ON THE THEORY OF INCREASED LIMITS AND EXCESS OF LOSS PRICING

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Since the time of Jeffrey Lange's paper on increased limits[1] in 1969 much has happened to the market for increased limits in the liability lines of insurance. Insureds, particularly commercial insureds, are now interested in purchasing liability coverage with limits in the millions of dollars. This reflects the concern of insureds about exposure to inflation which has greatly increased the magnitude of jury awards and settlements in recent years. The ability of the insurance industry to provide liability insurance for this market greatly depends on sensible pricing.

Currently, in liability insurance there is little experience on losses in excess of $500,000 per occurrence. Indeed the probability that a loss will exceed $500,000 has been quite small. Furthermore, because of the great statistical variation of large losses, there will always be a limit to the credibility of data for making increased limits factors, especially for high limits. Consequently, there will always be a need for judgment in the pricing of high limits and excess of loss coverage.

This paper presents the mathematical foundations of the pricing of increased limits and excess of loss coverage. The paper will attempt to tie together the various aspects of this area of insurance pricing in a logical, straightforward manner by means of a mathematical model. It is hoped that this model will be helpful in making pricing judgments or evaluating such judgments.

Section one presents the mathematical model of expected value pricing by considering frequency and severity separately. An insurance cost function is introduced into the model that should aid greatly in understanding the mathematics of insurance pricing. Such functions are defined for increased limits and excess of loss coverage and are used to derive increased limits factors in concise mathematical terms. A simple formula is found that relates

[1] J. T. Lange, "The Interpretation of Liability Increased Limits Statistics," *PCAS* Vol. LVI (1969), p. 163.

a set of increased limits factors to the loss severity distribution underlying them. From this formula a very useful and convenient test is developed that can identify pricing inconsistencies. The implications of trend in the model are investigated and a commonly used method of adjusting increased limits factors for trend is shown to be undesirable.

Section two considers risk and its relationship to pricing. The variance principle of risk loadings is used to determine a risk charge for increased limits and excess of loss pricing. An analysis of spreading the risk by layering of coverage and reinsurance is also presented and demonstrates a reduction in risk by layering.

Section three describes some ways to treat the many difficulties and practical problems in applying the theory of increased and excess of loss pricing, particularly in regard to obtaining severity distributions.

Section four discusses four related areas of pricing—three in liability insurance and one in property insurance. The mathematics of the leveraged effect of inflation are presented and the consistency test is applied to increased limits factors for aggregate and split limits. Finally, the potential value of the consistency test in other lines of insurance is shown by an application to a similar pricing problem in coinsurance.

The paper will treat only the pure loss element of ratemaking. There are many practical problems concerning expenses, particularly loss adjustment expenses, which cannot be resolved solely by this model.

1. EXPECTED VALUE PRICING

Traditional actuarial ratemaking is predicated on the estimation of expected, mean, or average values. As will be discussed later, these methods can be sufficient for most ratemaking problems. In this section, a general model of expected value pricing is presented and then applied to increased limits. In addition, a test of increased limits factors is developed. Excess of loss coverage is also considered along with an analysis of two different methods of trend adjustment. The next section deals with the determination of a risk loading appropriate for increased limits and excess of loss coverage.

The General Pricing Model

Let us describe the general insurance ratemaking or pricing problem in mathematical terms with the following definitions:

1. Let n be a random variable representing the number of accidents (occurrences) an insured will have over the course of one year (the usual policy period). This is the loss frequency variable.

2. Let x be a random variable representing the dollar amount of damage which the insured incurs given an accident has occurred. This is the loss severity variable.

3. Let g be a function of x representing the dollar amount of coverage[2] provided by the insurer for a loss of size x.[3] This is the insurer's cost function. If $F(x)$ is the cumulative distribution function of x then we can express $E[g(x)]$ in terms of $F(x)$ as follows:

$$\begin{aligned} E[g(x)] &= \int_0^\infty g(x)\, dF(x), \text{ or} \\ &= \int_0^\infty g(x) \cdot f(x)\, dx, \text{ where } f(x) = \frac{dF(x)}{d(x)} \end{aligned}$$

4. Let y be a random variable representing the total dollars of insured losses that an insured will have in one year. This is the pure premium variable.

While y is not easily expressed in terms of n and $g(x)$, we can express the expected value of y, $E[y]$, as

$$E[y] = E[g(x)] \cdot E[n] \tag{1}$$

Equation (1) is merely the mathematical expression for the division of the

[2] The function g represents those coverage provisions which depend only on the size of loss and which treat each loss individually and identically.

[3] As will be mentioned later, the size of a loss may depend on the amount of coverage. The present discussion assumes independence.

average pure premium into the average size of insured loss and the average frequency of loss. The derivation of this equation can be shown as follows:

1. Assume that the distribution of the size of each loss does not depend on how many losses occur during the year under each policy. That is, frequency and severity are independent.

2. Assume also that if more than one loss occurs in a year for a policy, then the size of each loss is independent of the size of any of the other losses.

3. Hence, if n insured losses occur during the year under a given policy, then the expected value of the sum of those losses is equal to n times the expected value of one such loss,

$$E[y \mid n] = n \cdot E[g(x)]$$

4. The expected value of y, the total dollars of insured losses incurred during the year for a given policy, is given by taking the expected value of $E[y \mid n]$ with respect to the random variable n.

5. Therefore,
$$\begin{aligned} E[y] &= E_n\,(E[y \mid n]) \\ &= E_n\,(n \cdot E[g(x)]) \\ &= E[n] \cdot E[g(x)] \end{aligned}$$

Increased Limits Coverage

In liability insurance, a policy generally covers such loss in full up to a specified maximum dollar amount that will be paid on any one loss. If k is such a policy limit then we can express the cost function, $g(x;k)$, for this coverage as

$$g(x;k) = \begin{cases} x, & 0 < x < k, k > 0 \\ k, & x \geq k \end{cases} \tag{2}$$

and

$$\begin{aligned} E[g(x;k)] &= \int_0^k x\, dF(x) + k \cdot \int_k^\infty dF(x) \\ &= \int_0^k x\, dF(x) + k \cdot [1 - F(k)] \end{aligned} \tag{3}$$

It is general practice to publish rates for some standard limit called the basic limit, b. Increased limits rates are expressed as a factor, $I(k)$, for a limit k to be applied to the basic limit pure premium rate. The mathematical expression for the increased limits factor is the ratio of expected total losses with k limit coverage to expected total losses with basic limit coverage. Thus, using equation (1) we have

$$I(k) = \frac{E[g(x;k)] \cdot E[n]}{E[g(x;b)] \cdot E[n]} = \frac{E[g(x;k)]}{E[g(x;b)]} \tag{4}$$

We can see that the increased limits factor is dependent only on loss severity and the cost function, but not on loss frequency. Note that $E[g(x;b)]$ is simply the average basic limits severity and will be hereafter referred to as ABLS. Consequently, if we know the appropriate loss severity distribution then we can use equations (3) and (4) to determine expected value increased limit factors for various limits.

