A MODEL FOR COMBINING TIMING, INTEREST RATE, AND AGGREGATE LOSS RISK

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Abstract:

The purpose of this paper is to develop a simple model for determining distributions of present value estimates of aggregate losses. Three random components of the model that will be described are aggregate losses, payout patterns, and interest rates.

In addition, this paper addresses the impact of timing and investment variability on risk margin/solvency requirements.

INTRODUCTION

The work product of the actuary has historically involved the estimation of expected or average values. Although expected values have been known to be imprecise, the quantification of the dispersion of actual losses around their forecast expected value has not been awarded much significance.

In more recent years with an increase in the number of companies employing self-insurance as a risk funding mechanism the evaluation of risk has taken on greater importance. For instance, the management of a small self-insured entity needs to be aware of the potential impact that variability in the retained losses can have on the entity's financial status. The management may want to fund the company's expected losses and also provide for a "risk margin" which can absorb shock losses.

A captive is frequently used as an alternative to self-insurance. Capital requirements for the captive must be sufficient to prevent insolvency of the captive in the event that actual loss experience proves to be much worse than it is expected to be. Currently, capital requirements for captives as well as for larger insurance companies are frequently determined by ad hoc rules rather than rigorous analytical methods.

Methods have been described by a number of authors[1,2] which can, and frequently are, used to determine "risk margins" on self-insured loss funding or capital requirements for companies. The most popular of these models analyzes the variability of aggregate losses. However, the impact of adverse loss experience on a company is also affected by the amount of investment income which can be earned on funds set aside to pay losses. If losses do not have to be paid for many years and interest rates are high, investment income can offset some of the bad loss experience.

The contribution of future investment income to risk margin requirements is frequently estimated by present valuing a selected percentile of an aggregate loss distribution. This method of estimation generally treats payout patterns and interest rates as if they are deterministic variables. In this paper, a simulation model is described which can be used to analyze the dispersion of the present value estimates of aggregate losses. This model will incorporate three components of variability into the estimate of discounted losses. These components are modelled as follows:

1. Aggregate losses are generated from an aggregate probability distribution based upon the "collective risk model."

2. Payment pattern estimates are developed from a regression. The mean square error of the regression's residual is then used to model

payment pattern variability.

3. A time series procedure is used to develop a stochastic model for interest rates.

The models described in this paper represent an approach to simultaneously analyzing timing risk, investment return uncertainty, and aggregate loss variability . For the purpose of illustrating this approach, relatively simple models have been used. However, many appropriate alternatives to these models exist and may be preferred by the analyst.

Two examples of applications of this model will be presented in this paper. In the first example, results of the model will be used to determine capital requirements for a captive insuring automobile liability. In the second example, the model capital requirements will be calculated for the medical malpractice line. These two examples are intended to illustrate the different impact that timing and interest rate risk have on different lines of business.

AGGREGATE LOSS DISTRIBUTIONS

The most commonly employed model for aggregate loss distributions is known as the "collective risk model." This model is described more thoroughly in numerous publications, but a brief review of some of the assumptions of the collective risk model follows:[3,4]

1. The number of losses in any time period is assumed to be generated by a random process.

2. Claim sizes are assumed to be independent and identically distributed.

3. The size of claims is assumed to be independent of the number of claims.

In order to generate an aggregate probability distribution using this model, the following algorithm is utilized:

1. Generate a random claim count, N, from a probability distribution.

2. Generate N random claim sizes from a claim size distribution.

3. Apply per occurrence limits for the program being analyzed to each claim generated.

4. Sum the claims and apply aggregate limits to this total loss amount.

The probability distributions from which the claim counts and claim sizes are generated are typically parametric functions, although distributions derived from empirical data can also be used. The claim count distributions most commonly employed are the Poisson and the negative binomial. When the expected claim count is small, it makes little difference which distribution is used to model the claim count generating process.[5] As the expected claim count increases, however, the choice of model for the claim count distribution has a more significant impact on the variability of the aggregate probability model. Claim counts distributed according to the negative binomial distribution may exhibit significantly greater variability than claims distributed according to a Poisson distribution. The negative binomial is believed to be an appropriate distribution for incorporating parameter variance into a model. That is, observed aggregate claim counts frequently are generated by a heterogenous population whose underlying "true" expected claims vary among the members of the population. The observed variability will be greater than would be expected of Poisson distributed claims. Thus, the negative binomial is an appropriate distribution for modelling claims whenever the population generating the claims is believed to be heterogenous. This is usually the case when the variance of historical claims exceeds the mean of historical claims.

