Extending the Asset Share Model: Recognizing the Value of Options in P&C Insurance Rates

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Abstract: In this paper we will present a refinement of the well-known asset share model for ratemaking. The new method for calculating premiums and premium relativities accounts for risk classification transition probabilities. The relationship between risk class transition and options on insurance coverage is discussed. Some simple examples will be worked which will demonstrate how risk class transition can cause problems for the traditional asset share model which are remedied by our extended asset share model. We will also show how the new method can be used to determine the price for insurance policies with the popular "accident forgiveness" feature.

Keywords. Ratemaking, risk classification, asset share model, option pricing

1. INTRODUCTION

Many P&C actuaries are familiar with the asset share model through the work of Sholom Feldblum, [1]. The method demonstrates how lower prices can be justified for more loyal customers by essentially crediting some of the expected profits on renewal terms to the first term. However, there is a disconnect between the theory and practice. The asset share model assumes a constant (or perhaps gradually inflating) premium over the life of the policy, whereas in practice the rate may change at policy renewal, sometimes drastically, as more information about the insured is revealed. In this paper we will present a method for calculating premiums and premium relativities that accounts for risk classification transition probabilities.

We begin in Section 2 with a discussion of the merits and shortcomings of the established asset share model. Section 3 will present a very simple example for ratemaking with two risk classes. Policies will be limited to one renewal, but the twist is that policies can change from one risk class to another. We will show that in carefully accounting for risk class transition, we arrive at different indicated rates than both the traditional method and the standard asset share model.

In Section 4 we will discuss options, and demonstrate that the difference between the extended asset share method developed here and the standard asset share model is that the assumption of constant premium inflation causes the standard model to include the cost of options on future coverage in the indicated prices. The extended model allows us to correctly price the "floating rate" policy, and separately determine the cost of the options.

In Section 5 we will present the full mathematical framework for the extended asset share model and in Section 6 we will apply the extended asset share model in two worked examples, including pricing insurance policies with the "accident forgiveness" feature popular in personal lines insurance. We will conclude in Section 7 with some additional considerations

2. THE STANDARD ASSET SHARE MODEL

Hopefully the reader is familiar with Feldblum's paper [1] on the asset share model which presents a method for pricing insurance contracts by looking at lifetime expected premium, loss and expense. Traditional methods view only a single contract term. The mechanics of the model are straightforward: for the first and all subsequent renewal terms estimate the losses and expenses expected to be incurred. An estimate of the probability of the policy renewing is also required. The required premium can then be calculated using the traditional premium formula:

$$P = \frac{F + L}{1 - U - V}$$

where P, F, and L represent the expected lifetime present value of premium, fixed expense and loss, respectively, while V is the variable cost ratio and U is the desired underwriting profit provision as a percent of premium. Feldblum uses examples to demonstrate how differing probabilities of renewal combined with decreasing loss costs as policy tenure increases can cause optimal premium relativities to be very different from loss relativities; these were traditionally assumed to be the same except for fixed expense loading.

2.1 Why the Asset Share Model is an Improvement

The asset share model is a major improvement over the traditional method of ratemaking. It has been well documented, in the paper by Wu and Lin [4], that average loss costs improve as policyholder tenure increases. The paper shows that this phenomenon is present in most lines of business, not just private passenger auto which was the subject of the Feldblum paper.

So why not just charge the traditional premium for each term of a policy? Premiums will be higher for the first term and decrease from there, with the company securing an expected profit for every term of a customer's time with the company. Consumer behavior gives insurers an opportunity for a better strategy: charge a lower premium to new customers when they are more sensitive to price, and charge higher than actuarial premiums for renewal terms when customers are likely to renew and less sensitive to price. Note that this is not necessarily irrational consumer behavior; it is simply a matter of preferring cash now (in the form of a lower new business policy premium) to a stream of contingent future cashflows (renewal discounts which are realized only if the policy is renewed). The asset share model determines the required premium under the better strategy.

2.2 Shortcomings of the Asset Share Model

Another well documented phenomenon in insurance is that past loss experience is indicative of future loss experience. Ratemaking schemes in personal lines, commercial lines and reinsurance businesses all charge higher premiums to policyholders who experience losses, or greater than expected losses. In the asset share model it is assumed that premium gradually inflates over time so that if a customer is charged a premium P at inception, they will be expected to pay $(1+r)^n * P$ for the n^{th} policy term, where r is the rate of inflation. In actuality, a cohort of policies that were identical at inception will, due to experience rating, be paying a wide range of premiums for their future renewal policies.

To see why this is problematic, think about how we would use the asset share model to determine optimum relativities for experience rating factors in personal lines. We would set up tables of expected loss, expenses, renewal probabilities and premium for two groups, "Low" and "High" expected future loss. There would be two independent variables, the premium for the low group and the premium for the high group. We could determine the correct premium for each and the differential between the two would be our new experience rating factor. Suppose the new factor amounts to a 20% increase in premium for "high" risk customers. The asset share model we set up is likely no longer valid because renewal rates should decrease with increasing premiums. Also, the expected inflation trend in the premium for the low group should increase because the small fraction of them that receive the "high risk" premium on renewal after having adverse experience will pay 20% higher premiums. It may be possible to iterate the process and come to an answer, but it seems that this may not be the best solution.

The assumption of constantly inflating premiums can lead us astray in other scenarios as well. Suppose we use the asset share model to determine optimal premium relativities for new drivers versus those with 5 years of experience, and we determine that premiums for new drivers need to be 30% higher over the life of the policy. What happens when the driver attains 5 years of experience and wishes to renew the policy? In most cases the company would need to charge the customer the lower "experienced driver" rate. The original asset share model is now invalidated because it was counting on profitable policy years six and beyond priced at the higher rate to subsidize expected underwriting losses on the first few years of the policy. Even if the new rate is consistent with an asset share model somehow recalculated for this risk, we can never go back in time and charge more for policy terms that are already over for which our rate was inadequate. This error would lead to the company systematically undercharging customers over their lifetimes and poor profitability.

2.3 Remedy for the Above Issues

The problems noted above are just symptoms of the fact that a group of policies that were identical in terms of risk classification as new customers will have a range of different characteristics at renewal and be charged different premiums. The solution is simply to track the movement of policies into different risk classes as they renew. Note that "risk classifications" need not be limited to different expected losses, but also renewal rates, loss cost trends or any other parameter that will have an effect on premium.

