ACTUARIAL APPLICATIONS IN CATASTROPHE REINSURANCE

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1. The pricing of catastrophe reinsurance treaties is much more of an art than an actuarial science. The parties involved are usually well informed and are free to bargain in the spirit of free and open competition. Reinsurance has also been favored with a high degree of integrity on the part of all participants.

One of the important contributions that the actuary can make to the reinsurance field is the maintenance of logical consistency among the various alternatives that may be considered at different stages of the negotiation process. It is quite common for modifications in terms to be discussed, such as altering the retention; changing the thickness of the layer; subdividing the layer into two or more strata. Although it may not be possible to claim that the various alternatives are actuarially equivalent in the strict sense, the actuary can help assure that they are at least logically consistent with each other. I would hasten to add that other positive contributions from the actuary in the reinsurance field would include: pricing estimates themselves; determining incurred but not reported and developmental reserves; assessing inflationary impact and evaluating financial aspects of his own company or of prospective reinsureds.

A great deal of sound thinking, together with innovation and the open acceptance of new ideas, is required in the reinsurance field. An actuary's training is very helpful in developing the type of individual needed. In reinsurance, very few pricing situations lend themselves to statistical or rating manual analysis. However, the maintenance of logical consistency within various reinsurance quotations is greatly aided by a mathematical model and appropriate study of the implied actuarial relationships. The purpose of this note is to study some of the relationships in the catastrophe reinsurance area.

2. Let there be a reinsurance treaty with an exposure to the reinsurer* L excess over a specified retention at a pure premium of P. Attention

* The author is most indebted to Matthew Rodermund for suggesting an improvement in the paper that led to this definition of L. Instead of considering the vertical slice of a layer which represents the amount at risk to the reinsurer, an alternative definition of L as the complete 100% thickness of the layer could also be used. The premiums thus determined would be on a 100% basis and would have to be modified to fit the terms of the specific cover and the reinsurer's portion.

will be focused on situations where it will be appropriate to assume that any loss which hits the cover will run all the way through it, that is, all losses will be total losses. While this assumption is not strictly true, the reinsurance company normally assesses treaties on this basis, and it is very nearly the true situation. If this assumption causes difficulty, it may be necessary to apply this model to narrow sub-layers of a given treaty.

Further, we shall consider here treaties which are unbalanced, that is, they attach at a high level such that the pure premium will be small with respect to the limit L. It will be further assumed that the Poisson distribution (with parameter m) is the appropriate mathematical model for the occurrence of claims, and we shall designate p_c as the probability of having exactly c claims.

Some of the functions of interest are:

(2.1) $p_c = m^c e^{-m}/c!$

And, in particular:

(2.2) $p_o = e^{-m}$

(2.3) $p_1 = m\, e^{-m}$

(2.4) $\sum_0^\infty p_c = 1$

(2.5) $\sum_1^\infty p_c = 1 - e^{-m}$

(2.6) $\sum_2^\infty p_c = 1 - e^{-m} - me^{-m}$

The last two functions are the probability of having one or more claims and two or more claims respectively. Also in general:

(2.7) Expected Losses = Expected Pure Premium

3. Let us first take the simple case of a treaty designed to cover one catastrophe event in the year with no agreement to reinstate the coverage if the insured utilizes it during the one year period. In this case we have:

$$P_1 = 0.p_o + L \sum_1^\infty p_c$$

or:

(3.1) $P_1 = L(1 - e^{-m})$

It follows that:

(3.2) $e^{-m} = 1 - P_1/L$

and:

(3.3) $m = -ln(1 - P_1/L)$

where ln is the natural logarithm.

Example A: Analyze a 90% of \$10 million cover which sells for a gross premium of \$1 million with no automatic reinstatement provision. It will be assumed that commisssion to the broker and the reinsurer's overhead and profit provisions total 18%. Therefore, $P_1 = 1(.82) = .82$ and $L = .90(10) = 9.0$. We can now note that the expected number of times the cover will be hit in a one year period by (3.3), is .09553. The probability of running claim free for a year is, by (2.2) and (3.2), .90889. From equation (2.6) we calculate the probability of hitting the cover two or more times to be .00428.

4. A more common case in reinsurance is for a catastrophe treaty to have an automatic reinstatement provision. In this case the insured will receive a reinstatement of the amount of cover he has utilized in hitting it, up to an amount L, so that the cover is immediately available again for a second event. In exchange for this reinstatement he pays a gross premium (and hence a pure premium) equal to the proportion of the cover which is reinstated times the proportion of time left to run in the contract year times the original gross premium (and hence times the original pure premium P). To develop the mathematics of this case, let f designate the mean portion of the year from the start of the contract to the occurrence of the first claim.

We may then write the following equations:

$$\text{Expected Losses} = 0.p_0 + Lp_1 + 2L \sum_{2}^{\infty} p_c$$

$$= Lme^{-m} + 2L(1 - e^{-m} - me^{-m})$$

or:

(4.1) $\text{Expected Losses} = 2L - 2Le^{-m} - Lme^{-m}$

(4.2) $\text{Expected Pure Premium} = P_2 + P_2\,(1 - f)$

Hence, by (2.7):

(4.3) $2P_2 - P_2f = 2L - 2Le^{-m} - Lme^{-m}$

This can be rearranged in many ways. To solve for m when P and L are given, the following form is helpful:
(4.4) $e^{-m}(2L + Lm) - 2(L - P_2) - P_2f = 0$

To obtain an expression for f, we note that for zero claims, $f = 1$; for one claim, $f = 1/2$; for two claims, $f = 1/3$; for three claims, $f = 1/4$ and so forth (we are assuming that a claim is equally likely to occur at any time during the year). Hence:

$$f = (1)e^{-m} + (1/2)me^{-m} + (1/3)m^2e^{-m}/2! + (1/4)m^3e^{-m}/3! + \ldots$$
$$= e^{-m}(1 + m/2! + m^2/3! + m^3/4! + \ldots)$$

(4.5) $f = (1 - e^{-m})/m$

Substituting this value in (4.4):

(4.6) $e^{-m}(2L + Lm + P_2/m) - P_2/m - 2(L - P_2) = 0$

Thus, the case of reinstatement requires an iterative process whereby m is approximated, substituted in (4.6) and the value improved through repeated trials of values of m.