As will be seen later, the compilation of a loss severity distribution from experience data can be very difficult and in some cases may not be feasible. Consequently, considerable judgment is needed to develop increased limits factors. In many instances it may be easier to make judgments in terms of specific increased limits factors rather than working with loss severity distributions. Therefore, it would be helpful to analyze the loss severity distribution underlying a given set of increased limits factors. The derivation of the necessary mathematical expression is as follows:

$$I(k) = \frac{E[g(x;k)]}{ABLS}, \text{ where } ABLS = E[g(x;b)]$$

$$= \frac{1}{ABLS} \cdot \left(\int_0^k x\, dF(x) + k[1 - F(k)] \right)$$

$$\frac{dI(k)}{dk} = \frac{1}{ABLS} \cdot \left(k \cdot \frac{dF(k)}{dk} + 1 - F(k) - k \cdot \frac{dF(k)}{dk} \right)$$

$$\frac{dI(k)}{dk} = I'(k) = \frac{1 - F(k)}{ABLS} \tag{5}$$

or

$$I'(k) = \frac{G(k)}{ABLS}, \text{ where } G(x) = 1 - F(x) \tag{6}$$

Solving for $F(k)$, the underlying severity distribution, we get

$$F(k) = 1 - ABLS \cdot I'(k) \tag{7}$$

In words, this result shows that the probability that a loss will be greater than k is equal to the product of the average basic limits severity and the rate of change in the increased limits factors at k. Thus we see that there is a loss severity distribution implicitly defined by any set of expected value increased limit factors. Note that the specific distribution is, as we might have suspected, a function of the average basic limits severity. In theory, $I(k)$ and therefore $I'(k)$ exist for all $k > 0$. However, in practice $I(k)$ is defined only for $k > b$ as the term "increased limits" implies. Consequently, any practical application of equation (7) to estimate $F(k)$ from a set of increased limits factors would be limited to sizes of loss greater than the basic limit.

Aside from deriving specific distributions we can also use this relationship to determine general properties of expected value increased limit factors from those of distribution functions.

1. As k approaches ∞, $F(k)$ will approach 1, $I'(k)$ will approach zero, and $I(k)$ will approach some constant. If $I(k)$ becomes constant for all k greater than some value M, then $I'(k) = 0$ and $F(k) = 1$ whenever $k > M$ because there is no probability of a size of loss greater than that value of k. This would imply no additional charge for higher limits.

2. Since $F(k)$ is monotonic increasing, $I'(k)$ will be monotonic decreasing. If $F(k)$ has a point of inflection, then so will $I'(k)$ at the same value of k. The converse of both statements also holds.

3. The probability density function, $f(k)$, can be expressed as follows:

$$f(k) = \frac{dF(k)}{dk} = -\,ABLS \cdot \frac{d^2I(k)}{dk^2} \tag{8}$$

Consequently,

$$I''(k) = \frac{d^2I(k)}{dk^2} = \frac{-\,f(k)}{ABLS}$$

Note that $I''(k)$ can never be positive since $f(k)$ and ABLS should always be positive. Consequently, to avoid the implication of negative probabilities, $I'(k)$ must be monotonically decreasing and $I(k)$ must be strictly increasing[4]. Also, any modes in $f(k)$ will correspond to inflection points in $I'(k)$.

[4] It is permissible for $I(k)$ to reach some limit and stay there for all larger values of k.

The Consistency Test

Using property (3) above, we can construct a "consistency" test for evaluating a given set of increased limits factors. The marginal premium per $1000 of coverage should decrease as the limit of coverage increases. If not, this implies negative probabilities. For example, consider the following set of increased limits factors for per occurrence limits between $25,000 and $10,000,000.

Per Occurrence Limit (in thousands of dollars)	Increased Limits Factor	Marginal Rate[5] per $1000 of Coverage
25	1.000	—
50	1.250	.0100
100	1.425	.0035
200	1.625	.0020
250	1.705	.0016
300	1.775	.0014
350	1.865	.0018*
400	1.915	.0010
500	1.975	.0006
750	2.175	.0008*
1000	2.400	.0009*
1250	2.575	.0007*
1500	2.700	.0005
1750	2.825	.0005
2000	2.950	.0005
2500	3.100	.0003
3000	3.300	.0004*
4000	3.600	.0003
5000	3.800	.0002
7500	4.300	.0002
10000	4.800	.0002

This set of increased limits factors is "inconsistent" at the indicated (*) limits of 350, 750, 1000, 1250, and 3000. These factors are very similar to factors actually in use until 1975.

[5] The Marginal Rate is the difference in increased limits factors between the given limit and the next lower limit, divided by the difference in the limits.

Aside from the mathematical interpretation of this consistency test, it has a very practical meaning. In general, it does not make sense to the insurance buyer to have to pay more for each additional $1000 of coverage since the probability of losses larger than some limit should be less than for a lower limit. Of course there can be anti-selection, that is where the existence of higher limits influences the size of the suit, award or settlement. However, this should not restrict the general applicability of the consistency test. Other applications of the consistency test will be described later in the paper.

Excess of Loss Coverage

In general, an excess of loss contract or non-proportional reinsurance arrangement covers losses greater than a given amount, r, the retention and has a maximum liability of j. Any loss, x, exceeding r is insured for the amount $x - r$, up to the maximum j. We can express the excess of loss cost function, $h(x;r,j)$, as follows:

$$h(x;r,j) = \begin{cases} 0, 0 < x \leq r \\ x - r, r < x < s, s = r + j \\ j, x \geq s \end{cases} \tag{9}$$

and therefore,

$$\begin{aligned} E[h(x;r,j)] &= \int_r^s (x - r)\, dF(x) + j[1 - F(s)] \\ &= \int_r^s x\, dF(x) - r[F(s) - F(r)] + j[1 - F(s)] \\ &= \int_r^s x\, dF(x) + s[1 - F(s)] - r[1 - F(r)] \end{aligned} \tag{10}$$

Consequently,

$$E[h(x;r,j)]^{6} = E[g(x;s)] - E[g(x;r)] \tag{11}$$

Note that the expected number of accidents, E[n], has not changed just because losses less than the retention, r, are not insured under an excess

[6] Equation (11) can also be derived by observing that $h(x;r,j) = g(x;s) - v(x;r)$.

contract. Consequently, the expected value pure premium for the excess contract, is given by

$$E[h(x;r,j)] \cdot E[n]$$

and can be expressed in terms of the basic limits pure premium, ABLS • E[n], as follows,

$$\begin{aligned} \frac{E[h(x;r,j)] \cdot E[n]}{ABLS \cdot E[n]} &= \frac{E[h(x;r,j)]}{ABLS} \qquad (12) \\ &= \frac{E[g(x;s)]}{ABLS} - \frac{E[g(x;r)]}{ABLS} \\ &= I(s) - I(r) \end{aligned}$$

This proves mathematically that the expected value pure premium for an excess contract is equal to the difference in expected value pure premiums of two "first dollar" contracts.

Trend

Inflationary pressures, both economic and social, increase the size of losses over time. Inflationary trends can have substantial effects on pricing increased limits and excess of loss coverages. These effects are difficult to evaluate since the limits and retentions remain fixed while loss severity is shifting[7].

