DISCUSSION BY ALBERT J. BEER

With the increased importance of utilizing quantitative analysis in risk management decision-making, Miss Wilkinson's paper should provide our profession with a valuable use of the concept of probable maximum loss (PML), a term that has been a fixture of the insurance vernacular for decades. Previously, underwriters have used the PML, or other related tools, to establish the range for the "working layer of coverage." While it was always acknowledged that a larger loss was possible, the PML estimated the expected maximum loss potential for the risk, with the exposure beyond the PML being treated as a catastrophe. Today, the dramatic increase in the amount of risk retained by insureds has made the pricing of large accounts more complex, since the "buffer" of the working layer is no longer available. It is at these extreme values that the author's work with order statistics may provide a variety of applications.

Before I discuss the results of the paper, I would like to resolve what I perceive to be an ambiguity in the treatment of PML as defined by the author. In my opinion, any discussion of PML is unclear without a quantification of the term "probable." If a pair of dice are rolled, is it reasonable to say the total will "probably" be less than eight ($p = 21/36$); less than ten ($p = 30/36$); or, less than twelve ($p = 35/36$)? How certain of an outcome must one be in order to say it is probable? It is precisely this subjectivity that leads to the potential conflict between the insured and the carrier which is alluded to by the author. This dilemma could easily be resolved by quantifying the term "probable." McGuinness[1] accomplishes this by means of a reference to a "*stated proportion of all cases*" which will be equaled or exceeded by the PML. This concept is similar to the confidence coefficient of a one-sided confidence interval. With these ideas in mind, I would suggest that the PML could be redefined as follows:

Definition: PML_α is that amount (or proportion of total value) which will equal or exceed $100\alpha\%$ of all losses that are incurred.

For example, $PML_{.95}$ would represent that amount which would be expected to equal or exceed 95% of the losses incurred by the risk.

If the PML_α is so defined, an insured and underwriter who agree on the underlying loss distribution would arrive at the same PML_α. It is true that the respective risk aversion and risk acceptance levels would certainly affect the degree of satisfaction each would have at various α levels. However, at any fixed α point, there would be technical agreement on PML_α. The "negotiation"

[1] John S. McGuinness, "Is Probable Maximum Loss (PML) A Useful Concept?" *PCAS* LVI, 1969, p. 31.

on the appropriate price for risk transfer would at least have a common starting point.

Miss Wilkinson's definition of PML as the "worst loss likely to happen" does not include any quantification of the term "likely." Therefore, as is noted in the paper, the PML estimates that appear in Exhibit III are not approximating the same quantities. For example, the nth sample order statistic $X_{(n)}$ is intended to be an estimator for the upper bound of the loss variate X. Therefore, $X_{(n)}$ is more closely related to the maximum *possible* loss. Clearly, this is not the same concept McGuinness had in mind when he discussed the generalized PML. It may be noted that my suggested definition of PML_α allows for this degenerate case by choosing $\alpha = 1.00$. (Of course, it may not be technically possible to derive a $\text{PML}_{1.00}$ if the distribution has no finite upper bound.) In contrast, in a situation with 100 losses, using $X_{(95)}$ as an estimate for $k_{.95}$, the 95th percentile, is equivalent to approximating $\text{PML}_{.95}$. I will try to demonstrate that the results displayed in Exhibit III are much more consistent than they appear.

Throughout this discussion an attempt will be made to provide more general results derived from the author's excellent foundation. I hope these additional comments help to clarify any imprecision in the PML concept.

General Results Concerning $X_{(n)}$

This section concisely presents the theory upon which most of the remainder of the paper is based. In addition to the results which appear, the corresponding distribution for $X_{(r)}$ could be given by:

$$f_{x(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!}(F_x(x))^{r-1} f(x)\,(1 - F_x(x))^{n-r}.$$

The reason for introducing this more general result is to allow for the derivation of properties of $X_{(r)}$ similar to those presented for $X_{(n)}$. In particular, it may be shown that the order statistics from a uniform distribution over (0,1), with $u_{(r)} = F_x(x_{(r)})$, have a beta distribution with parameters $a = r$, $b = n - r + 1$.

Therefore, $E(u_{(r)}) = r/(n + 1)$

$\text{Var}(u_{(r)}) = r(n - r + 1)/((n + 1)^2(n + 2))$ for $r = 1,2, \ldots, n$.