A helpful analogy is the binomial model of stock movements from basic finance. A stock starts at time $t = 0$ at price S_0, and at time 1 it increases to price $S_0\ e^{r+s}$ with probability p or decreases to price $S_0\ e^{r-s}$ with probability $1 - p$.

For insurance contract pricing purposes we will need several additional pieces of information at each "node": premium, loss, expense and probability of moving into each of the next nodes. Since the policy may not renew, the probabilities need not add to 1. We will refer to this method as either the extended asset share model, or the option model.

3. SIMPLE EXAMPLE

Let's illustrate a simple example to show how the extended model works. Importantly, we'll also show that the extended model indicates a different premium amount and relativities even in the simplest of cases.

The example is set up as follows:

- Insurer sells annual policies, and the rate plan has two risk classes: high and low.
- Policies renew 80% of the time after 1 year, and none renew after 2 years.
- Premium P collected and expense $20\% \ * \ P$ paid at time 0 and 1
- Fixed expense 10 paid at time 0
- Losses of $L_{low}\ =\ 50$ or $L_{high}\ =\ 70$ paid at times 1 and 2, depending on the policy's risk classification
- There is a 10% chance after 1 year that a low risk policy will be reclassified as high risk and charged the corresponding premium; all high risk policies remain high risk; renewal probability is independent of reclassification

- Cash flows discounted at 5% interest, and the company desires a 5% UW profit on premium

The goal is to determine what premium to charge for a low risk policy and a high risk policy, under the assumption that the rate can vary only by a policy's risk classification at the beginning of each term and is independent of the original risk classification.

3.1 Traditional Ratemaking

We will begin with traditional ratemaking, which uses the premium loading formula:

$$P = \frac{F + L}{1 - U - V}$$

Taking all inputs literally we would get:

$$P_{low} = (10 + 50)/(1 - 20\% - 5\%) = 80$$

$$P_{high} = (10 + 70)/(1 - 20\% - 5\%) = 106.67$$

The obvious problem with this is that if this premium is also applied at renewal, the customer must pay again for the fixed expenses, even though they are incurred only at the initial policy purchase. The most elementary way to correct for this would be to amortize fixed expenses over the policy's expected lifetime. In this example we have an expected lifetime of 1.8, so we would replace the fixed expenses of 10 in the formulas above with 10/1.8 = 5.56 and determine premiums of:

$$P_{low} = (5.56 + 50)/(1 - 20\% - 5\%) = 74.08$$

$$P_{high} = (5.56 + 70)/(1 - 20\% - 5\%) = 100.75$$

Note that the loss relativity is $70/50 = 1.400$ and the implied premium relativity is $100.75/74.08 = 1.3600$.

3.2 Standard Asset Share Model

Next we will use the standard asset share model. We will again use the premium loading formula, but the inputs will be expected present value of lifetime premium, fixed expense and loss. We also need to have an estimate of premium and loss trend, for which we will need some additional assumptions.

The classification ratemaking example in Feldblum's paper uses the same premium and loss trend for both classes, so we will do the same here. Suppose the company does not currently charge different rates for the two classes, but the book of business is made up mostly of low risk policies. The observed premium trend will then be 0%, since at renewal the average insured's premium stays the same. The loss trend however could be approximated at 4%, since the typical insured is low risk and their loss cost increases by 40% in 10% of cases. Below, we calculate the premium for each class of policyholders by balancing the expected present value of premium to the expected present value of loss and expense.

$$EPV(P_{low}) = P_{low} * (1 + \frac{0.8}{1.05}) =$$
$$(10 + \frac{50}{1.05} + \frac{0.8 * 50 * 1.04}{1.05^2})/(1 - 20\% - 5\%)$$
$$\Rightarrow P_{low} = 72.16$$
$$EPV(P_{high}) = P_{high} * (1 + \frac{0.8}{1.05}) =$$
$$(10 + \frac{70}{1.05} + \frac{0.8 * 70 * 1.04}{1.05^2})/(1 - 20\% - 5\%)$$
$$\Rightarrow P_{high} = 97.99$$

The premium relativity implied here is $97.99/72.16 = 1.3580$. Notice that the premiums here are lower than for the traditional ratemaking case. That's because losses are discounted to time 0 at 5% in the asset share model. Also notice that the premium relativity is lower, so that premiums for the low and high risk classes are proportionally closer to each other. That's because when losses are discounted, a larger portion of the premium goes toward the time 0 fixed expense which is equal for the two risk classes.

3.3 Extended Asset Share/Option Model

Now we will calculate premiums using the extended asset share/option model. Like the traditional asset share model, we will use the expected present value of premium, loss and expense, accounting for probability of renewal. The difference in this solution is that we will explicitly recognize the effect of low risk policies moving to the high risk class at time 1.

We first calculate the premium for high risk policies:

$$EPV(P_{high}) = P_{high} * (1 + \frac{0.8}{1.05}) =$$
$$(10 + \frac{70}{1.05} + \frac{0.8 * 70}{1.05^2})/(1 - 20\% - 5\%)$$
$$\Rightarrow P_{high} = 96.46$$

Notice that we actually arrived at a lower indicated premium than in the standard asset share model. The difference is that before we had determined a 4% average loss trend and applied that to all policies, whereas here we recognize that high risk policies have zero loss trend.

For low risk class policies, the calculation differs from the standard model in the renewal term expected losses and premium. Since there is a 10% chance that the risks are reclassified as high risk, we have:

$$EPV(L_2 \mid risk\ class\ =\ low\ at\ t = 0)\ =\ (0.8\ *\ 0.9\ *\ 50\ +\ 0.8\ *\ 0.1\ *\ 70)/1.05^2$$
$$=\ 37.73$$

Note that, not coincidentally, this is exactly the same result as for the standard asset share model because this process is what we observed in order to determine the 4% overall loss trend.

For premium, we also recognize that the 10% of low risk policies that are reclassified will also be charged the high risk class premium upon renewal. Following the same logic as above we have:

$$EPV(P_2 \mid risk\ class\ =\ low\ at\ t = 0)\ =\ (0.8\ *\ 0.9\ *\ P_{low}\ +\ 0.8\ *\ 0.1\ *\ 96.46)/1.05$$

where we have used the premium calculated above for high risk policies. Putting it all together:

$$P_{low}\ +\ (0.8\ *\ 0.9\ *\ P_{low}\ +\ 0.8\ *\ 0.1\ *\ 96.46)/1.05\ =$$
$$(10\ +\ \frac{50}{1.05}\ +\ 37.73)/(1 - 20\%\ -\ 5\%)$$
$$\Rightarrow\ P_{low}\ =\ 71.06$$

This implies a premium relativity of 1.3574, slightly less than in the standard asset share case. Also, the premiums for both classes are lower. They are lower for high risk policies because we are not incorrectly applying a loss trend factor, and they are lower for the low risk class policies because the higher expected losses in term two are paid for by increased premiums on those policies that transition to the high risk class group. In the standard asset share case, the increased second term losses were partially prepaid by higher premiums in the first term, even on policies that remain low risk.