We can investigate the mathematical aspects of trend in terms of a transformation of the loss severity variable. Let's assume that the economic and social values that produce a loss of size x are changing such that a loss of size x' will be produced by the new values after a fixed period of time (one year). This can be described mathematically by equating the probability of a loss size less than or equal to x at a given point in time, with the probability of a loss size less than or equal to x' one year later. If $F(x')$ is the cumulative distribution function of x', then

$$F(x') = F(x) \qquad (13)$$

[7] An alternative method of setting retentions is described by Ferguson, R. E., "Non-proportional Reinsurance and the Index Clause," *PCAS* Vol. LVI, (1974), p. 141. The method attempts to reduce the excess carrier's pricing problem caused by inflation.

Let $\alpha(x)$ represent the transformation that describes the relationship between x' and x.

$$x' = \alpha(x) \tag{14}$$

Assuming that $\alpha(x)$ is monotonic, we find

$$F(\alpha(x)) = F(x) \tag{15}$$

Also, since $x = \alpha^{-1}(x')$, we can write

$$F(x') = F(\alpha^{-1}(x')) \tag{16}$$

In the simple case each loss is increased by the same multiplicative factor, a, which is greater than one:[8]

$$\begin{aligned} x' &= \alpha_1(x) \\ &= ax \end{aligned}$$

Here we have $\alpha_1^{-1}(x') = x'/a$, thesefore using equation (16) we find

$$F_1(x') = F(x'/a)$$

Now we would like to know what the trended increased limits factor for policy limit k, $I_1(k)$, should be. Starting from equation (3) using x' and $F_1(x')$,

$$\begin{aligned} E[g(x';k)] &= \int_0^k x'\, dF_1(x') + k[1 - F_1(k)] \\ &= \int_0^k x'\, dF(x'/a) + k[1 - F(k/a)] \end{aligned}$$

Letting $u = x'/a$,

$$\begin{aligned} E[g(x';k)] &= a \cdot \int_0^{k/a} u\, dF(u) + k[1 - F(k/a)] \qquad (17) \\ &= a \cdot E[g(x;k/a)] \end{aligned}$$

[8] This type of trend and its relationship to basic limits trend are studied by Finger, R. J., "A Note on Basic Limits Trend Factors", *PCAS* Vol. LXIII, (1976), p. 106.

Consequently, applying the development of equation (4) to the trended severity we get,

$$I_1(k) = \frac{E[g(x';k)]}{ABLS_1} \tag{18}$$

$$= \frac{a \cdot E[g(x;k/a)]}{a \cdot E[g(x;b/a)]}$$

$$= \frac{I(k/a)}{I(b/a)}$$

For excess of loss coverage,

$$I_1(s) - I_1(r) = \frac{I(s/a) - I(r/a)}{I(b/a)} \tag{19}$$

Also note that differentiating equation (18) gives

$$I_1'(k) = \frac{1}{a} \cdot \frac{I'(k/a)}{I(b/a)} \tag{20}$$

$$= \frac{1}{a} \cdot \frac{G(k/a)}{E[g(x;b/a)]} \text{ (from equation (6))}$$

There is another more commonly used approach to updating increased limits factors for trend. The procedure considers separately:

1) the trend in average severity for basic limits, t_b, and

2) the trend in average severity for increased limits, t_i.

If $ABLS_2$ is the average basic limits severity after one year of inflation, then

$$ABLS_2 = t_b \cdot \text{ABLS}$$

Every "layer or loss" in excess of basic limits is similarly inflated by t_i where such a layer is defined by the excess portion of the increased limits factor, I(k) − 1.

For purposes of comparison with the first trend method or other methods, we would like to know the transformation, $\alpha_2(x)$, implied by this second trend procedure.

We can express this procedure as follows:

$$I_2(\mathrm{k}) - 1 = \mathrm{t_l} \cdot (\mathrm{I(k)} - 1) \tag{21}$$

hence,

$$I_2(\mathrm{k}) = 1 + \mathrm{t_l} \cdot (\mathrm{I(k)} - 1) \tag{22}$$

also,

$$I'_2\ (\mathrm{k}) = \mathrm{t_l} \cdot \mathrm{I'(k)} \tag{23}$$

$$= \mathrm{t_l} \cdot \frac{\mathrm{G(k)}}{\mathrm{ABLS}} \text{ (from equation (6))}$$

and also from equation (6) we know

$$I'_2\ (\mathrm{k}) = \frac{G_2(\mathrm{k})}{ABLS_2}\text{, where } G_2(\mathrm{k}) = 1 - F_2(\mathrm{k})$$

thus solving for $G_2(\mathrm{k})$ we find

$$G_2(\mathrm{k}) = \frac{ABLS_2}{\mathrm{ABLS}} \cdot \mathrm{t_l} \cdot \mathrm{G(k)} \tag{24}$$

$$= \mathrm{t_b} \cdot \mathrm{t_l} \cdot \mathrm{G(k)}$$

Now using equation (15) we see that

$$F_2(\alpha_2(\mathrm{x})) = \mathrm{F(x)}$$

Therefore,

$$1 - F_2(\alpha_2(\mathrm{x})) = 1 - \mathrm{F(x)}$$

$$G_2(\alpha_2(\mathrm{x})) = \mathrm{G(x)}$$

$$\alpha_2(\mathrm{x}) = G_2^{-1}(\mathrm{G(x)})$$

But from equation (24) we find

$$G_2^{-1}\ (\mathrm{x}) = \mathrm{G}^{-1}\left(\frac{\mathrm{x}}{\mathrm{t_b} \cdot \mathrm{t_l}}\right)$$

Hence,

$$\alpha_2(\mathrm{x}) = \mathrm{G}^{-1}\left(\frac{\mathrm{G(x)}}{\mathrm{t_b} \cdot \mathrm{t_l}}\right) \tag{25}$$

We see that $\alpha_2(x)$ is defined in terms of the original severity distribution. In order to see what kind of function $\alpha_2(x)$ is, we can make some assumptions about the severity distribution.

1. If the severity distribution is exponential,

$$G(x) = \exp(-\beta x)$$

$$G^{-1}(x) = \frac{-\ln^{x}}{\beta}$$

$$\alpha_2(x) = x + \frac{1}{\beta} \cdot \ln(t_b \cdot t_i), \text{ where } \frac{1}{\beta} \cdot \ln(t_b \cdot t_i) \text{ is a constant.}$$

2. If the severity distribution is Weibull,

$$G(x) = \exp(-x^B/A)$$

$$G^{-1}(x) = (-A \cdot \ln x)^{1/B}$$

$$\alpha_2(x) = (x^B + A \cdot \ln(t_b \cdot t_i))^{1/B}$$

3. If the severity distribution is lognormal, a general solution is not available. However, $\alpha_2(x)$ can be computed using numerical approximation techniques.

4. If the only form of the severity distribution is given by a set of increased limits factors represented by $I(x)$, then

$$I'(x) = \frac{G(x)}{ABLS}$$

$$G(x) = ABLS \cdot I'(x)$$

$$G^{-1}(x) = I'^{-1}(x/ABLS)$$

$$\alpha_2(x) = I'^{-1}\left(\frac{I'(x)}{t_b \cdot t_i}\right)$$

Exhibit I gives numerical examples of $\alpha_2(x)$ for the exponential ($\beta = 2.54 \times 10^{-5}$), the Weibull ($A = 42.1898$, $B = .42045$) and the lognormal ($\mu = 8.9146$, $\sigma = 1.7826$) loss severity distributions where $t_b = 1.08$ and $t_i = 1.20$.