Additionally, the first approximations displayed in the paper as (4) and (5) can be extended to:

$$E(X_{(r)}) \doteq F_{xx}^{-1}(r/(n + 1))$$

$$\text{Var}(X_{(r)}) \doteq r(n - r + 1)/((n + 1)^2(n + 2))\,(f_x[F_x^{-1}(r/(n + 1))])^{-2}.$$

These results form the basis of the author's initial three estimates of PML. Using the generalized forms above (with $r = 100\,\alpha$), estimates for our PML_α may be computed as follows:

Method	Estimates for : $\mathrm{PML}_{.90}$	$\mathrm{PML}_{.95}$	$\mathrm{PML}_{.99}$	$\mathrm{PML}_{1.00}$
1) $X_{(r)}$ from sample data	\$331,179	\$434,449	\$563,899	\$576,525
2) $E(X_{(r)})$	344,158	404,453	516,532	589,468
3) $E(X_{(r)}) + 2(\mathrm{Var}\,(X_{(r)}))^{1/2}$	399,632	482,839	662,380	803,420

Methods 2 and 3 assume an underlying lognormal distribution with μ = \$212,521 and σ = \$110,506.

It may be noted that the $\mathrm{PML}_{1.00}$ estimates are those derived in the paper under the author's definition of PML.

Using $X_{(r)}$ *As An Estimate for the PML*

Although this is obviously the most convenient approach, it relates only to the data that are available from reported claims and may not be an accurate indication of the underlying exposure in the future. For example, immature loss history may not show any losses in excess of a few thousand dollars. Should the PML be chosen to be the largest claim paid to date, or the largest reported claim, or some other choice?

From another point of view, suppose $X_{(99)}$ = \$400,000 and the largest claim $X_{(100)}$ = \$2,000,000. Is the \$2,000,000 loss catastrophic and, by definition, not probable? Clearly $X_{(n)}$ alone should not be used in any of these cases and judgment would play a critical role in the choice of an appropriate PML.

I would also add that, technically, this method could have been described as distribution-free in Exhibit III since it requires no assumption regarding the underlying probability distribution.

Distribution-Free Bounds for $E(X_{(n)})$

The advantage of a reliable distribution-free bound for any variable is obvious. Hopefully, some work may be done in the future to test the sensitivity of this bound with regard to accuracy for various distributions.

The clever use of the Schwartz inequality was a novel application to this realm of actuarial science. In fact, this same technique may be used to derive the generalized result:

$$E(X_{(r)}) \leq \mu + \sigma \left[\frac{B(2r-1, 2n-2r+1)}{(B(r, n-r+1))^2} - 1 \right]^{1/2}$$

where $B(a,b,) = \dfrac{\Gamma\,(a)\,\Gamma\,(b)}{\Gamma\,(a+b)}$.

Assuming the same lognormal distribution as mentioned above, the following bounds may be computed:

$$
\begin{aligned}
E(X_{(90)}) &\leq \$531{,}509\\
E(X_{(95)}) &\leq 590{,}319\\
E(X_{(99)}) &\leq 756{,}736\\
E(X_{(100)}) &\leq 988{,}044
\end{aligned}
$$

General Results For Quantiles

The introduction of the pth quantile technique is a useful concept for quantifying the meaning of "probable" in PML. Based upon the discussion of order statistics both in the paper and above, it is easily seen that a reasonable estimator of k_p is the rth sample order statistic $x_{(r)}$ where: i) $F_x\,(k_p) = p$
ii) $r = np$ for p fixed.

It is interesting to note that this sample quantile estimate $X_{(r)}$ is asymptotically distributed as a normal variate; i.e.

$$X_{(r)} \to N(k_p,\, p(1-p)/n\,f^2\,(k_p)$$

for $r = np$, as n increases with p fixed.

The author has provided the technique for approximating the appropriate moments of this distribution by differentiating the Taylor series; namely,

$$
\begin{aligned}
E(X_{(r)}) &= E(F_x^{-1}\,(u_{(r)})) \doteq F_x^{-1}\,(E(u_{(r)}) = F_x^{-1}\,(r/(n+1))\\
&\qquad\qquad\qquad\qquad\qquad = F_x^{-1}\,(np/(n+1))\\
&\qquad\qquad\qquad\qquad\qquad \to F_x^{-1}\,(p) = k_p\\
\mathrm{Var}\,(X_{(r)}) &= \mathrm{Var}\,(F_x^{-1}\,(u_{(r)})) \doteq \mathrm{Var}\,(u_{(r)})\,(f_x\,(E(F_x^{-1}\,(u_{(r)}))^{-2}\\
&= \frac{r(n-r+1)}{(n+1)^2(n+2)}\,(f_x\,(F_x^{-1}\,(E(u_{(r)}))^{-2}\\
&= \frac{np(n-np+1)}{(n+1)^2(n+2)}\,(f_x\,(F_x^{-1}\,(E(np/(n+1)))^{-2}\\
&\to \frac{p(1-p)}{n}\,\frac{1}{f_x^{-2}\,(k_p)}
\end{aligned}
$$

This analysis demonstrates the theory behind the intuitive appeal of using $X_{(r)}$ as an estimate of k_p, which can be interpreted as PML_p as defined above.