3.4 Reclassification and Trend

The simplicity of this example allows us to closely examine what went wrong with the standard asset share model. The problem was our calculation and application of trend. For loss trend we saw that for 10% of policies loss costs increased 40% at renewal, so assuming that most of our book of business was low risk we would have observed a 4% annual loss trend. This trend is a function of risk class loss costs and transition probability. For a good paper describing this phenomenon, see [7]. In reality, there will also be a general inflation factor which impacts losses. The advantage that the extended model gives us is that we can explicitly break the average trend down into its two components: risk classification transition and inflation.

4. OPTIONS

4.1 Implicit Options in the Standard Asset Share Model

The standard asset share model assumes that premium will stay the same, or perhaps gradually inflate over the life of the policy, while the extended model recognizes that the rate can change at renewal. Even if all loss costs, expenses, renewal rates and trends were identical these two policies, one where the rate is the same or gradually inflates at renewal and one that has no premium restriction, would have different values.

Let's refer back to the example. What would be the indicated premium for a policy where the renewal rate was the same as the new business rate, even if the risk is reclassified at renewal? The rate would be the same for the high risk class policy as we calculated above for the extended asset share model. The expected lifetime premium, where the nominal premium is the same in both terms, is set equal to the loaded expected present value of fixed expenses and loss costs. For the high risk policies we have:

$$EPV(P_{high}) \;=\; P_{high} \;*\; (1 + \tfrac{0.8}{1.05}) \;=$$
$$(10 \;+\; \tfrac{70}{1.05} \;+\; 0.8 \;*\; \tfrac{70}{1.05^2})/(1 - 20\% \;-\; 5\%)$$
$$\Rightarrow \; P_{high} \;=\; 96.46$$

This is the same as in the option model because there is no risk reclassification for high risk policies in our example.

For the low risk class, we recognize the transition probability and corresponding high risk loss costs in the second term, but we do not recognize any increase in premium for the second term because there is none, as we are assuming. Borrowing from above we have:

$$EPV(P_{low}) \;=\; P_{low} \;*\; (1 + \tfrac{0.8}{1.05}) \;=$$
$$(10 \;+\; \tfrac{50}{1.05} \;+\; 37.73)/(1 - 20\% \;-\; 5\%)$$
$$\Rightarrow \; P_{low} \;=\; 72.16$$

Note that this is the same premium we obtained in the standard asset share model. That is because here the expected loss in the second term given that there is a second term is 50 * 1.04. This is the same term appearing in the calculation for the standard asset share model since we estimated a 4% loss trend. Also, when we calculated the standard asset share model we had assumed a 0% premium trend, which is exactly what the premium trend would be for an insurance policy with a guaranteed renewal rate as we have here.

So we obtain the same premium for a guaranteed rate contract as we get for a traditional contract priced using the standard asset share model. What should the difference in price be between an insurance contract with a guaranteed renewal price and one without? The difference will be the value of the implicit option that comes with a guaranteed renewal rate; the option, but not obligation, for the insured to renew the policy for a predetermined price.

Let's determine the value of this option in our example from the perspective of the customer. If they start out classified as low risk, the price for insurance using the extended asset share model is 71.06. The price has an 8% chance of increasing to 96.46 at renewal (10% chance of our company's premium increasing and an 80% chance that the policy renews), otherwise it remains at 71.06. This is exactly like a simple example from option pricing using the binomial stock price model.

The possible values of the insurance contract at time 1 are 96.46 with probability 8%, 71.06 with probability 72%, and 0 with probability 20%. The strike price of the call option is the low risk price which is 71.06. Since the option is purchased at time zero and the payoff is at time one we must discount, so the value of the option is:

$$C = 0.08 * (96.46 - 71.06) / 1.05 = 1.935$$

The reader may note that things don't seem to add up. We said that the guaranteed rate contract is a standard contract plus an option, so the value of the guaranteed rate contract should be the value of the standard contract plus the value of the option, but we have:

$$71.06 + 1.935 = 73.00 \neq 72.16$$

The disconnect is that since the rate for the guaranteed rate contract is fixed for the life of the policy, the price of the option is amortized and paid for in equal amounts in the first and second terms. The expected NPV of a dollar of premium charged to a low risk customer with a guaranteed price contract is:

$$1 + \frac{0.8}{1.05} = 1.762$$

so the price per term for the option is:

$$1.935/1.762 = 1.10$$

and we have:

$$71.06 + 1.10 = 72.16$$

So the price of the guaranteed rate contract is the sum of the price of the standard contract and the amortized cost of the option.

4.2 The Renewal Rate Change Option

It's also possible to view a standard insurance contract as a guaranteed rate contract where the policyholder has sold the implicit call option back to the insurer. How would that transaction work exactly? Suppose the policy has been reclassified as high risk before renewing. The policyholder can renew at the guaranteed price. The insurer will then choose to exercise the call option, since the price of insurance exceeds the option's strike price of the guaranteed rate. The option seller, the customer, must pay the insurer the difference between the strike price (low risk policy premium) and the asset price (high risk policy premium), the asset being an insurance policy. The net payment upon renewal by the customer is the high risk class price, as is the case in a standard insurance contract. This is a convenient way of viewing insurance options as insurance company assets; renewal rate change options.

4.3 Business Relevance

The above is not just abstract theory. We have shown that in a relatively simple example, we are able to charge actuarially justified lower prices for insurance coverage. We are also able to price the value of options on insurance coverage. In any insurance contract where the price at renewal is not completely predetermined, the insurer holds a renewal rate change option of some value, but there are many examples of true insurance options that are being sold to insurance customers in the marketplace today.

In personal lines, some companies offer special features that act to reduce or eliminate rate increases upon renewal. One such feature is "accident forgiveness". In this case, the company will decline to reclassify the policy as higher risk even though there has been a claim, meaning that the premium would be increased under the standard rating plan. The company in this case is selling an option to the customer, or equivalently, failing to buy back the renewal rate change option on a fixed rate contract. The option is a bit more complex however because other types of rate changes would still be allowed.