EXHIBIT 1

INFLATION BY SIZE OF LOSS UNDER $\alpha_2(x)$

Size of Loss (x)	Exponential $\alpha_2(x)$	Exponential $\alpha_2(x) \div x$	Weibull $\alpha_2(x)$	Weibull $\alpha_2(x) \div x$	Lognormal $\alpha_2(x)$	Lognormal $\alpha_2(x) \div x$
25,000	35,207	1.408	35,207	1.408	35,207	1.408
50,000	60,207	1.204	64,870	1.297	66,241	1.324
100,000	110,207	1.102	121,796	1.217	126,686	1.266
200,000	210,207	1.051	232,102	1.160	244,968	1.224
250,000	260,207	1.040	286,392	1.145	303,432	1.213
300,000	310,207	1.034	340,330	1.134	361,603	1.205
350,000	360,207	1.029	393,997	1.125	419,549	1.198
400,000	410,207	1.025	447,447	1.118	477,303	1.193
500,000	510,207	1.020	553,840	1.107	592,368	1.184
750,000	760,207	1.013	817,784	1.090	878,140	1.170
1,000,000	1,010,207	1.010	1,079,853	1.079	1,162,097	1.162
1,500,000	1,510,207	1.006	1,600,654	1.067	1,726,564	1.151
2,000,000	2,010,207	1.005	2,118,657	1.059	2,288,093	1.144
2,500,000	2,510,207	1.004	2,634,833	1.053	2,847,429	1.138
3,000,000	3,010,207	1.003	3,149,689	1.049	3,405,168	1.135
4,000,000	4,010,207	1.002	4,176,574	1.044	4,517,525	1.129
5,000,000	5,010,207	1.002	5,200,716	1.040	5,626,732	1.125
7,500,000	7,510,207	1.001	7,753,427	1.033	8,388,452	1.118
10,000,000	10,010,207	1.001	10,298,950	1.029	11,144,829	1.114

From the examples, it appears that $\alpha_2(x)$ will generally produce higher trends for small sizes of loss and lower trends for the large losses. Intuitively, it seems that $\alpha_2(x)$ might not be as good a representation of real loss trends as using the same trend for all sizes of loss, represented by $\alpha_1(x)$. In fact, it is more likely that the reverse of $\alpha_2(x)$ is true, i.e., lower trend for small losses, higher trend for large. The reason for these results with $\alpha_2(x)$ stems from the assumption made that *all* excess layers should receive the *same* trend factor. If indeed there is a difference between basic and excess trend, then should not different excess layers be trended differently? This is the contradiction implied by $\alpha_2(x)$.

Consequently, it is preferable to use $\alpha_1(x)$ rather than $\alpha_2(x)$ to adjust for trend.

2. THE CHARGE FOR RISK

A major problem with expected value pricing is that it fails to appropriately charge for the risk of being in the insurance business. Premium rates are usually determined from the expected pure premium, a provision for expenses, and a loading for profit and contingencies. For most lines of insurance, the profit and contingency loading presumably compensates for risk. However, the loading is usually low, and it therefore will only be adequate for relatively low risk lines or coverages. A more volatile line or coverage needs an additional safety loading or risk charge for the added risk in order for it to be on a par with the other lines or coverages. Consequently, while the general provision for profit and contingencies may be sufficient for most lines of insurance, it can be seriously deficient for a high risk line or coverage.

For the purpose of this paper, the meaning of risk will be associated with the degree of uncertainty in the pure premium. It is assumed that one who is averse to risk will desire stability and certainty. Given the choice between insuring ten individuals with $1,000,000 limits each or one insured at $10,000,000, a risk-averse actuary should argue for the ten separate policies in order to reduce the likely variation from the expected losses.[9] However, there should be some risk charge that would make such an actuary indifferent between the two choices based on some rational and objective criteria. Even though attitudes and preferences towards risk can be highly subjective, some measure of risk is desired to establish a reasonable standard for determining such a risk charge. In an article on risk and ratemaking, Lange[10] suggests a measure of risk based on the concept of variance. The discussion that follows will attempt to apply this idea to increased limits and excess of loss pricing.

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[9] An additional or alternative criteria in such a situation is to consider the probability of ruin as a basis for actuarial decision making. Utility theory presents yet another decision rule. See Bühlmann, H., *Mathematical Methods in Risk Theory*, Springer-Verlag (1970), p. 85-87, and Freifelder, L. R., *A Decision Theoretic Approach to Insurance Ratemaking*, Irwin (1976), p. 36-56.

[10] J. T. Lange, "Application of a Mathematical Concept of Risk to Property-Liability Insurance Ratemaking," *Journal of Risk and Insurance*, Vol. XXXVI, p. 383.

Sources of Risk

There are two main sources of risk associated with insurance ratemaking. First, variation between actual losses and expected losses can be the result of the stochastic or random nature of the frequency and severity of insurance losses. Freifelder[11] calls this the "process risk". Second, such variation can also result from an inability to estimate expected losses accurately.[12] This is appropriately termed the "parameter risk" by Freifelder. A major cause of difficulty in estimating expected losses in some types of insurance is the occurrence of catastrophes such as hurricanes, tornadoes, earthquakes, etc. Inflationary trends also have a substantial impact in estimating expected losses. For a line of insurance, changes in the mix of business among various classes, coverages and types of insureds can affect expected losses. A small independent insurer is faced with sampling error in estimating expected losses. Incorrect ratemaking data is always a potential problem. Finally, claims practices, underwriting practices, social attitudes, and judicial or legislative climate can undergo drastic and rapid changes which can not always be anticipated to adjust expected losses adequately.

While parameter risk can be substantial, the determination of a risk charge to compensate for this risk is very difficult and is beyond the scope of this paper. In an area such as a catastrophe cover for hurricanes, floods, etc. the parameter risk can be quite large and cannot be ignored. However in many applications the parameter risk should be minimal. This paper will only study the effects of the process risk and develop appropriate risk charges for such risk.

Variance as a Measure of Risk

The source of risk used in this paper for the determination of risk charges for various liability limits is the chance or random variation in the pure premium, i.e. the process risk. As will be shown, this source produces a substantial, measurable difference in risk charge by limit of liability. If we define the measure of this risk as the standard deviation of the pure premium as Lange[13] suggests, we can analyze the properties of risk and risk charges for increased limits and excess of loss coverages. However, the variance of the pure premium is felt to be a more appropriate measure be-

[11] Freifelder, *op. cit.*, p. 70-71.

[12] This second source of risk can also be considered to include errors in estimating any of the moments or parameters which determine the form or shape of the frequency and severity distributions.

[13] Lange, *op. cit.*, p. 386.

cause it satisfies the three basic ratemaking axioms advanced by Freifelder[14] and has some weighty theoretical advantages discussed by Bühlmann[15]. Also, as will be shown, it permits the development of risk adjusted increased limits factors from the severity distribution alone.