Distribution-Free Confidence Interval For k_p

The clarity of this section is enhanced by the interesting heuristic explanation of the result:

$$P(X_{(r)} < k_p < X_{(s)}) = \sum_{i=r}^{s-1} \binom{n}{i} p^i (1 - p)^{n-i}$$

as a binomial distribution.

As a technical note, the equation which appears in the paper as:

$$P(X_{(r)} < k_p < X_{(s)}) = P(F_x(X_{(s)}) < p) - P(F_x (X_{(r)}) < p)$$

may be expressed as:

$$P(X_{(r)} < k_p < X_{(s)}) = [1/B(s,n - s + 1)] \int_0^p x^{s-1} (1 - x)^{n-s}\, dx$$

$$- [1/B(r,n - r + 1)] \int_0^p x^{r-1} (1 - x)^{n-r}\, dx.$$

These integrals may be evaluated by means of an Incomplete Beta Function Table, a method which appears more efficient than actually calculating the various binomial probabilities.

Since the distributions of $X_{(r)}$, and hence k_p, are severely skewed, similar results for $p = .99$ and $p = 1.00$ are not practical. However, I performed the related calculations for $p = .90$, $\alpha = .10$ with the following results:

$$P(X_{(83)} < k_p < X_{(94)}) = .887349$$
$$P(X_{(85)} < k_p < X_{(95)}) = .902531$$
$$P(X_{(86)} < k_p < X_{(96)}) = .903715$$
$$P(X_{(87)} < k_p < X_{(97)}) = .868286$$

By minimizing $s - r$ and $X_{(s)} - X_{(r)}$, we would choose the upper bound for $k_{.90}$ as $X_{(95)} = \$434,449$.

Summary

The author's results as displayed in Exhibit III are not as disparate as they may appear at first glance if the various methods are recognized for what they are designed to produce. By allowing PML_α to be defined as I have suggested above, the consistency of the techniques proposed in the paper are better demonstrated as follows.

Method	Estimates for : PML$_{.90}$	PML$_{.95}$	PML$_{.99}$	PML$_{1.00}$
1) $X_{(r)}$	$331,179	$434,449	$563,899	$576,525
2) $E(X_{(r)})$	344,158	404,453	516,532	589,468
3) $E(X_{(r)}) + 2(\mathrm{Var}\ (X_{(r)}))^{1/2}$	399,632	482,839	662,380	803,420
4) Upper bound for $E(X_{(r)})$	531,509	590,319	756,736	988,044
5) $X_{(r)}$ as an estimate of k_p	331,179	434,449	563,899	576,525
6) Upper bound for k_p	434,449	563,899	—	—

I believe this type of analysis would be extremely informative to individuals charged with determining proper retention limits on a per occurrence basis as well as to the underwriter and actuary who must set a price for the related excess coverage. In addition, these methods could similarly be used on an aggregate basis to help select appropriate stop-loss thresholds. In one sense this latter approach would imply the existence of a new concept which is the aggregate analog to the PML$_\alpha$. Perhaps this new term could be defined as:

Definition: The Probable Maximum Aggregate Loss at the α level (PMAL$_\alpha$) is that amount (or proportion of total value) which will equal or exceed the accumulation of all losses to the risk during a fixed period of time with probability 100α %.

For example, if PMAL$_{.95}$ = $1,000,000, you would expect the aggregate loss over a particular period to be less than $1,000,000 ninety-five percent of the time. Expressed differently, it could be stated that the actual aggregate loss for the risk is expected to exceed $1,000,000 five percent of the time, or once every twenty similar periods.

Conclusion

Ms. Wilkinson has provided the literature with a number of valuable techniques for analyzing and determining estimates of the Probable Maximum Loss. The clarity of presentation and the numerous intuitive explanations are excellent pedagogical methods to utilize in the discussion of a term (PML) as familiar to the non-technician as it is to the actuary.

My suggested generalizations were introduced only to present further applications of the author's ideas as well as to, hopefully, clarify what I perceived to be an ambiguity in the definition of PML. As mentioned above, these generalized results actually give the results of Exhibit III in the paper a greater semblance of consistency than it may seem to display initially.

With regard to further study of this topic, I would be very interested in seeing more work done analyzing the accuracy of these estimates. In particular,