Example B: Analyze a treaty for 95% of \$5 million with a gross premium of \$1 million with one automatic reinstatement. It will be assumed that there is no brokerage commission and that profit and overhead are 12%. We thus have $L = .95(5) = 4.75$ and $P_2 = 1(.88) = 0.88$. Substituting these in (4.6) we write $e^{-m}(9.50 + 4.75m + .88/m) - .88/m - 7.74 = V$. As a first approximation, $m = -ln(1 - .88/4.75) = -ln(.81474) = .205$ by (3.3). Since the formula used for the approximation does not include consideration of the reinstatement coverage, it can be assumed that this approximation of m is a little too high so we shall start off with an initial value, $m_1 = .20$. This value is substituted in the equation for V to produce a value V_1. A second estimate designated m_2 is then made and the process repeated. Thereafter, improved estimates of m may be obtained by linear interpolation or extrapolation of the values of V until we reach a value of m where V equals zero.

In this example, the following values are obtained:

$m_1 = .20$	$V_1 = +.01815$
$m_2 = .204$	$V_2 = +.00103$
$m_3 = .204267$	$V_3 = -.00011$
$m_4 = .204244$	$V_4 = -.00001$

The process has converged quite rapidly indicating that the function V must be very nearly a straight line in the vicinity of m. We terminate after obtaining m_4 and conclude that the conditions of this treaty imply $m = .20424$.

5. Suppose now that we are interested in determining the relationship between the pure premium P_1 for a one event, no reinstatement cover and the pure premium P_2 for a similar cover involving one automatic reinstatement at pro-rata of the gross premium (and hence the pure premium).

From (3.1):

(5.1) $P_1 = L(1 - e^{-m})$

From (4.6):

(5.2) $P_2 = (mL)(2 - 2e^{-m} - me^{-m})/(e^{-m} + 2m - 1)$

To express P_2 in terms of P_1 and L, which will be given, use is made of (5.1), (3.2) and (3.3) in rewriting (5.2) as:

(5.3) $P_2 = L(ln(1 - P_1/L))[2 - (1 - P_1/L)(2 - ln(1 - P_1/L))]/[P_1/L + 2\ ln(1 - P_1/L)]$

Example C: If the reinsured in Example A now asked for an automatic reinstatement at pro-rata of premium, what is the new pure premium? $P_1 = .82$, $L = 9.0$. Thus by (5.3, $P_2 = .82057$.

It is interesting to note that the ratio

$$P_2/P_1 = .82057/.82$$
$$= 1.0007/1.000$$

Example D: In the Example B case, we observed that $L = 4.75$ and $P_2 = 0.88$ resulting in $m = .20424$. What is that treaty worth without the automatic reinstatement? By (5.1):

$P_1 = 4.75(1 - e^{-.20424})$
$= .87748$

Also note that:

$P_2/P_1 = .88/.87748$
$= 1.0029/1.000$

As a corollary, it is of interest to calculate the expected pure premium income under Example B. This would be the mean amount one would expect to actually collect over the long run. By (4.2), (4.5), (3.3) and (3.1) one would expect to collect $P_2[2 - (P_1/L)/(-\ln(1 - P_1/L))] = .96405$

Naturally, this same result could be obtained from (4.1) since the identity of (4.1) and (4.2) was used in determining m.

6. As an extension of the theory, consider the quotation of a premium for a third event cover when a company has hit its original cover once. The original cover has one automatic reinstatement and the company desires the additional protection of the third event cover for the remainder of the year. One would simply apply the appropriate equation to the conditions of the original cover to determine the implied m and then apply it to the future cover.

Example E: What would the pure premium be for a third event cover to be effective for the last half of the year when the reinsured in Example B has hit the cover in the middle of the policy period and desires the additional coverage for the last six months? We know from the solution of Example B that $m = .20424$. On pure theory, m for the last half of the year is $.10212$.*

To evaluate the pure premium, we have:

$$P = 0.p_0 + 0.p_1 + L \sum_{2}^{\infty} p_1$$
$$= L(1 - e^{-m} - me^{-m})$$
$$= 4.75\,(1 - e^{-.10212} - .10212e^{-.10212})$$
$$= .02314$$

* Practical considerations such as the effective and expiration dates of the treaty with respect to the hurricane season would affect this "pure theory". Also, since the reinsured has recently hit the cover, his new premium would undoubtedly be based on a higher value for m. A possible approach might be to re-price the original cover in light of the current experience and thus determine a new value for m.

In dollars, then, the insured would pay a pure premium of $23,140. Note how this would represent a mimimum which the reinsurer could use but which would undoubtedly be far below the final bargaining premium. It is apparent that if the reinsured makes similar actuarial calculations, he will be in a much improved bargaining position when the negotiations begin.

7. Many other interesting applications can be made of these simple concepts in order to keep the pricing of covers in a rational relationship to one another. Perhaps future theoretical developments will enable us to use more sophisticated models than the simple Poisson — for example the Pareto or the negative binominal.