The formula for premium determination (excluding expenses) with a safety or contingency loading proportional to risk[16] is

$$\text{Premium} = E[y] + \lambda \cdot \text{Var}[y] \tag{26}$$

Where E[y] is the pure premium and Var[y] is the variance of the pure premium variable. The factor λ must be selected judgmentally. This can be done on the basis of the relative magnitude of Var[y] compared to E[y].

The pure premium variance can be expressed in terms of frequency and severity (assuming independence) as follows:[17]

$$\text{Var}[y] = E[n] \cdot \text{Var}[g(x)] + \text{Var}[n] \cdot E[g(x)]^2 \tag{27}$$

Since

$$\text{Var}[g(x)] = E[g(x)^2] - E[g(x)]^2 \text{ we can write}$$

$$\text{Var}[y] = E[n] \cdot E[g(x)^2] + (\text{Var}[n] - E[n] \cdot E[g(x)]^2 \tag{28}$$

In most cases, the frequency variance, Var[n], will be greater than the expected frequency, E[n]. Therefore at a minimum we should have

$$\text{Var}[y] = E[n] \cdot E[g(x)^2] \tag{29}$$

Note that if the frequency of loss distribution is Poisson, equation (29) is exactly right. In addition, the second moment of the severity of insured losses $E[g(x)^2]$, can be many times larger than the square of the first moment, $E[g(x)]^2$, particularly for excess of loss coverage, since the severity distribution has a long tail. Consequently, if we can assume that the ratio of $E[g(x)^2]$ to $E[g(x)]^2$ will be substantially greater than the ratio of Var[n] − E[n] to E[n], then equation (29) should be adequate for determining risk charges. Further work is needed to test this assumption, how-

14 Freifelder, *op. cit.*, p. 36-56.

15 Bühlmann, *op. cit.*, p. 89-92.

16 This is known as the "variance principle of premium calculation" as discussed by Bühlmann, *op. cit.*, p. 85-87.

17 A. L. Mayerson, D. A. Jones, and N. L. Bowers, Jr., "On the Credibility of the Pure Premium," *PCAS* Vol. LV, (1968), p. 175.

ever it will be used as a first approximation to illustrate the inclusion of risk charges in increased limits factors.

The $E[g(x)^2]$ formulas for the cost functions considered in this paper, $g(x;k)$ and $h(x;r,j)$ as defined in equations (2) and (9) respectively, are given below.

$$E[g(x;k)^2] = \int_0^k x^2\,dF(x) + k^2[1 - F(k)] \tag{30}$$

$$\begin{aligned} E[h(x:r,j)^2] &= \int_r^s (x-r)^2\,dF(x) + (s-r)^2[1 - F(s)],\ s = r + j \\ &= \int_r^s x^2\,dF(x) - 2r \cdot \int_r^s x\,dF(x) + r^2[1 - F(r)] \\ &\quad + (s^2 - 2rs)\,[1 - F(s)] \\ &= \int_r^s x^2\,dF(x) - 2r\left\{\int_r^s x\,dF(x) + s[1 - F(s)] \right. \\ &\quad \left. - r[1 - F(r)]\right\} - r^2[1 - F(r)] + s^2[1 - F(s)] \\ &= \int_r^s x^2\,dF(x) - 2r \cdot E[h(x;r,j)] - r^2[1 - F(r)] \\ &\quad + s^2[1 - F(s)] \\ &= E[g(x;s)^2] - E[g(x;r)^2] - 2r \cdot E\,[h(x;r,j)] \end{aligned} \tag{31}$$

The examples in Exhibit II will demonstrate premium determination including risk charge using equations (26), (1) and (29) for different retentions and policy limits. The assumptions used for Exhibit II are:

1. The expected frequency is the same for each insured. Also, the frequency variance is equal to the expected frequency. The $E[n]$ will be set at 0.10.

2. Insureds are also homogeneous with respect to severity and the severity distribution is given by a lognormal distribution[18] with

[18] The formulas used for approximating the necessary values from the lognormal distribution are given in the Appendix.

parameters $\mu = 8.9146$ and $\sigma = 1.7826$. This distribution has a relatively high coefficient of variation ($\sqrt{\mathrm{Var}[x]}/E[x]$) and therefore is highly skewed. It should illustrate the potential magnitude of the risk charges.

3. A risk charge of 5% of the expected value pure premium will be assumed adequate for \$25,000 policy limits (zero retention). This produces a λ factor of 2.559×10^{-6}.

The increased limits factors from the same severity distribution as used in Exhibit II are shown in Exhibit III both on an expected value basis and risk adjusted. Note that since equation (29) was used to estimate the pure premium variance, the risk adjusted increased limits factors, $I_r(k)$, do not depend on the frequency of loss.

$$\begin{aligned} I_r(k) &= \frac{\text{Premium for policy limit k}}{\text{Premium for basic limit b}} \\ &= \frac{E[n] \cdot E[g(x;k)] + \lambda \cdot E[n] \cdot E[g(x;k)^2]}{E[n] \cdot \mathrm{ABLS} + \lambda \cdot E[n] \cdot E[g(x;b)^2]} \\ &= \frac{E[g(x;k)] + \lambda \cdot E[g(x;k)^2]}{\mathrm{ABLS} + \lambda \cdot E[g(x;b)^2]} \end{aligned} \tag{32}$$

It is important to note that it is not appropriate to determine the risk adjustment for excess of loss coverage from the risk adjusted increased limits factors. This will be discussed further in the next section.

Risk Reduction by Layering

The large risk associated with high limits coverage can be significantly reduced by "vertical" layering. This type of layering can be effected by two methods. The first is by insuring through two or more carriers[19], one carrier providing "first-dollar" coverage and the others excess of loss coverage. The second is through the use of non-proportional reinsurance. In this discussion it will be assumed that the insurance coverage is being provided to a large homogeneous group of insureds.

[19] At some point the number of carriers involved in providing coverage for one insured cannot be increased without expense considerations offsetting the risk reduction.

EXHIBIT II

PREMIUM DETERMINATION INCLUDING RISK CHARGE

Retention	Policy Limits	Expected Value Pure Premium	Pure Premium Variance	Risk Charge ($\lambda \times$ Variance)	Premium (before expenses)
0	25,000	1,113	2.175×10^7	56	1,169
0	50,000	1,579	5.563×10^7	142	1,721
0	100,000	2,083	12.834×10^7	328	2,411
0	300,000	2,811	38.790×10^7	993	3,804
0	500,000	3,074	59.192×10^7	1,515	4,589
0	1,000,000	3,335	95.916×10^7	2,454	5,789
0	1,300,000	3,406	112.144×10^7	2,870	6,276
0	1,500,000	3,439	121.405×10^7	3,107	6,546
0	2,000,000	3,495	140.658×10^7	3,599	7,094
0	3,000,000	3,552	168.506×10^7	4,312	7,864
0	4,000,000	3,581	188.114×10^7	4,814	8,395
300,000	1,000,000	595	37.646×10^7	963	1,558
500,000	1,000,000	365	25.686×10^7	657	1,022
1,000,000	1,000,000	160	12.711×10^7	325	485
2,000,000	1,000,000	57	4.963×10^7	127	184
3,000,000	1,000,000	28	2.561×10^7	66	94