Few other distributions have been used to model property and casualty claims although many other discrete distributions with potential applications in insurance exist.

In the examples used in this paper, a negative binomial distribution is used to model claim counts. Appendix I describes the procedures used to incorporate this model into a simulation.

A commonly used distribution for modelling claim severity is the lognormal distribution with the Pareto frequently employed to model excess losses, or the tail of the distribution.[6] Other distributions, such as the Weibull, are also used to model claim severity and are easily utilized in simulation. Hogg and Klugman discuss the selection and fitting of distributions from empirical severity data.[7]

As with claim count distributions, the populations from which claim severities are derived are frequently, if not almost always, heterogenous. Claim severity distributions are frequently modelled as if the parameters of the distribution are known and do not vary over a population. In fact, the parameters are unknown, having been estimated from sample data, and the "true" parameters frequently vary within the population. Methods for modelling this variability are described by Meyers and Schenker.[8]

The illustrations used in this paper will employ a log Student's t distribution for severity. Appendix V describes this distribution.

Beard, Pentikäinen, and Pesonen[9] point out that as the expected number of claims increases, a simulation becomes quite time consuming. This consideration is particularly relevant to the model used in this paper due to the additional computational time involved in simulating payment patterns and interest rates. If the expected claim count is large they suggest that aggregate losses be modelled with a probability distribution fit to aggregate loss data. The model they use is the normal power approximation although other models for aggregate loss distributions have been proposed. These include the lognormal, generalized gamma, and generalized beta distributions.[10,11]

TIMING RISK

The timing of payments has frequently been recognized as the result of a random process, though deterministic payment patterns are usually used in discounting losses.

Recently, some attention has been focused upon the use of parametric models of loss development. These models relate the percentage of losses paid to the time period they are paid in using regression functions and probability distributions.[12,13,14] For these parametric models, theoretical probability distributions exist which describe the variability of actual parameters around an expected value. For instance, if the maximum likelihood method is used to estimate the parameters of development data, the variability of the actual parameters can be modelled using a normal distribution.[15] In addition, functions using the parameters, such as the probability density function and cumulative probability distribution have a normal distribution. Therefore, when using the maximum likelihood method, random payments could be generated using the normal distribution.

In practice, the use of maximum likelihood can be difficult for the nontechnical actuary, therefore regression was chosen as the methodology to generate payment patterns. A disadvantage of the regression method is that the payment pattern from the model may not sum to one. This problem is addressed in a later section of the paper.

For any regression, the distribution of actual values for the dependent variable, Y_t, around a forecast value, $\hat{Y}_t$, follows a t distribution with parameters μ, σ^2, and v.

Where,

$$\hat{\mu} = \text{estimate of } \mu = \hat{Y}_t$$

$$\sigma^2 = \text{Variance of the forecast at time t}$$

This variance is directly proportional to the squared deviation between the actual value for x_t and its mean. For a regression with only one independent variable the following is true:[16]

$$\hat{\sigma}^2 = \left(1 + \frac{1}{N} + \frac{(x_t - \bar{x})^2}{\sum_{i=1}^{N} (x_i - \bar{x})^2} \right) \cdot \text{MSE}$$

$$\text{where} \quad \text{MSE} = \frac{\sum_{i=1}^{N} (Y_i - \hat{Y}_i)^2}{V}$$

N = the number of observations

v = degrees of freedom = N–k–1

k = the number of independent variables in the model

When there is more than one independent variable in the model, the estimated variance of the forecast, $\hat{\sigma}^2$, is defined as follows.[17]

$$\hat{\sigma}^2 = \left(1 + x_t^T (X^T X)^{-1} x_t \right) \cdot \text{MSE}$$

Where X represents the matrix of data and x_t is an array of actual values for one period, for the independent variables in the regression. The details of this calculation are presented in Appendix III for a two variable regression. However, to simplify the programming the following approximation to $\hat{\sigma}^2$ has been found to produce reasonable results although the variance is understated at the extremes of the distribution of x values.