Another interesting product in personal lines is perpetual deposit fire insurance. For this type of policy, the insured places a deposit with the insurer and in return receives "free" fire insurance coverage. The deposit is returned when the policy terminates, no matter what the loss experience has been. This product is a swap between interest payments and insurance payments. One could imagine some bells and whistles added, much like has been done in life insurance, that incorporate complex option-like features such as minimum investment returns.

In commercial insurance and reinsurance, there are many types of policy features which involve the implicit selling of options. Some common features are policy extensions, see [5], and multi-year coverage. Multi-year coverage is the most obvious example of options in insurance because it is precisely the guaranteeing of future insurance coverage availability at a predetermined price. Different contracts may have features which oblige the insured to purchase coverage in the future years, which would make these contracts more like forward contracts.

An interesting example in reinsurance is a contingently automatically renewing policy. How it works is that if the financial result (e.g. loss ratio or combined ratio) remains within a certain range, then the reinsurer is obligated to renew the treaty at a predetermined price. If the cedent has the right but not obligation to buy at the strike price, then this reinsurance contract contains options on future reinsurance coverage with a trigger. It may also contain compound options if the renewed treaty again includes options.

Another area where insurance options are present is in the regulation of insurance rates. There are many laws which limit cancellation of policies, limit the magnitude of rate changes, and prevent or slow rate changes. Some of the most hotly debated regulations are those which either allow or disallow certain risk classifications. These are all examples of options which are required to be included in the regulated contracts, or limitations on the exercise of renewal rate change options.

5. GENERAL FRAMEWORK FOR THE EXTENDED ASSET SHARE/OPTION MODEL

We will now present the full mathematical framework for applying the extended asset share model in the general case. The general case is an extension of the simple example in a number of ways. There can be an unlimited number of risk classes and policy terms. Risk classes can transition to and from any other class. The expected loss for any risk class can be anything in any policy term, independent of previous or future terms and of other risk classes. The same will be true for renewal rates and transition probabilities.

The mechanics are essentially just linear algebra, with each dimension representing a risk class. Premium, losses and fixed expenses will be represented by vectors, or series of vectors. We track the movements of risks through the risk classes with a series of matrices. These matrices will represent the transition probabilities between each risk class and probability of termination at each renewal. We will continue to use the simple premium loading formula, with expected present lifetime value for all inputs.

Given the loss cost and fixed expense vectors, the risk class transition matrices, the discount factors, premium trend and the variable expense and profit loads, the method will solve for the current premium level and rate relativities which result in the expected present value of lifetime premium equaling the loaded expected present value of loss costs and expenses simultaneously for all risk classes.

Suppose there are n distinct risk classifications. Let R be an n-dimensional vector such that $R[i]$

is the premium relativity for the i^th^ risk class. R is fixed and does not vary over time. We can require that $R[1] = 1.00$. Let $P(t)$ be a premium index over time (a real function, not a vector), so that $P(t)$ is the expected average rate level independent of risk class transitions. Basically, if a company assumes it will take a 5% rate increase each year, then we would have

$$P(n) = P(0) * 1.05^n$$

We do not need to assume or specify a $P(0)$, as the method will solve for the correct current average rate. Let L_t be a series of n-dimensional vectors so that $L_t[i]$ is the expected loss costs for risk class i at time t. Let F be an n-dimensional vector of fixed expenses so that $F[i]$ is the fixed expense, assumed to all be incurred at time 0, for risk class i. Let U and V be the usual profit and variable expense provisions in the premium. These are real numbers, independent of time and risk class. Let $v(t)$ be a series of discount factors, such that the present value of \$1 at time t is $v(t)$ at time 0. Obviously we must have $v(0) = 1$. Finally, let A_t be a series of n-dimensional square matrices such that the probability that a risk in class i at time 0 is in class j at time t is $A_t[i,j]$. The matrices are cumulative transition probability matrices; it is equivalent to define a series of incremental transition probability matrices.

Many readers will notice that under the above construction the policies' risk classification forms a Markov Chain. It may be possible to put more of this exposition into the language already developed in the extensive literature on Markov models, but the goal of this paper is to build on the asset share model as developed in the actuarial literature.

5.1 Premium Formula

We use the standard premium loading formula and as usual equate premiums with loaded loss plus fixed expense, but we use the expected present lifetime value for all quantities as in the standard asset share model:

$$\sum_{n=0}^{\infty} P(n) * v_n * A_n(R) = \frac{1}{1-U-V}\left[F + \sum_{n=0}^{\infty} v_n * A_n(L_n)\right] \tag{1}$$

Let's first break down the left hand side to see why it equals the expected present lifetime value of premium. More precisely, it is an n-dimensional vector, such that the i^th^ entry is the expected lifetime present value of premium for a risk in class i at time 0. $P(n)$ is the average "rate" at time n, and v_n is the discount factor to time zero. The i^th^ entry of $A_n(R)$, which is a matrix applied to a vector, is the inner product of the ith row of the matrix A_n with the vector R. Since A_n is the cumulative transition probability matrix, the rows represent the distribution among the risk classes at time n of risks starting in the class corresponding to that row (as we have defined it). That means the i[th] entry of $A_n(R)$ is the average premium relativity factor at time n for risks in class i at time 0, weighted by the probability that the risk is in each class at time n. So when we sum over all time periods n (the initial new business policy purchase and all subsequent renewals), the i[th] entry on the left hand side is the expected present lifetime value of premium for risks in class i at time 0, taking into account discounting, probability of renewal, premium trend and the difference in premium due to future expected risk classification changes.

What about the right hand side? Likewise, it is a vector. The i[th] entry equals the expected present value of lifetime losses and fixed expense, loaded for profit and variable expense. The profit and expense portions are clear, so we will discuss the sum. Again, v_n is the discount factor to time 0. The i[th] entry of $A_n(L_n)$ is the average loss costs at time n for a risk that was in class i at time 0. That's because the i[th] entry is the inner product of the i[th] row of A_n, which is the risk class distribution at time n of risks starting in class i at time 0, with the vector L_n, which is the vector of loss costs by risk class at time n. Therefore the right hand side is a vector representing the loaded expected present value of lifetime expense and loss costs for each risk class.