EXHIBIT III

INCREASED LIMITS FACTORS INCLUDING RISK CHARGE

Policy Limit	Increased Limits Factors: Expected Value	Risk Adjusted
25,000 (basic limit)	1.000	1.000
50,000	1.419	1.472
100,000	1.872	2.062
300,000	2.526	3.254
500,000	2.762	3.926
1,000,000	2.996	4.952
1,500,000	3.090	5.600
2,000,000	3.140	6.068
3,000,000	3.191	6.727
4,000,000	3.217	7.181

The risk reduction can be demonstrated mathematically by comparing the pure premium variance with and without layering. First consider one insurer providing high limits coverage. With policy limit equal to k, his cost function is given by $g(x;k)$. The variance in the pure premium without layering, $Var[y_0]$, from equation (27) would be:

$$Var[y_0] = E[n] \cdot Var[g(x;k)] + Var[n] \cdot E[g(x;k)]^2 \tag{33}$$

Next suppose the same coverage is layered between two insurers (or an insurer and reinsurer) where the bottom layer has limit r. The cost functions for the two layers are as follows:

First layer: $g(x;r)$

Second layer: $h(x;r,j)$, where $j = k - r$

Since $g(x;k) = g(x;r) + h(x;r,j)$ we see that the expected value pure premiums for the two layers sum to the non-layered pure premium.

$$E[n] \cdot E[g(x;k)] = E[n] \cdot E[g(x;r)] + E[n] \cdot E[h(x;r,j)]$$

Again using equation (27), the pure premium variances for the two individual carriers[20] are:

[20] The carriers must be entirely separate entities operating from different capital bases. Layering coverage between subsidiaries or affiliates will not produce the desired risk reduction.

First Layer:

$$\mathrm{Var}[y_1] = E[n] \cdot \mathrm{Var}[g(x;r)] + \mathrm{Var}[n] \cdot E[g(x;r)]^2 \tag{34}$$

Second Layer:

$$\mathrm{Var}[y_2] = E[n] \cdot \mathrm{Var}[h(x;r,j)] + \mathrm{Var}[n] \cdot E[h(x;r,j)]^2 \tag{35}$$

For the purpose of comparison, the pure premium variance without layering is needed in terms of the two layers. We can express both $E[g(x;k)]^2$ and $\mathrm{Var}[g(x;k)]$ in terms of $g(x;r)$ and $h(x;r,j)$ as follows:

$$\begin{aligned} E[g(x;k)]^2 &= (E[g(x;r)] + E[h(x;r,j)])^2 \\ &= E[g(x;r)]^2 + 2 \cdot E[g(x;r)] \cdot E[h(x;r,j)] \\ &\quad + E[h(x;r,j)]^2 \end{aligned} \tag{36}$$

and

$$\begin{aligned} \mathrm{Var}[g(x;k)] &= \mathrm{Var}[g(x;r)] + \mathrm{Var}[h(x;r,j)] \\ &\quad + 2 \cdot \mathrm{Cov}[g(x;r),h(x;r,j)] \end{aligned} \tag{37}$$

where

$$\begin{aligned} \mathrm{Cov}[g(x;r),h(x;r,j)] &= E[g(x;r) \cdot h(x;r,j)] \\ &\quad - E[g(x;r)] \cdot E[h(x;r,j)] \end{aligned} \tag{38}$$

However, since

$$\begin{aligned} g(x;r) \cdot h(x;r,j) &= \begin{cases} 0, & 0 \le x \le r \\ r(x - r), & r < x < k,\ k = j + r \\ r \cdot j, & k \le x \end{cases} \\ &= r \cdot h(x;r,j) \end{aligned} \tag{39}$$

we find that

$$\mathrm{Cov}[g(x;r),h(x;r,j)] = (r - E[g(x;r)]) \cdot E[h(x;r,j)] > 0 \tag{40}$$

and therefore, equation (37) becomes

$$\begin{aligned} \mathrm{Var}[g(x;k)] &= \mathrm{Var}[g(x;r)] + \mathrm{Var}[h(x;r,j)] \\ &\quad + 2 \cdot E[h(x;r,j)] \cdot (r - E[g(x;r)]) \end{aligned} \tag{41}$$

Substituting equations (36) and (41) into equation (33), we see that the pure premium variance without layering exceeds the variance with layering by an amount, $R(r,j)$, given by

$$\begin{aligned} R(r,j) &= Var[y_0] - Var[y_1] - Var[y_2] \\ &= 2 \cdot E[n] \cdot E[h(x;r,j)] \cdot (r - E[g(x;r)]) \\ &\quad + 2 \cdot Var[n] \cdot E[g(x;r)] \cdot E[h(x;r,j)] \\ &= 2 \cdot E[h(x;r,j)] \cdot (r \cdot E[n] + E[g(x;r)] \cdot (Var[n] - E[n] \end{aligned} \tag{42}$$

If we assume as before that $E[n] \simeq Var[n]$, then equation (42) simplifies to

$$R(r,j) = 2 \cdot r \cdot E[n] \cdot E[h(x;r,j)] \tag{43}$$

which is just twice the retention times the expected value pure premium of the second layer. This is the reduction in the variance, consequently to get the reduction in the risk charge we multiply by the λ factor[21]. Since there is no reduction in expected value pure premium by layering, the dollar reduction in risk charge is equal to the dollar reduction in premium by layering.

Exhibit IV shows that this reduction by layering can be substantial. The examples in Exhibit IV use the same assumptions as in Exhibit II.

3. APPLICATIONS

The principal applications of the pricing model described in this paper require knowledge of a specific loss severity distribution. The only exception to this is the consistency test. Of course, the development of a severity distribution from experience data is not without difficulties. Special data gathering techniques are required to produce individual losses ranked by size of loss. Loss development also poses certain problems in working with severity distributions. Some approaches to treating these difficulties are outlined below.

One approach to compiling an empirical size of loss distribution is to use all reported claims from a few recent accident (or policy) years. It is very likely that this distribution of immature claim values will change considerably as these claims develop. Some claims with high estimates may be settled for a small amount or adjudicated as no liability. Others which seem unmeritorious initially may ultimately result in very large awards or settlements. Consequently, each open claim has a probability distribution

[21] It is assumed that the same λ factor is appropriate for both carriers.

EXHIBIT IV

RISK REDUCTION BY LAYERING

Total Coverage	First Layer Limit (Second Layer Retention)	Second Layer Expected Value Pure Premium	Premium (before expenses) without layering	Premium Reduction with layering	
1,300,000	300,000	595	6,276	914	(14.6%)
1,500,000	500,000	365	6,547	934	(14.3%)
2,000,000	1,000,000	160	7,094	819	(11.5%)
2,000,000	500,000	421	7,094	1,077	(15.2%)
3,000,000	1,000,000	217	7,864	1,111	(14.1%)
3,000,000	2,000,000	57	7,864	583	(7.4%)
4,000,000	1,000,000	246	8,395	1,259	(15.0%)
4,000,000	2,000,000	86	8,395	880	(10.5%)

of its ultimate value. Hachemeister[22] describes a technique for estimating such loss development distributions conditioned on the age of the claim and its estimated value. The method can also be used to estimate a distribution for unreported and reopened claims. The actual procedure for adjusting a severity distribution for loss development using the Hachemeister technique will be left to the interested reader.