$$\hat{\sigma}^2 \approx \left(1 + \frac{1}{N} \right) \cdot \text{MSE}$$

An example is derived from a regression with two independent variables. To fit this curve, historic incremental paid losses were expressed as a percentage of estimated ultimate losses. Exhibit I presents a payment pattern to which a curve was fit. The curve which was fit to the data was

$$f(t) = e^{-ab^t t^c}$$

Where t is the midpoint of a development period. This curve was found to have the best fit as compared to a number of

Payment Pattern Triangle
Cumulative Percent Paid

Accident	Evaluation Point (Months From Inception)																
Year	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102
1978		29.66%	48.24%	55.70%	59.16%	63.43%	73.67%	81.86%	85.24%	88.70%	94.92%	96.73%	96.81%	97.08%	97.49%	99.91%	99.91%
1979	7.03%	24.13%	41.53%	46.93%	49.89%	70.85%	72.60%	79.99%	90.18%	94.42%	95.25%	98.69%	99.40%	99.73%	99.73%	99.73%	99.73%
1980	9.47%	29.41%	44.49%	52.94%	55.87%	72.02%	81.30%	85.73%	91.35%	92.62%	93.14%	94.05%	98.72%	98.75%	98.81%		
1981	9.01%	24.17%	34.35%	41.06%	55.45%	67.92%	82.33%	88.15%	89.87%	94.71%	97.49%	97.99%	98.03%				
1982	6.34%	23.95%	37.50%	47.95%	52.04%	55.80%	62.56%	77.92%	84.61%	89.67%	90.51%						
1983	5.27%	23.15%	39.46%	51.97%	59.91%	66.77%	70.70%	78.38%	85.71%								
1984	7.26%	29.72%	46.95%	58.69%	66.61%	72.41%	77.10%										
1985	7.71%	26.22%	39.92%	49.93%	56.45%												
1986	7.03%	23.41%	38.54%														

Payment Pattern Triangle
Interval Percent Paid

Accident	Evaluation Point (Months From Inception)																
Year	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102
1978			18.58%	7.46%	3.46%	4.27%	10.24%	8.18%	3.38%	3.46%	6.22%	1.81%	0.09%	0.27%	0.41%	2.43%	0.00%
1979	7.03%	17.10%	17.41%	5.40%	2.97%	20.96%	1.74%	7.40%	10.18%	4.24%	0.84%	3.44%	0.71%	0.33%	0.00%	0.00%	0.00%
1980	9.47%	19.94%	15.08%	8.45%	2.93%	16.15%	9.28%	4.44%	5.61%	1.27%	0.53%	0.91%	4.67%	0.03%	0.06%		
1981	9.01%	15.16%	10.18%	6.71%	14.39%	12.47%	14.40%	5.83%	1.71%	4.84%	2.78%	0.49%	0.04%				
1982	6.34%	17.61%	13.55%	10.45%	4.08%	3.77%	6.75%	15.36%	6.69%	5.05%	0.85%						
1983	5.27%	17.88%	16.31%	12.51%	7.94%	6.85%	3.93%	7.68%	7.33%								
1984	7.26%	22.46%	17.23%	11.74%	7.92%	5.80%	4.69%										
1985	7.71%	18.51%	13.70%	10.01%	6.52%												
1986	7.03%	16.38%	15.13%														
Avg	7.39%	18.13%	15.24%	9.09%	6.28%	10.04%	7.29%	8.15%	5.82%	3.77%	2.24%	1.66%	1.38%	0.21%	0.16%	1.21%	0.00%

similar curves. (Note that $f(t)$ is intended to be an analog of a probability density function.)

$f(t)$ = the percentage of losses paid in an interval, divided by the interval size

This model is fit as follows.

$$Y = \ln(-\ln(f(t))) = a + b\, t + c \ln(t)$$

Thus this model can be fit by regressing $\ln(-\ln(f(t)))$ on t and $\ln(t)$. The parameters of the model can be estimated using commonly available statistical software. Exhibit II shows the parameters which were estimated using Lotus 1-2-3. The columns of the spreadsheet display the raw data used to fit the payment pattern model as well as the transformed values used for the regression. The regression facility of Lotus 1-2-3 was used to estimate the parameters of the curve. The parameters were then used to calculate fitted values for the dependent variable, Y, and for the percent of losses paid for each observation. Also shown, in Exhibit II, is the significant F statistic. The calculation of the Durbin-Watson test statistic for autocorrelation is shown in Exhibit III. The Durbin-Watson statistic was calculated as follows:

Payout Pattern Regression
f(t) = exp^[-A*(B^t)*(t^C)]