Despite the summation, the left hand side is just a matrix applied to the premium relativity vector. To get premium alone on the left hand side we apply the inverse of the matrix (assuming it has one):

$$P(0) * R = \frac{1}{1-U-V}\left(\sum_{n=0}^{\infty} \frac{P(n)}{P(0)} * v_n * A_n\right)^{-1} \left(F + \sum_{n=0}^{\infty} v_n * A_n(L_n)\right) \qquad (2)$$

5.2 Return to Traditional Ratemaking

We can recreate the traditional ratemaking formula from the extended asset share model premium formula above by making some simplifying assumptions for the parameters. Suppose that losses gradually inflate over time with premium trending in step. Also suppose losses trend with general inflation, which is expected to match the risk free investment return. In the notation above we would have:

$$\frac{P(n)}{P(0)} = \frac{L_n[i]}{L_0[i]} = {v_n}^{-1}, \forall\, i$$

The premium formula then simplifies to:

$$P_0 * R = \frac{1}{1-U-V}\left(\sum_{n=0}^{\infty} A_n\right)^{-1} \left(F + \sum_{n=0}^{\infty} A_n(L_0)\right)$$

Distributing the matrix on the right side to the fixed expense and loss terms separately, we can cancel the matrix and its inverse:

$$P_0 * R = \frac{1}{1-U-V}\left[\left(\sum_{n=0}^{\infty} A_n\right)^{-1}(F) + L_0\right]$$

Now premium and loss are unaffected by the matrix sum. Supposing again that we ignored risk class transitions, then each A_n would be a diagonal matrix, with entries equal to the renewal probability for each risk class. We would then have that the sum of A_n over all n would be a diagonal matrix with entries equal to the expected lifetime at time 0 of the policy given its risk class. The i^{th} entry of the $(\sum_{n=0}^{\infty} A_n)^{-1}(F)$ vector would then be the fixed expense for the i^{th} risk class divided by the expected policy lifetime for risk class i. This is the same expense amortization that we have seen previously. So what we have shown is that traditional ratemaking with fixed expense amortization is the equivalent of the extended asset share model with simplified premium and loss trend assumptions.

6. IN-DEPTH EXAMPLES

We will work through two more complex examples to demonstrate the potential of the extended asset share model. First we will broaden the assumptions in the two risk class case we analyzed at the beginning of the paper, and explore how changes in the parameters affect the premium relativities. Then we enhance the example by adding a "medium risk" class with slightly higher expected losses but the same premium as the low risk class. This will demonstrate how to use the extended asset share/option model to determine the proper premium to charge for "accident forgiveness" or other anti-experience rating policy features. We will again show that the indicated premium will be the sum of the standard premium plus the amortized cost of the options.

6.1 Two Risk Class Case

We can use the extended asset share model premium formula to calculate premiums and relativities for a more complex two risk class example. The low risk class will be given all the characteristics of more desirable underwriting risks such as decreasing loss costs and longer expected lifetime, while the high risk group will be given less desirable characteristics. After calculating the premium, we will compare the derivative of the premium with respect to the various parameters. The assumptions are as follows:

- Insurer sells annual policies, and the rate plan has two risk classes: high and low

- Low risk policies renew with 90% probability at the end of each term, indefinitely
- High risk policies renew with 70% probability at the end of each term, indefinitely
- Premium P collected and expense $20\% * P$ paid at times 0, 1, 2, etc.
- Fixed expense 10 paid at time 0
- Losses of $L_{low}(0) = 50$ or $L_{high}(0) = 70$ paid at time 1 for the first term
- Losses inflate at 1% per year for low risk policies and 3% per year for high risk policies
- At the end of each and every year there is a 10% chance that a low risk policy will be reclassified as high risk and charged the corresponding premium; all high risk policies remain high risk
- Low risk class policies renew with only a 70% probability at the time of reclassification to high risk
- Premiums are expected to inflate at 4% per year.
- Cash flows discounted at 5% interest, and the company desires a 5% UW profit on premium

The only area of ambiguity is that we have said low risk policies renew 90% of the time, transition to high risk 10% of the time, and renew 70% of the time after transition (due to their aversion to rate increases). We will assume the 90% renewal rate is the total renewal rate so that at renewal 10% do not renew, 7% (=10% * 70%) transition to high risk and renew, and the remaining 83% remain low risk and also renew.

Notice that we have assumed premium trend will outpace loss trend on policies that start today. Accounting for this phenomenon of improving loss costs over policy tenure was the original motivation behind the asset share model.

The starting point is the extended asset share premium formula (2) where we had isolated the premium on the left hand side. All necessary parameters are defined above, and the solution is an exercise in linear algebra. Detail on the calculations can be found in Appendices 1 and 3 and were completed using Sage, the Python-based open source mathematics software.

As a point of reference, let's determine what premiums we would charge if our ratemaking method was traditional with expense amortization:

$$P_{low} = (50 + \tfrac{10}{10})/(1 - .2 - .05) = 68.00$$
$$P_{high} = (70 + \tfrac{10}{3.33})/(1 - .2 - .05) = 97.33$$

where 10 and 3.33 are the expected lifetimes of low and high risk policies, respectively, given a

90% and 70% probability of renewal.

The extended asset share model determines the optimal premiums as follows:

$$P_{low} = 56.22$$

$$P_{high} = 90.92$$

The method indicates discounts relative to the traditional premium, just as the standard asset share model would. The discount is higher for low risk policies because their loss trend is more greatly outpaced by premium trend.

It is more interesting here to determine the sensitivity of the low risk premium to the new parameter we have introduced with the extended model: risk classification transition probability. Also, because we are recognizing the chance for reclassification, the low risk premium may also depend on the parameters of the high risk class.

In fact, it isn't even obvious a priori whether the derivative of P_{low} is positive or negative with respect to transition probability. Let us present seemingly valid arguments for both positions. We will call the transition probability inclusive of renewal probability, 7% in our example here, q:

- $\partial P_{low}/\partial q < 0$: Since P_{low} is smaller than P_{high} and total lifetime profit is set at 5% of premium, the dollar amount of required profit on low risk policies is lower than for high risk policies. When policies transition to high risk, they are priced to a higher dollar value of profit, so the extra profit made on transitioning low risk policies subsidizes the price for low risk policies that do not transition.
- $\partial P_{low}/\partial q > 0$: Since high risk policies have short lifetimes, the profit must be front loaded in the first few terms. When low risk policies transition, it is into the later terms of the high risk class which have low profitability or may even be unprofitable. The low risk price will need to be raised to subsidize the relative loss on transitioning policies.

