DISCUSSION BY COSTANDY K. KHURY

Once again Mr. Simon favors the *Proceedings* with a new direction that should receive significant attention in the future. The importance of maintaining logical consistency among various reinsurance alternatives can hardly escape either the reinsurer of the reinsured. Perhaps the accent which this paper has placed on the concept will serve to underscore the degree of care that every primary insurer must exercise in assuring logical consistency among its own various plans—whether they concern deductible options, excess coverage plans, etc.

There are three assumptions underlying the treatment of the problem as posed by Mr. Simon:

(1) The Poisson distribution is the appropriate mathematical model for the occurrence of claims.

(2) The subject treaties are unbalanced, that is, they attach at a <u>high</u> limit such that the pure premium is <u>small</u> in relation to the size of the cover. (Emphasis added)

(3) All covered losses are total losses, that is, any loss which penetrates the cover will run all the way through it.

Assumption (1) presents no problem in that the principles espoused in the paper are fundamentally independent of the Poisson distribution—as specifically implied by the last sentence of the paper. Also, there is considerable literature in support of this particular choice. The use of another distribution would impact only upon the methodology and would leave the basic principles intact.

Assumption (2) presents some interpretation problems. First, what constitutes a "high" point of attachment is certainly different for different reinsureds. Also, what constitutes a "small" pure premium in relation to the size of the cover is a matter on which different observers could easily disagree. With each of these two characterizations open to debate—in the absence of specific definition criteria—the combination thereof is in turn open to compound interpretation problems inasmuch as they are not independent of each other. Close examination of the paper failed to yield the specific point(s) at which this assumption became operative. It appears that this assumption provides essentially a restatement of what a catastrophe is interpreted to be in relation to assumption (3) and [therefore] to the subsequent methodology. In fact, if the paper is re-read without benefit of this assumption, then the methodology is unaffected provided assumption (3) is accepted without further qualification. The matter of including or excluding this assumption fundamentally rests, at first blush, with the degree of aesthetic elegance sought in bridging from assumptions to conclusions. In this same regard, a proposition is strongest only when the set of assumptions has been reduced to an absolute minimum. More on this assumption later.

Due to the impact that assumption (3) has on molding the main body of the paper, it is in fact the focal point of this discussion. The implications of assuming all losses to be total losses (in most casualty insurance situations) are numerous. Among them:

(1) If consecutive reinsurers, at all layers, adopted the same philosophy, then once a loss penetrates the first layer it must be unlimited. A loss must stop somewhere, *a priori*.

(2) If a cover of size L excess of a specified retention requires a pure premium P; then a cover of size kL must imply a pure premium of kP for all $k > 0$. This linear movement of the pure premium could cause the cost of reinsurance to become prohibitive for $k > 1$ and inadequate for $k < 1$. A competing reinsurer may be able to capitalize on this feature by making appropriate adjustments for successive layers of equal thickness, etc.

These conditions, as well as others not mentioned above, can be eliminated at once provided a size-of-loss distribution is incorporated in the various formulations in the paper. An illustration of how such a distribution could be utilized is set forth below in terms of an arbitrary but fixed size-of-loss distribution.

Given a reinsurance treaty with a limit C excess over a specified retention c. The cedent's recovery is at a ratio r such that a loss of size x with $x > c$ generates reinsurance recoverable of $r(x - c)$ subject to the maximum rC. Let the size-of-loss random variable be x with a probability density function $f(x)$. In this discussion x may be assumed to be continuous without loss of generality. Let $g(n)$ be the distribution representing the model for the occurrence of exactly n claims during a specified time interval Δt and each of which is in excess of c. It should be noted here that $g(n)$ and $f(x)$ are independent. Finally, let $R(x)$ be the reinsurance recoverable by the cedent given the occurrence of a loss of size x:

$$R(x) = \begin{cases} 0 & \text{if } \quad c \geq x \geq 0 \\ r(x-c) & \text{if } C+c \geq x > c \\ rC & \text{if } \quad x > C+c \end{cases}$$

At first the development will track a "no-reinstatement" assumption during a time interval Δt. Two key probabilities of occurrence of claims are:

$$g(0) \text{ and } g(n \geq 1)$$

The reinsurer's expectation of loss (the pure premium $\overline{P}_1$)[1] is given by:

$$\overline{P}_1 = 0 \cdot g(0) + * \cdot g(n \geq 1)$$

with * yet to be determined.