(1) Year	(2) Cumulative Percent Paid	(3) Interval Percent Paid	(4) Maturity	(5) Interval Width	(6) f(t) Col.(3)/(5)	(7) Y [Ln(-Ln(f(t)))]	(8) t Midpoint	(9) Ln(t) [LN(Col.(8))]	(10) Fitted Y	(11) Fitted Percent In Interval
1978	29.660%	29.66%	12	12	2.47%	130.84%	6	1.7918	1.96%	23.48%
1978	48.240%	18.58%	18	6	3.10%	124.55%	15	2.7081	2.51%	15.08%
1978	55.700%	7.46%	24	6	1.24%	147.87%	21	3.0445	2.28%	13.68%
1978	59.160%	3.46%	30	6	0.58%	164.01%	27	3.2958	1.90%	11.38%
1978	63.430%	4.27%	36	6	0.71%	159.84%	33	3.4965	1.48%	8.88%
1978	73.670%	10.24%	42	6	1.71%	140.38%	39	3.6636	1.09%	6.57%
1978	81.860%	8.19%	48	6	1.36%	145.72%	45	3.8067	0.77%	4.61%
1978	85.240%	3.38%	54	6	0.56%	164.46%	51	3.9318	0.51%	3.08%
1978	88.700%	3.46%	60	6	0.58%	164.01%	57	4.0431	0.32%	1.95%
1978	94.920%	6.22%	66	6	1.04%	151.93%	63	4.1431	0.19%	1.16%
1978	96.730%	1.81%	72	6	0.30%	175.85%	69	4.2341	0.11%	0.65%
1978	96.810%	0.08%	78	6	0.01%	218.86%	75	4.3175	0.06%	0.34%
1978	97.080%	0.27%	84	6	0.05%	204.20%	81	4.3944	0.03%	0.17%
1978	97.490%	0.41%	90	6	0.07%	198.63%	87	4.4659	0.01%	0.07%
1978	99.910%	2.42%	96	6	0.40%	170.71%	93	4.5326	0.01%	0.03%
1978	99.910%	0.00%	102	6	0.00%	274.77%	99	4.5951	0.00%	0.01%
1979	7.030%	7.03%	6	6	1.17%	149.22%	3	1.0986	1.18%	7.09%
1979	24.130%	17.10%	12	6	2.85%	126.92%	9	2.1972	2.35%	14.08%
1979	41.530%	17.40%	18	6	2.90%	126.43%	15	2.7081	2.51%	15.08%
1979	46.930%	5.40%	24	6	0.90%	154.98%	21	3.0445	2.28%	13.68%
1979	49.890%	2.96%	30	6	0.49%	166.99%	27	3.2958	1.90%	11.38%
1979	70.850%	20.96%	36	6	3.49%	121.02%	33	3.4965	1.48%	8.88%
1979	72.600%	1.75%	42	6	0.29%	176.43%	39	3.6636	1.09%	6.57%
1979	79.990%	7.39%	48	6	1.23%	148.09%	45	3.8067	0.77%	4.61%
1979	90.180%	10.19%	54	6	1.70%	140.50%	51	3.9318	0.51%	3.08%
1979	94.420%	4.24%	60	6	0.71%	159.99%	57	4.0431	0.32%	1.95%
1979	95.250%	0.83%	66	6	0.14%	188.45%	63	4.1431	0.19%	1.16%
1979	98.690%	3.44%	72	6	0.57%	164.12%	69	4.2341	0.11%	0.65%
1979	99.400%	0.71%	78	6	0.12%	190.80%	75	4.3175	0.06%	0.34%
1979	99.730%	0.33%	84	6	0.05%	201.56%	81	4.3944	0.03%	0.17%
1979	99.730%	0.00%	90	6	0.00%	274.77%	87	4.4659	0.01%	0.07%
1979	99.730%	0.00%	96	6	0.00%	274.77%	93	4.5326	0.01%	0.03%
1979	99.730%	0.00%	102	6	0.00%	274.77%	99	4.5951	0.00%	0.01%
1980	9.470%	9.47%	6	6	1.58%	142.28%	3	1.0986	1.18%	7.09%
1980	29.410%	19.94%	12	6	3.32%	122.50%	9	2.1972	2.35%	14.08%
1980	44.490%	15.08%	18	6	2.51%	130.39%	15	2.7081	2.51%	15.08%
1980	52.940%	8.45%	24	6	1.41%	144.99%	21	3.0445	2.28%	13.68%
1980	55.870%	2.93%	30	6	0.49%	167.18%	27	3.2958	1.90%	11.38%