There are some obvious flaws in both of these arguments, but either could be valid under certain assumptions for this simple example. For the parameters we have assumed we have:

$\partial P_{low}/\partial q = -0.28$ (q expressed as a percentage)

That means that keeping all other parameters the same, a 1 point increase in transition probability (inclusive of renewal rate) should result in a reduction in premium of 28 cents for all low risk policies. That makes sense in terms of argument number 1: since premium trend outpaces loss trend for high risk policies, they become more profitable as time goes on and that additional profit subsidizes the price of all low risk policies. If we change the parameters of the example so that high risk loss trend is 4% and premium trend is only 2%, then we see that:

$\partial P_{low}/\partial q \;=\; 0.19$ (q expressed as a percentage)

In this case argument number 2 is correct. Since high risk loss trend outpaces premium trend, later terms of high risk policies are less profitable and may even result in loss. Those terms are precisely when low risk classes are expected to transition, so those losses must be subsidized by higher rates for all low risk policies.

To give some context to the values above we will show the derivative of the low risk premium with respect to some of the other variables. The reader should keep in mind that a 1 point change in trend is a much larger variation proportionally than a $1 change in expected loss, or a 1 point change in renewal probability. To put these measures on an even basis of deviations from the mean, we should look at derivatives of the natural log of the quantities. These can be interpreted as the percentage change in low risk class premium for every percent change in the independent variable. To obtain the figures below from the raw derivatives calculated in Appendix 1 one simply needs to scale by the ratio of the quantities using:

$$dln(y)/dln(x) = (dy/y)/(dx/x) = dy/dx * x/y$$

We have:

$\partial ln(P_{low})/\partial ln(q) \;=\; -0.04$

$\partial ln(P_{low})/\partial ln(p) \;=\; -0.94$ (p = low risk renewal and non-transition probability)

$\partial ln(P_{low})/\partial ln(L_{low}) \;=\; 0.99$

$\partial ln(P_{low})/\partial ln(r_0) \;=\; 3.91$ (r_0 = low risk class loss trend)

$\partial ln(P_{low})/\partial ln(r_1) \;=\; 1.73$ (r_1 = high risk class loss trend)

The above shows that the calculated premium is much more highly leveraged on loss trends than the other variables. While the risk class transition probability appears relatively unimportant, the high risk class loss trend has more leverage on the low risk class premium than either the low risk renewal rate or the low risk loss costs. There must be some significant second order effects as well, since if the transition probability were zero, then the high risk loss trend would have no effect on low risk premium.

6.2 Anti-Experience Rating

As mentioned above, a popular feature of personal lines insurance is “accident forgiveness” or anti-experience rating. A private passenger auto rating plan typically segments risks finely by accident and driving violation history. What would typically happen after an accident is that at renewal the policy would be reclassified based on the new information and charged a higher premium. The promise made by companies offering anti-experience rating to select groups of customers is that in the event of an accident, the company will decline to reclassify the risk and the renewal premium will remain the same. This is a very good example of an option, or more precisely a call option held by the customer and purchased from the insurer.

We will calculate the correct premium for policies with anti-experience rating options using the extended asset share model. We will utilize the previous example with some modifications as follows:

- There is a third risk class, medium or "med" as a variable subscript
- $L_{med} = 55$ and loss trend over time is 2%
- Low risk policies transition to medium risk only with the same 10% probability at the end of each and every year
- Medium risk policies are charged the same premium as low risk, i.e. their slightly higher expected loss is "forgiven"
- Medium risk policies renew with 95% probability, including the term in which they transition
- Medium risk policies transition to high risk with probability 25% at each renewal, otherwise they remain medium risk

We will try to answer the following simple questions: What price would we charge to each of the three classes, low, medium and high, if we ignore anti-experience rating? Then, given that medium risk policies are "forgiven", and charged the same price as low risk policies, what should that price be? What is the value of the call options offered to the low risk policies? How do these quantities relate?

It is worth taking a moment to think about how difficult it would be to answer these questions using just the traditional ratemaking formula or the standard asset share model. Traditional ratemaking clearly falls short, since it only considers one policy period of loss costs. If we add fixed expense amortization, then it would actually indicate a lower price for low risk policies with anti-experience price guarantees, since the expected policy lifetime is longer. Although it is possible for a policy feature which increases insurer costs but also increases policy tenure to pay for itself in increased lifetime profit, the traditional formula is not capable of truly identifying these opportunities and this example is not one of them.

The standard asset share model incorporates persistency rates and trends, so it gives a better chance for modeling this example. The option would affect premium trend since premiums increase less because of option exercise, and customer persistency since customers offered below market prices will be more likely to renew. The standard asset share model would be able to pick up these effects; it's just a question of how accurate it would be. Since the two effects are likely to be small and in opposite directions the analyst may only be confident that the premium should be different, but unsure whether it should be higher or lower.

Another problem with the standard asset share model is that it assumes constant trends throughout time. Since the low risk policies are guaranteed the same rate at the first renewal, the premium trend due to risk reclassification is 0%. There will still be the 4% overall annual premium inflation we assumed. Premium trend will increase for subsequent renewals as policies transition to high risk and are charged higher premiums. Eventually, all policies will be classified as high risk and premium trend will again be 0% attributable to reclassification, plus premium inflation. This clearly cannot be modeled properly with a constant trend. Likewise the standard asset share model assumes a constant loss trend, but in this example we have the same phenomenon of changing trends throughout time.

Even if we were to recognize the changing loss and premium trend throughout time in the standard asset share model, how would we calculate those trends? What is the average premium increase between the third and fourth renewal for policies starting out as low risk at time zero? It is a function of the transition matrix and the premiums charged, but we haven't determined the proper premiums yet. The same complications arise with persistency rates. We seem to be doomed before we even begin.

This is a relatively simple problem when using the extended asset share model, solved in just a few lines of code which are detailed in Appendix 2. The correct premiums are:

$$P_{low} \text{ (no option)} = \$54.98$$

$$P_{med} \text{ (no option)} = \$64.25$$

$$P_{low} \text{ (with option)} = P_{med} \text{ (with option)} = \$57.65$$

$$P_{high} = \$90.92$$

Note that for policies starting out as high risk, there is no difference between this example and the previous one where we only had high and low risk policies. All trends and persistency rates are the same, so the extended asset share model has indicated the same premium. The premiums indicated for the low risk class, either with or without the option, are not comparable with the previous example because we have changed the parameters. In particular, the expected lifetime is different and the rate of transition to high risk is different.

