of the relation for σ^2:

$$\ln(1+x) \approx x,$$

$$\text{so} \qquad \sigma^2 \approx [\sigma_y/(1+y)]^2.$$

The price for a put option, which is actually the contract of interest here, is given by put-call parity as

$$\text{put price} = \text{call price} + PV(E) - P_0.$$

Here, $PV(E)$ equals P_0 since we want the exercise price to be the same as the value which would result from growth at the risk-free rate. Hence the put price equals the call price, and for either option the

$$\text{option cost} = P_0\Phi(\sigma\sqrt{t}/2) - P_0\Phi(-\sigma\sqrt{t}/2)$$

so the

$$\begin{aligned}\text{option rate} &= \Phi(\sigma\sqrt{t}/2) - \Phi(-\sigma\sqrt{t}/2)\\ &= \sqrt{\frac{2}{\pi}}\int_0^{\sigma\sqrt{t}/2} \exp\left(-\frac{z^2}{2}\right) dz.\end{aligned}$$

The exponential may be expanded as $(1 - z^2/2)$ and integrated to get the approximation of Equation 2.14 for t equal one. For the order of magnitude of numbers used here this approximation is actually rather good.

APPENDIX B

As stated in Appendix A, the probability density function for the investment value (which is 1 + return) is lognormal with parameters μt and $\sigma\sqrt{t}$. That is,

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi t}} \exp\left\{\frac{-(\ln(x)-\mu t)^2}{\sigma^2 t}\right\}.$$

The investment hedged with the option to time t has the characteristics (r_f is the risk-free rate):

$$\begin{aligned} \text{investment} &= x \qquad \text{for} \quad x \geq (1+r_f)^t \\ &= (1+r_f)^t \qquad \text{for} \quad x < (1+r_f)^t. \end{aligned}$$

What is needed are the moments of the investment, in particular its mean and standard deviation.

Define

$$\begin{aligned} Fn &= \int_0^{(1+r_f)^t} x^n f(x)\,dx \\ &= \Phi(\zeta - n\sigma\sqrt{t})\exp\{n\mu t + n^2\sigma^2 t/2\}, \end{aligned}$$

where $\zeta = \sqrt{t}(\ln(1+r_f)-\mu)/\sigma$. In general,

$$\begin{aligned} \text{moment}(n) &= \int_0^{\infty} \text{investment}^n f(x)\,dx \\ &= (1+r_f)^{nt}\int_0^{(1+r_f)^t} f(x)\,dx + \int_{(1+r_f)^t}^{\infty} x^n f(x)\,dx. \end{aligned}$$

Using the results for Fn above, the moment of order n of the investment is

$$\begin{aligned} \text{moment}(n) &= (1+r_f)^{tn}F_0 + \exp\{n\mu t + n^2\sigma^2 t/2\} - Fn \\ &= (1+r_f)^{tn}\Phi(\zeta) + \exp\{n\mu t + n^2\sigma^2 t/2\} \\ &\quad \times [1-\Phi(\zeta - n\sigma\sqrt{t})]. \end{aligned}$$

The mean value is just moment(1) and the variance of the investment is $\{\text{moment}(2) - \text{moment}(1)^2\}$. The standard deviation σ_i of the investment is, of course, the square root of the variance.

APPENDIX C

Derivation of Equation 2.5: Substitute for L and F in Equation 2.4:

Equation 2.1 may be solved for L as

$$L = (1 + r_f)(P - R). \tag{C.1}$$

Substitute F from Equation 2.2 and L from Equation C.1 into Equation 2.4:

$$\begin{aligned}(1 + y)A &= (1 + r_f)(P + A) - (1 + r_f)(P - R)\\ &= (1 + r_f)A + (1 + r_f)R.\end{aligned}$$

Solving for R gives Equation 2.5.

Derivation of Equation 2.20: Equation 2.16 can be written

$$(1 + r)F = P + A = A + L/(1 + r_f) + R,$$

from Equation 2.1. Rearranging to solve for R, and subsequently using Equations 2.6 for F and 2.19 for A,

$$\begin{aligned}R &= (1 + r)F - A - L/(1 + r_f)\\ &= (1 + r)\frac{S}{1 + r_f} - \frac{1}{1 + y}\left\{\frac{1 + i}{1 + r_f}S - L\right\} - L/(1 + r_f)\\ &= \frac{S}{1 + r_f}\left[(1 + r) - \frac{1 + i}{1 + y}\right] + L\left[\frac{1}{1 + y} - \frac{1}{1 + r_f}\right]\\ &= \frac{1}{(1 + r_f)(1 + y)}\{S[(1 + y)(1 + r) - (1 + i)] - L[y - r_f]\}.\end{aligned}$$

Derivation of Equation 2.22: Equation 2.18 can be written

$$F = \frac{(1 + y)A + L}{1 + i}.$$

Substituting for F in Equation 2.21 gives

$$A^2\sigma_y^2 = [(1 + y)^2A^2 + 2AL(1 + y) + L^2]\frac{\sigma_i^2}{(1 + i)^2} + \sigma_L^2.$$

Multiplying through by the denominator and collecting terms,

$$0 = A^2[(1+y)^2\sigma_i^2 - \sigma_y^2(1+i)^2] \\ + 2AL(1+y)\sigma_i^2 + L^2\sigma_i^2 + \sigma_L^2(1+i)^2.$$

This is Equation 2.22. If there is a correlation ρ_{iL} between investment and loss, then this equation becomes

$$0 = A^2[(1+y)^2\sigma_i^2 - \sigma_y^2(1+i)^2] \\ + 2A(1+y)\sigma_i[L\sigma_i - \sigma_L\rho_{iL}(1+i)] \\ + L^2\sigma_i^2 + \sigma_L^2(1+i)^2 - 2L\sigma_i\sigma_L(1+i)\rho_{IL}.$$

Derivation of Equation 2.24: By substituting for F from Equation 2.16 into Equation 2.18, we get

$$(1+y)A = (1+i)\frac{P+A}{1+r} - L.$$

Multiplying through by the denominator and using Equation 2.1 for P,

$$A(1+y)(1+r) = (1+i)\left(R + \frac{L}{1+r_f} + A\right) - L(1+r).$$

Rearranging terms,

$$(1+i)R = A[(1+y)(1+r) - (1+i)] \\ + L\left[(1+r) - \frac{1+i}{1+r_f}\right].$$

Equation 2.24 for R results immediately.