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EXHIBIT II

Payout Pattern Regression

f(t) = exp^[-A*(B^t)*(t^C)]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Year	Cumulative Percent Paid	Interval Percent Paid	Maturity	Interval Width	f(t) Col.(3)/(5)	Y [Ln(-Ln(f(t)))]	t Midpoint	Ln(t) [LN(Col.(8))]	Fitted Y	Fitted Percent In Interval
1980	72.020%	16.15%	36	6	2.69%	128.51%	33	3.4965	1.48%	8.88%
1980	81.300%	9.28%	42	6	1.55%	142.77%	39	3.6636	1.09%	6.57%
1980	85.730%	4.43%	48	6	0.74%	159.10%	45	3.8067	0.77%	4.61%
1980	91.350%	5.62%	54	6	0.94%	154.13%	51	3.9318	0.51%	3.08%
1980	92.620%	1.27%	60	6	0.21%	181.77%	57	4.0431	0.32%	1.95%
1980	93.140%	0.52%	66	6	0.09%	195.31%	63	4.1431	0.19%	1.16%
1980	94.050%	0.91%	72	6	0.15%	187.05%	69	4.2341	0.11%	0.65%
1980	98.720%	4.67%	78	6	0.78%	158.02%	75	4.3175	0.06%	0.34%
1980	98.750%	0.03%	84	6	0.01%	229.29%	81	4.3944	0.03%	0.17%
1980	98.810%	0.06%	90	6	0.01%	222.03%	87	4.4659	0.01%	0.07%
1981	9.010%	9.01%	6	6	1.50%	143.47%	3	1.0986	1.18%	7.09%
1981	24.170%	15.16%	12	6	2.53%	130.24%	9	2.1972	2.35%	14.08%
1981	34.350%	10.18%	18	6	1.70%	140.52%	15	2.7081	2.51%	15.08%
1981	41.060%	6.71%	24	6	1.12%	150.26%	21	3.0445	2.28%	13.68%
1981	55.450%	14.39%	30	6	2.40%	131.65%	27	3.2958	1.90%	11.38%
1981	67.920%	12.47%	36	6	2.08%	135.42%	33	3.4965	1.48%	8.88%
1981	82.330%	14.41%	42	6	2.40%	131.61%	39	3.6636	1.09%	6.57%
1981	88.150%	5.82%	48	6	0.97%	153.38%	45	3.8067	0.77%	4.61%
1981	89.870%	1.72%	54	6	0.29%	176.72%	51	3.9318	0.51%	3.08%
1981	94.710%	4.84%	60	6	0.81%	157.28%	57	4.0431	0.32%	1.95%
1981	97.490%	2.78%	66	6	0.46%	168.17%	63	4.1431	0.19%	1.16%
1981	97.990%	0.50%	72	6	0.08%	195.87%	69	4.2341	0.11%	0.65%
1981	98.030%	0.04%	78	6	0.01%	226.34%	75	4.3175	0.06%	0.34%
1982	6.340%	6.34%	6	6	1.06%	151.51%	3	1.0986	1.18%	7.09%
1982	23.950%	17.61%	12	6	2.94%	126.09%	9	2.1972	2.35%	14.08%
1982	37.500%	13.55%	18	6	2.26%	133.25%	15	2.7081	2.51%	15.08%
1982	47.950%	10.45%	24	6	1.74%	139.88%	21	3.0445	2.28%	13.68%
1982	52.040%	4.09%	30	6	0.68%	160.71%	27	3.2958	1.90%	11.38%
1982	55.800%	3.76%	36	6	0.63%	162.38%	33	3.4965	1.48%	8.88%
1982	62.560%	6.76%	42	6	1.13%	150.09%	39	3.6636	1.09%	6.57%
1982	77.920%	15.36%	48	6	2.56%	129.89%	45	3.8067	0.77%	4.61%
1982	84.610%	6.69%	54	6	1.11%	150.33%	51	3.9318	0.51%	3.08%
1982	89.670%	5.06%	60	6	0.84%	156.35%	57	4.0431	0.32%	1.95%
1982	90.510%	0.84%	66	6	0.14%	188.27%	63	4.1431	0.19%	1.16%
1983	5.270%	5.27%	6	6	0.88%	155.50%	3	1.0986	1.18%	7.09%
1983	23.150%	17.88%	12	6	2.98%	125.65%	9	2.1972	2.35%	14.08%
1983	39.460%	16.31%	18	6	2.72%	128.24%	15	2.7081	2.51%	15.08%
1983	51.970%	12.51%	24	6	2.09%	135.34%	21	3.0445	2.28%	13.68%