The second part of the problem is to determine the value of the options embedded in the low risk class policy with forgiveness, and to relate it to the price we determined for the policy with and without the options. We know the option is a call option held by the insured. What is the strike price? Because the low policyholder is guaranteed a price at renewal of $57.65 we might guess that to be the strike price, but the $57.65 is the price for another term of insurance coverage plus another call option on the price of insurance at the next renewal. The $57.65 really represents $54.98 for insurance coverage, plus $2.67 for call options with strike price $54.98.

We can easily determine the value of the option with exercise at the first renewal. There is a 10% chance that they will be reclassified as medium risk, and in that event a 95% chance of renewing for a combined 9.5% chance of option exercise. The "market price" for medium risk insured is what the company would like to charge, $64.25. After incorporating the 4% premium trend and 5% cash flow discounting, the present value of this option at time zero is:

$$0.095 * (64.25 - 54.98) * e^{0.04-0.05} = \$0.8719$$

That's a lot less than the $2.67 we've determined should be charged to provide low risk policies with the option, so where is the disconnect? We need to consider options provided on future renewals as well. What is the value of the option to a policy that has already moved to the medium classification? The market price is $64.25 and we are offering them an in-the-money option to purchase insurance at renewal for $54.98. We have assumed they have a 95% total persistency rate with a 25% chance of being reclassified as high risk and a 70% rate of renewal if that happens. The probability of option exercise is 95% - 25% * 70% = 77.5%. With discounting and trending, the NPV of this option at the beginning of a term is:

$$0.775 * (64.25 - 54.98) * e^{0.04-0.05} = \$7.1128$$

This is a lot more than the difference in premium we determined for adding the option to a low risk policy. Recall the simple example from Section 3. There was only one option provided for the life of the policy since all policies terminated at the second renewal, but the cost of the option was spread over both the first and second term premiums. We will show the same for this example, that the $2.67 extra charged for the low and medium risk policies with forgiveness is the amortized cost of options offered over the life of the policy.

The lifetime value of the options can be calculated directly using the binomial model from basic finance. The value of the option is the expected present value of the payoff, which is the difference between the strike price and the market price at time of option exercise, times the probability of option exercise. At the nth renewal, the option pays off for all low risk class policies transitioning to medium risk, and all low risk policies that transitioned to medium risk in the 1st though n-1st renewal and remain medium risk. The reader can verify the following expression with increased decimal precision as the lifetime present value of the option payoffs:

$$(64.24538 - 54.98420) \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} 0.805^{n-1-k} * 0.095 * 0.775^{k} * e^{-0.01n}$$

Evaluating either algebraically or numerically, we find that the lifetime present value of options offered to a low risk policy at time zero is approximately $18.4379.

From the Sage output (see Appendix 2), we have that the present value of $1 charged to low and medium risk policies over their lifetime, until termination or transition to high risk, is $6.92. Again increasing decimal precision, we find that the present value of the $2.67 charge is equal to the above:

$$2.66569 * 6.91675 = \$18.4379$$

So what we have shown is that the extended asset share model calculates the premium required for an insurance policy with accident forgiveness as the price of a standard policy plus the amortized cost of the options.

7. ADVANCED CONSIDERATIONS AND DISCUSSION

As we have seen, there are a number of advantages with the extended asset share/option model. We have been able to more precisely price traditional insurance contracts based on expected lifetime profit, and we can also price more complex policy features such as options on future coverage prices. This paper is meant to be an introduction of the method, not the final word. Below are some issues worth further consideration.

7.1 Dynamic Persistency

The biggest shortcoming of the method as presented here is that renewal rates are not independent of premium levels in reality. Obviously, the higher the price, the lower the probability of renewal, especially if the price is high relative to competitors. In the development above our transition matrices A_n are not functions of $R[i]$, which effectively assumes independence so that the premium can be isolated on the left hand side of the equation. This was a problem we identified with the original asset share model. There is nothing stopping us from defining A_n as being a matrix of functions of the premium relativities, $R[i]$, but this would require more complex numerical methods for determining the optimal premium relativities.

7.2 More Sophisticated Premium Formulas

Throughout we assumed that the premium was given by a simple loading formula. What the asset share model was able to do for us was more precisely determine the effect of risk reclassification on the optimal premiums. In practice, it may be desirable to use one of the many other premium formulas, see [6], such as IRR or RORAC with capital allocation. One could extend the current method by determining an allocated capital vector C, one amount for each risk class. The desired rate of return times the allocated capital would then be an expense accounted for on the right hand side of the premium equation.

7.3 Regulatory Constraints

A final issue worth discussing is the constraint on rates that may prevent the use of the extended (or even traditional) asset share models. It is conceivable that two risks could have identical expected loss costs, expenses and renewal rates but be charged different premiums. For example, it could be that one risk is much more likely to transition to a state that is not able to be charged actuarial rates and cannot be canceled. Our method would call for that risk's current rate to be increased to pay for expected future losses. This phenomenon may violate some state insurance laws, and even "Actuarial Standard Of Practice Number 12, Risk Classification" could be interpreted as prohibiting this.

The disconnect is that, as we mentioned in our discussion of insurance options, there is an implicit option being sold with the policy since the insured will have the right but not obligation to buy at a predetermined price which is less than "market price" (the rate the company would like to charge) in case of transition to the subsidized risk class. A potential defense of the method to the above criticism would be that the price difference reflects a true difference in costs, with that difference being the cost of the option. Just as options on stock prices or commodities are real and quantifiable, options on insurance coverage are real and quantifiable as I hope this paper has demonstrated. It would therefore be actuarially appropriate and in line with the ASOP to charge an increased premium for this additional cost.

If the company were prohibited from explicitly charging for legally mandated insurance options bundled with the policy, then a solution would be to selectively offer a policy with a longer term to the risks with lower transition probability to the subsidized class. The rate for the standard policy would be increased to cover the option value, and all policyholders who qualified would opt for the longer term policy whose rate would be relatively lower because the implicit options are cheaper.