There are many other problems in dealing with empirical distributions. Data on individual losses usually come from different policies with different policy limits, causing a bias in the distribution. The credibility of the distribution, especially at the high end, is another area for concern. The use of a theoretical distribution (lognormal, Weibull, etc.) can help considerably in dealing with these problems. One can fit a theoretical distribution to the empirical one and use the fitted distribution for pricing. In a recent paper, Finger[23] fitted a lognormal distribution to medical malpractice data using an empirical procedure based on the particular properties of the lognormal parameters.

In the absence of reliable empirical data it is not unreasonable to assume a theoretical severity distribution to use for pricing. The selection of a particular distribution can be made on the basis of the analytical properties of a distribution such as the mean, variance, coefficient of variation, skewness, etc. Even if the selection of a distribution were based on a subjective evaluation of the resulting increased limits factors, this would be an improvement over selecting factors directly without regard to the loss severity implications.

If a loss severity distribution is available from experience data or by assumption, then the formulas presented in this paper have the following applications.

1) The computation of expected value increased limits factors.

2) The adjustment of the severity distribution and the increased limits factors for trend, where trend is assumed to have the same multiplicative effect on each loss size.

3) The computation of risk charges by limit of liability.

[22] C. A. Hachemeister, "Breaking Down the Loss Reserve Process," presented at the CAS Loss Reserve Symposium (September, 1976).

[23] R. J. Finger, "Estimating Pure Premiums by Layer—An Approach", *PCAS* LXIII (1976), p. 34.

4) The calculation of the reduction in risk charge afforded by "layering" coverage.

5) The computation of the expected value pure premium and risk charge for excess of loss coverage.

If increased limits factors are computed by means other than those described in this paper, it is possible that such factors will produce inconsistencies in the pricing of increased limits and excess of loss coverage. The consistency test described in this paper can be used to evaluate a set of increased limits factors and point out the particular factors that are inconsistent with the rest.

4. RELATED TOPICS

The following are other areas of insurance pricing where the theories developed in this paper, particularly the consistency test, can be applied.

Leveraged Effect of Inflation

The concept of the leveraged effect of inflation is discussed thoroughly by Ferguson[24]. This concept can be expressed analytically in terms of what has been defined in this paper. What we are looking for is the change in the expected value pure premium for excess of loss coverage. Assuming an inflationary trend that has the same multiplicative effect on each size of loss, as defined by $\alpha_1(x) = ax$, the leveraging effect is controlled by the retention. The following formulas can be useful in analyzing the effects of inflation for excess of loss coverage.

1. Average increase in losses with fixed upper limit.

$$\frac{E[g(x';k)]}{E[g(x;k)]} = a \cdot \frac{E[g(x;k/a)]}{E[g(x;k)]}$$

$$= a \cdot \frac{I(k/a)}{I(k)}$$

2. Average increase in excess losses with fixed upper limit.

$$\frac{E[h(x';r,j)]}{E[h(x;r,j)]} = \frac{E[g(x';s)] - E[g(x';r)]}{E[g(x;s)] - E[g(x;r)]}, \; s = r + j$$

$$= a \cdot \frac{E[g(x;s/a)] - E[g(x;r/a)]}{E[g(x;s)] - E[g(x;r)]}$$

$$= a \cdot \frac{I(s/a) - I(r/a)}{I(s) - I(r)}$$

[24] Ferguson, *op. cit.*

3. Average increase in excess losses with no upper limit.

$$\frac{E[x'] - E[g(x';r)]}{E[x] - E[g(x;r)]} = a \cdot \frac{E[x] - E[g(x;r/a)]}{E[x] - E[g(x;r)]}$$

The expected value increased limits factors that were computed in the previous examples from the lognormal distribution can be adjusted for inflation using equation (18). Exhibit V shows the effects of inflation for various retentions given an overall inflation of 9% ($a = 1.09$).

The examples in this exhibit indicate somewhat small leveraged effects. This is primarily the result of the specific severity distribution used. Some other distribution could exhibit significantly higher leveraged effects. However, the author has not attempted to study this further. The conclusion from this is that while inflation may cause very serious pricing problems for excess of loss coverage, such problems may not always be as severe as they first appear.

Aggregate Limits

A maximum limitation on the total amount of insured losses for all accidents/occurrences is generally referred to as an aggregate limit. Such a limit usually applies for a one year policy period and can be used in conjunction with a per accident/occurrence limit. Aggregate limits are intended to restrict the exposure to multiple large losses or an excessive frequency of losses. The theoretical pricing structure of aggregate limits and aggregate excess coverage (excess of aggregate limits, also known as stop-loss reinsurance) will not be discussed in this paper. However, the theory does permit the application of a consistency test. The test described previously can be used by analyzing the marginal rate per $1,000 of accident/occurrence limit keeping the aggregate limit constant and also the marginal rate per $1,000 of aggregate limit keeping the accident/occurrence limit constant. Thus, if increased limits factors are displayed in a table where the columns indicate an accident/occurrence limit and the rows indicate an aggregate limit, then each row and each column of increased limits factors should be tested separately in the same manner as a per accident table of factors.

EXHIBIT V

LEVERAGED EFFECT OF INFLATION
(OVERALL INFLATION OF 9%)

Retention	Increased Limits Factors: before inflation adjustment	Increased Limits Factors: after inflation adjustment	Average Increase in Losses Limited to Retention	Leveraged Effect: Average Increase in Losses in Excess of Retention: Limited to $1,000,000	Leveraged Effect: Average Increase in Losses in Excess of Retention: Unlimited
25,000	1.000	1.000	3.8%	10.3%	11.3%
50,000	1.419	1.432	4.8	11.2	12.2
100,000	1.872	1.905	5.7	12.2	13.4
300,000	2.526	2.604	7.1	14.2	15.5
500,000	2.762	2.862	7.6	15.2	16.7
1,000,000	2.996	3.121	8.1	16.7	18.3
2,000,000	3.140	3.282	8.5	18.4	20.1

Per Person, Per Accident Limits

Liability coverage can also be defined by dual or "split" limits. In general, such limits provide for a maximum amount of insured loss for each person injured in an accident in addition to a maximum amount for each accident. To extend the pricing model to this type of coverage would require the introduction of another random variable. This random variable would represent the number of persons injured in an accident. It would also be necessary to change the loss severity variable to a per person basis rather than per accident.

Obviously such changes would complicate the model considerably unless further assumptions are made. It is not clear what advantages split limits have over the single per accident limit other than to further restrict coverage. The elimination of split limits coverage would aid greatly in the pricing of increased limits, both in the evaluation of experience data and in the mathematical model.

The application of the consistency test to evaluate split limits increased limits factors is similar to aggregate limits. Given a table of factors with the columns indicated the per person limit and the rows indicating the per accident limit, the test would be applied to each row and to each column separately.

Property Insurance — Coinsurance Pricing

All aspects of coinsurance including pricing are discussed very thoroughly by Head[25]. However, in discussing the relationships between the rates for different coinsurance requirements Head requires that no premium reversals[26] exist between two coinsurance requirements and that coinsurance rates should decrease at a declining rate with added coverage. The consistency test can be adapted to coinsurance pricing and provide a further check on coinsurance rates.