Given that a loss has occurred, then $f(x)$ would yield the following probabilities:

$$Pr(C + c \geq x > c) = \int_c^{C+c} f(x)dx) \text{and}$$

$$Pr(x > C + c) = \int_{C+c}^{\infty} f(x)dx$$

These, in turn, give immediate rise to the following conditional probabilities:

$$Pr(C + c \geq x > c \mid x > c) = \int_c^{C+c} f(x)dx \Bigg/ \int_c^{\infty} f(x)dx$$
$$= Q(c, C + c, f(x)), \text{ and}$$

$$Pr(x > C + c \mid x > c) = \int_{C+c}^{\infty} f(x)dx \Bigg/ \int_c^{\infty} f(x)dx$$
$$= Q(C + c, \infty, f(x))$$

[1] This assumed equivalence reflects Mr. Simon's choice of not including a separate risk loading.

Accordingly, the reinsurer's conditional expectation of loss is given by:

$$E(R(x) \mid C + c \geq x > c) = \int_c^{C+c} r(x-c)f(x)dx \Bigg/ \int_c^{C+c} f(x)dx$$

$$E(R(x) \mid x > C + c) = rC$$

Hence, * is given by:

$$* = [Q(c, C + c, f(x)). \int_c^{C+c} r(x-c)\, f(x)dx \Bigg/ \int_c^{C+c} f(x)dx]$$
$$+ Q(C + c, \infty, f(x)). rC$$

And the pure premium $\overline{P}_1$, resolves to:

$$\overline{P}_1 = * \cdot g(n \geq 1)$$

If the expression obtained by Mr. Simon in (3.1) is recast into this new format (where rC, the maximum possible single loss reinsurance recoverable under the contract, is substituted for the expected average loss derived via the size-of-loss distribution when the loss is less than total), it would appear as follows:

$$P_1 = \{Q(c, C + c, f(x)) \cdot rC + Q(C + c, \infty, f(x)) \cdot rC\} \cdot g(n \geq 1)$$

Thus, the additional (built-in) pure premium due to assumption (3) is given by:

$$(P_1 - \overline{P}_1) = (rC - [\int_c^{C+c} r(x-c)f(x)dx \Big/ \int_c^{C+c} f(x)dx]) \cdot$$
$$Q(c, C + c, f(x)) \cdot g(n \geq 1)$$

Trivially, this expression is always non-negative.

It should be noted here that the expression given above for $\overline{P_1}$, given the same circumstances of example A, is equally usable as a vehicle for extracting the Poisson parameter by iteration since the introduction of a specific size-of-loss distribution did not produce any additional unknowns.

Also, since $(P_1 - \overline{P_1}) \geq 0$, it follows immediately that the Poisson parameter m, as implied by example A, is at its natural maximum only if the size-of-loss distribution ultimately implies $(P_1 - \overline{P_1}) = 0$ and is overstated whenever $(P_1 - \overline{P_1}) > 0$.

Now turning to example B, the probabilities of occurrences are:

$$g(0), g(1), \text{ and } g(n \geq 2)$$

and the pure premium $\overline{P_2}$ is given by:

$$\overline{P_2} = 0 \cdot g(0) + * \cdot g(1) + 2 \cdot * \cdot g(n \geq 2)$$

$$= * \cdot [g(1) + 2\, g(n \geq 2)] \text{ with } * \text{ as defined above.}$$

Once again, recasting Mr. Simon's definition expression for P_2 in the above format would produce:

$$P_2 = \{Q(c, C + c, f(x)) \cdot rC + Q(C + c, \infty, f(x)) \cdot rC\} \cdot [g(1) + 2g\,(n \geq 2)]$$

Thus the additional (built-in) pure premium due to assumption (3) is given by:

$$(P_2 - \overline{P_2}) = (rC - [\int_c^{C+c} r(x - c) f(x) dx \Big/ \int_c^{C+c} f(x) dx]) \cdot Q(c, C + c, f(x)) \cdot [g(1) + 2g(n \geq 2)]$$

and again this expression is trivially non-negative. Note that $(P_2 - \overline{P_2})$ compounds the original difference term $(P_1 - \overline{P_1})$. As was true for example A, the expression for $\overline{P_2}$ is equally usable for the determination of the Poisson parameter since the introduction of the size-of-loss distribution did not produce any additional unknowns. Finally, $(P_2 - \overline{P_2}) \geq 0$ implies that m as derived from $\overline{P_2}$ will never exceed m as derived from P_2.