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Author Biography

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Appendix 1. Code for Example of Section 6.1

Below is a copy of the Sage code used to calculate the results in Section 6.1:

```
sage: a,b = var('a,b')
sage: p,q,r,s = var('p,q,r,s')
sage: v,d = var('v,d')
sage: A = matrix(2, [p,q,r,s])
sage: D = matrix(2, [a,0,0,b])
sage: x,y,z,w = var('x,y,z,w')
sage: S = matrix(2, [x,y,z,w])
sage: I = matrix(2, [1,0,0,1])
sage: T = v*I + v*A*S*D - S
sage: g = var('g')
sage: g = solve([ e == 0 for e in T.list() ], S.list(), solution_dict = True)
sage: S_solve = S.subs(*g)
sage: reset('e')
sage: M_left = I-v*d*A
sage: M_right = M_left*S_solve
sage: l_1,l_2 = var('l_1,l_2')
sage: L = vector([l_1,l_2])
sage: P_1,P_2,U,V, F_1,F_2 = var('P_1,P_2,U,V, F_1,F_2')
sage: P =vector([P_1,P_2])
sage: F = vector([F_1, F_2])
sage: P = (M_left*S_solve*L + M_left*F)/(1-U-V)
sage: example = [{l_1:50, l_2:70, F_1:10, F_2:10, v:e^-0.05, d:e^0.04, a:e^0.01, b:e^0.03, U:0.05, V:0.20,
sage: p:0.83, q:0.07, r:0, s:0.7}];
sage: P.subs(*example)
(56.2179531176269, 90.9236721460172)
sage: P[0].diff(q).subs(*example)/100
-0.281473365369170
sage: example2 = [{l_1:50, l_2:70, F_1:10, F_2:10, v:e^-0.05, d:e^0.02, a:e^0.01, b:e^0.04, U:0.05, V:0.20,
sage: p:0.83, q:0.07, r:0, s:0.7}];
sage: P[0].diff(q).subs(*example2)/100
0.188685376430719
sage: P[0].diff(p).subs(*example)
-63.9320830023675
sage: P[0].diff(l_1).subs(*example)
1.11622969670737
sage: P[0].diff(a).subs(*example)
217.553423159780
sage: P[0].diff(b).subs(*example)
94.3772606495679
sage: P[0].diff(l_2).subs(*example)
-0.0149466720869281
sage: P[0].diff(d).subs(*example)
-359.635554630114
sage: P[0].diff(s).subs(*example)
-3.26752863057154
sage: P[0].diff(v).subs(*example)
-2.68704335235142
```

Appendix 2. Code for Example of Section 6.2

Below is a copy of the Sage code used to calculate the results in Section 6.2:

```
sage: v,d = var('v,d')
sage: A = matrix(3, [.805,.095,0, 0,.775,.175, 0,0,.7])
sage: D = matrix(3, [e^0.01,0,0,0,e^0.02,0,0,0,e^0.03])
sage: s1,s2,s3,s4,s5,s6,s7,s8,s9 = var('s1,s2,s3,s4,s5,s6,s7,s8,s9')
sage: S = matrix(3, [s1,s2,s3,s4,s5,s6,s7,s8,s9])
sage: I = matrix(3, [1,0,0,0,1,0,0,0,1])
sage: v =e^-0.05
sage: d =e^0.04
sage: T = v*I + v*A*S*D - S
sage: g = var('g')
sage: g = solve([ e == 0 for e in T.list() ], S.list(), solution_dict = True)
sage: S_solve = S.subs(*g)
sage: reset('e')
sage: M_left = (I-v*d*A).inverse()
sage: L = vector([50,55,70])
sage: P_1,P_2,P_3, U, V= var('P_1,P_2,P_3, U, V')
sage: P =vector([P_1,P_1,P_3])
sage: F = vector([10,10,10])
sage: M = matrix(2,3,[1,0,0,0,0,1])
sage: U = 0.05
sage: V= 0.2
sage: G1 = M*(M_left*P - (S_solve*L + F)/(1-U-V))
sage: G1
(6.91675209043679*P_1 + 1.12370402721951*P_3 - 500.921305786324, 3.25769915424678*P_3 -
296.201975547860)
sage: solns = solve( [e == 0 for e in G1.list()], [P_1, P_3], solution_dict = True)
sage: [[s[P_1].n(25), s[P_3].n(25)] for s in solns]
[[57.64989, 90.92367]]
sage: P2 =vector([P_1,P_2,P_3])
sage: G2 = (M_left*P2 - (S_solve*L + F)/(1-U-V))
sage: solns2 = solve( [e == 0 for e in G2.list()], [P_1, P_2, P_3], solution_dict = True)
sage: [[s[P_1].n(25), s[P_2].n(25), s[P_3].n(25)] for s in solns2]
[[54.98420, 64.24538, 90.92367]]
```

Appendix 3. Matrix Math

Here we will bridge the extended asset share model premium formula in the paper with the code in the previous two appendices.

We begin with the premium side matrix sum, which is simpler. In formula (2) of Section 5.1, the extended asset share premium formula where we have isolated the premium on the left hand side, the right hand side contained the term:

$$\left(\sum_{n=0}^{\infty} \frac{P_n}{P_0} * v_n * A_n\right)^{-1}$$

Since the incremental transition probabilities are the same for all time periods, we have $A_n = A^n$. Likewise, $v_n = v^n$ for the present value factors and $d_n = d^n$ for the premium trend. The familiar geometric series formula can then be used:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Since the sum is inverted on the right side of the formula, that is where we get the term:

$$M_{left} = 1 - v * d * A$$

The matrix sum for the loss is a little trickier. In the general case, we let L_n be a series of independent vectors. We cannot just sum the matrices because they are being applied to a series of vectors, not just one as in the premium sum:

$$\sum_{n=0}^{\infty} v_n * A_n(L_n)$$

In these examples however, we assume constant loss inflation trends for each risk class, so we have:

$$L_n = D^n(L_0);\ D = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}$$

So the problem is now to simplify the matrix sum:

$$S = \sum_{n=0}^{\infty} v^{n+1} * A^n * D^n$$

Here we use $v_n = v^{n+1}$ since losses are paid at the end of the period, instead of the beginning as premiums are.

We cannot simply apply the geometric series however because the matrices A and D do not necessarily commute, i.e. $A^n * D^n \neq (A * D)^n$. We do however have the following which produces a simple identity for S:

$$v * I + v * ASD = v * I + v * A \left(\sum_{n=0}^{\infty} v^{n+1} * A^n * D^n \right) D$$

$$= v * I + \sum_{n=0}^{\infty} v^{n+2} * A^{n+1} * D^{n+1}$$

$$= v * I + \sum_{n=1}^{\infty} v^{n+1} * A^n * D^n$$

$$= \sum_{n=0}^{\infty} v^{n+1} * A^n * D^n = S$$

We let Sage solve this equation numerically.