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[25] G. L. Head, *Insurance to Value*, Irwin (1971).
[26] *Ibid.*, p. 116.

Consider the following set of factors which relate the rates for various coinsurance requirements to the 80% coinsurance rate. These factors have no premium reversals and produce rates that decrease at a decreasing rate for increasing coinsurance requirements.

Coinsurance Requirement	Coinsurance Factor
100%	.90
90	.94
80	1.00
70	1.07
60	1.15
50	1.28
40	1.50

Next suppose a full value amount of $100,000 and an 80% coinsurance rate of $1.00 per $100 of insurance. The amount of insurance and the premium for the various coinsurance requirements would be:

Coinsurance Requirement			Marginal Premium[28]
Percent	Amount of Insurance	Premium[27]	per $1,000 of Coverage
100%	$100,000	$900	$5.40*
90	90,000	846	4.60
80	80,000	800	5.10
70	70,000	749	5.90
60	60,000	690	5.00*
50	50,000	640	4.00*
40	40,000	600	—

For this pricing to be consistent, the marginal premium per $1,000 of coverage (amount of insurance) should decrease as the coverage increases. This example shows inconsistencies for coinsurance requirements of 100%,

[27] Premium = Coinsurance Percentage $\times \dfrac{\$100{,}000 \text{ (amount of full value)}}{\$100 \text{ (exposure base)}}$

$\times$ Coinsurance Factor $\times$ $1.00 (80% coinsurance rate).

[28] The Marginal Premium is the premium difference between the given amount of insurance and the next lower amount of insurance, divided by the difference in the amounts of insurance.

60%, and 50% as indicated by *. It is important to note that this result is caused solely by the coinsurance factors, i.e., the same inconsistencies will be indicated regardless of the full value amount or the 80% coinsurance rate. The coinsurance factors used in this example are similar to factors in actual use at the time this paper was written.

5. CONCLUSION

Through the use of a mathematical model, the pricing of increased limits and excess of loss coverage can be analyzed both in theory and in practical application. The model presented in this paper gives a mathematical statement of the pricing problem. The complete solution to this problem requires actual data, judgment and some further study.

The key element to the model is the size of loss distribution. Unfortunately, there is not very extensive knowledge about such distributions, either empirical or theoretical. Techniques must be developed and refined for the collection and evaluation of size of loss data. Moreover, new theoretical distributions must be found that can simulate the many possible types of severity distributions. The treatment of loss adjustment expense is also very important because these expenses are related to the existence and severity of a loss. This relationship must be defined and fit into the model in order to create increased limits factors for actual use.

Other areas where research is needed are a more realistic approach to adjusting for the effects of inflation by size of loss, the detection and implications of anti-selection, the classification of insureds into homogeneous groups with similar severity characteristics, the development of a risk charge for parameter risk, and a pricing model for split and aggregate limits.

Acknowledgements

The lognormal parameters used in the examples were developed by Gary Patrik from medical malpractice closed claim data. Mr. Patrik's work in this area also aided the author in defining the mathematical pricing model. J. Ernest Hansen contributed the original thinking of risk reduction by layering in mathematical terms. The impetus and motivation for the research behind this article was due to Charles Walter Stewart.

APPENDIX

The Lognormal Distribution

For the purpose of this paper, an evaluation of the following integrals is required for various values of x, where f(t) is the lognormal probability density function.

$$\int_0^x f(t)\,dt$$

$$\int_0^x t \cdot f(t)\,dt$$

$$\int_0^x t^2 \cdot f(t)\,dt$$

These three integrals can be evaluated by means of a transformation to values of the normal cumulative distribution function, $\Phi(x)$.

From the definition of the lognormal distribution[29], we know that

$$f(x) = \frac{d}{dx}\,\Phi\left(\frac{\ln x - \mu}{\sigma}\right), \quad \Phi(x) = \int_0^x \frac{1}{\sqrt{2\pi}} \cdot \exp(-\tfrac{1}{2}t^2)\,dt$$

$$= \frac{1}{\sqrt{2\pi} \cdot \sigma \cdot x} \cdot \exp\left\{-\tfrac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right\}$$

Consequently,

$$\int_0^x f(t)\,dt = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

The derivation of the formulas for the remaining two integrals follows.

$$\int_0^x t \cdot f(t)\,dt = \int_0^x \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left\{-\tfrac{1}{2}\left(\frac{\ln t - \mu}{\sigma}\right)^2\right\} dt$$

$$= \int_{-\infty}^{\frac{\ln x - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp(\sigma y + \mu) \cdot \exp(-\tfrac{1}{2}y^2)\,dy\,,$$

$$\left\{\begin{array}{l} y = \dfrac{\ln t - \mu}{\sigma} \\[2ex] dy = \dfrac{1}{\sigma t}\,dt \\[2ex] t = \exp(\sigma y + \mu) \end{array}\right.$$

[29] J. Aitchinson and J. A. C. Brown, *The Lognormal Distribution*, Cambridge University Press (1957).

$$= \frac{1}{\sqrt{2\pi}} \exp(\tfrac{1}{2}\sigma^2 + \mu) \cdot \int_{-\infty}^{\frac{\ln x - \mu}{\sigma}} \exp(-\tfrac{1}{2}(y - \sigma)^2)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \exp(\tfrac{1}{2}\sigma^2 + \mu) \cdot \int_{-\infty}^{-\sigma + \frac{\ln x - \mu}{\sigma}} \exp(-\tfrac{1}{2}z^2)\, dz\,, \begin{cases} z = y - \sigma \\ dz = dy \end{cases}$$

$$= \exp(\tfrac{1}{2}\sigma^2 + \mu) \cdot \Phi\left(-\sigma + \frac{\ln x - \mu}{\sigma}\right)$$

$$\int_0^x t^2 \cdot f(t)\, dt = \int_0^x \frac{1}{\sqrt{2\pi} \cdot \sigma} t \cdot \exp\left\{-\tfrac{1}{2}\left(\frac{\ln t - \mu}{\sigma}\right)^2\right\} dt$$

$$= \int_{-\infty}^{\frac{\ln x - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp(2\sigma y + 2\mu) \cdot \exp(-\tfrac{1}{2}y^2)\, dy\,,$$

$$\begin{cases} y = \dfrac{\ln t - \mu}{\sigma} \\ dy = \dfrac{1}{\sigma t}\, dt \\ t = \exp(\sigma y + \mu) \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} \exp(2\sigma^2 + 2\mu) \cdot \int_{-\infty}^{\frac{\ln x - \mu}{\sigma}} \exp(-\tfrac{1}{2}(y - 2\sigma)^2)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \exp(2\sigma^2 + 2\mu) \cdot \int_{-\infty}^{-2\sigma + \frac{\ln x - \mu}{\sigma}} \exp(-\tfrac{1}{2}z^2)\, dz\,,$$

$$\begin{cases} z = y - 2\sigma \\ dz = dy \end{cases}$$

$$= \exp(2\sigma^2 + 2\mu) \cdot \Phi\left(-2\sigma + \frac{\ln x - \mu}{\sigma}\right)$$