Now turning to the critical relationships of $\overline{P_2}$ to $\overline{P_1}$ and P_2 to P_1:

$$\overline{P_2} \,/\, \overline{P_1} = [* \cdot g(1) + 2 \cdot * \cdot g(n \geq 2)] \,/\, * \cdot g(n \geq 1)$$

$$= [g(1) + 2\, g(n \geq 2)] \,/\, g(n \geq 1)$$

$$= [g(1) + g(n \geq 2) + g(n \geq 2)] \,/\, [g(1) + g(n \geq 2)]$$

$$= 1 + [g(n \geq 2) \,/\, g(n \geq 1)]$$

Using Mr. Simon's definition expression for P_1 and P_2 (with L replaced by C), we have:

$$P_2 / P_1 = [Cp_1 + 2C \sum_{i=2}^{\infty} p_i] \Big/ C \sum_{i=1}^{\infty} p_i$$

$$= [p_1 + 2 \sum_{i=2}^{\infty} p_i] \Big/ \sum_{i=1}^{\infty} p_i$$

$$= [p_1 + \sum_{i=2}^{\infty} p_i + \sum_{i=2}^{\infty} p_i] \Big/ [p_1 + \sum_{i=2}^{\infty} p_i]$$

$$= 1 + [\sum_{i=2}^{\infty} p_i \Big/ \sum_{i=1}^{\infty} p_i]$$

which is the same expression as derived above as $1 + [g(n \geq 2) / g(n \geq 1)]$. Therefore, the relationships $\overline{P_2} / \overline{P_1}$ and P_2 / P_1 are independent of the size of the cover and dependent only upon the all important point of attachment of the treaty and therefore, on the particular selection of the distribution for the occurrence of claims in excess of the retention. Perhaps the preceding discussion in connection with assumption (2) as to what constitutes a "high" point of attachment could now be viewed as requiring a definition of the term "high" inasmuch as it turns out to be the controlling feature in comparing the various reinstatement alternatives.

As a generalization of the preceding findings, any two proposals involving reinstatements could be compared. In this manner the examples given in Mr. Simon's paper (e.g., A and C) can be treated as special cases of the larger class of problems they represent.

If P_{i+1} *and* P_{j+1} represent two options with i and j reinstatements respectively, then:

$$P_{i+1} / P_{j+1} = [\sum_{u=0}^{i} u\, g(u) + (i+1)\, g(n \geq i+1)] \Big/$$

$$[\sum_{u=0}^{j} u\, g(u) + (j+1)\, g(n \geq j+1)]$$

One other interesting generalization can be obtained in connection with the reinsured wishing unlimited reinstatements. That is, P_∞ at the inception of the time interval Δt is given by:

$$P_\infty = * \cdot \sum_{u=o}^{\infty} u\, g(u)$$

and in case of the Poisson distribution, this expression is reduced to the compact, and perhaps obvious, form:

$$P_\infty = * \cdot \sum_{u=o}^{\infty} u\, g(u) = * \cdot \sum_{u=o}^{\infty} u \cdot [m^u \; e^{-m} / u\,!] = * \cdot m$$

In conclusion, it should be acknowledged that it is the author's exclusive privilege to select his underlying assumptions and proceed within the guidelines provided thereby. In this discussion it is hoped that additional light has been shed on this problem in terms of reducing the set of working assumptions, a generalization of the problem and its solution, and finally a more direct approach to the comparison of reinstatement options. Once again, we should be grateful to Mr. Simon for opening the doors to this extremely exciting and glamorous area of actuarial research.