EXHIBIT II

Payout Pattern Regression
f(t) = exp^[-A*(B^t)*(t^C)]

(1) Year	(2) Cumulative Percent Paid	(3) Interval Percent Paid	(4) Maturity	(5) Interval Width	(6) f(t) Col.(3)/(5)	(7) Y [Ln(-Ln(f(t)))]	(8) t Midpoint	(9) Ln(t) [LN(Col.(8))]	(10) Fitted Y	(11) Fitted Percent In Interval
1983	59.910%	7.94%	30	6	1.32%	146.44%	27	3.2958	1.90%	11.38%
1983	66.770%	6.86%	36	6	1.14%	149.77%	33	3.4965	1.48%	8.88%
1983	70.700%	3.93%	42	6	·0.66%	161.51%	39	3.6636	1.09%	6.57%
1983	78.380%	7.68%	48	6	1.28%	147.21%	45	3.8067	0.77%	4.61%
1983	85.710%	7.33%	54	6	1.22%	148.27%	51	3.9318	0.51%	3.08%
1984	7.260%	7.26%	6	6	1.21%	148.49%	3	1.0986	1.18%	7.09%
1984	29.720%	22.46%	12	6	3.74%	118.94%	9	2.1972	2.35%	14.08%
1984	46.950%	17.23%	18	6	2.87%	126.70%	15	2.7081	2.51%	15.08%
1984	58.690%	11.74%	24	6	1.96%	136.96%	21	3.0445	2.28%	13.68%
1984	66.610%	7.92%	30	6	1.32%	146.50%	27	3.2958	1.90%	11.38%
1984	72.410%	5.80%	36	6	0.97%	153.45%	33	3.4965	1.48%	8.88%
1984	77.100%	4.69%	42	6	0.78%	157.93%	39	3.6636	1.09%	6.57%
1985	7.710%	7.71%	6	6	1.29%	147.12%	3	1.0986	1.18%	7.09%
1985	26.220%	18.51%	12	6	3.09%	124.66%	9	2.1972	2.35%	14.08%
1985	39.920%	13.70%	18	6	2.28%	132.96%	15	2.7081	2.51%	15.08%
1985	49.930%	10.01%	24	6	1.67%	140.94%	21	3.0445	2.28%	13.68%
1985	56.450%	6.52%	30	6	1.09%	150.90%	27	3.2958	1.90%	11.38%
1986	7.030%	7.03%	6	6	1.17%	149.22%	3	1.0986	1.18%	7.09%
1986	23.410%	16.38%	12	6	2.73%	128.12%	9	2.1972	2.35%	14.08%
1986	38.540%	15.13%	18	6	2.52%	130.30%	15	2.7081	2.51%	15.08%

Regression Output:

Constant		1.714138
Std Err of Y Est		0.178048
R Squared		0.737556
No. of Observations		96
Degrees of Freedom		93
X Coefficient(s)	0.0186	-0.2546
Std Err of Coef.	0.0016	0.0439
F Statistic		130.7

EXHIBIT III

Test of Residuals for Autocorrelation With Durbin Watson Statistic

Observation Number	Regression Residual	Residual^2	(e(t)-e(t-1))^2	Observation Number	Regression Residual	Residual^2	(e(t)-e(t-1))^2	Observation Number	Regression Residual	Residual^2	(e(t)-e(t-1))^2
1	0.061	0.004	0.000	33	-0.361	0.130	0.183	65	-0.069	0.005	0.026
2	0.058	0.003	0.043	34	0.067	0.005	0.001	66	-0.229	0.053	0.002
3	-0.149	0.022	0.013	35	0.097	0.009	0.009	67	-0.186	0.034	0.037
4	-0.262	0.069	0.010	36	0.000	0.000	0.014	68	0.006	0.000	0.077
5	-0.160	0.026	0.070	37	-0.120	0.014	0.030	69	0.284	0.081	0.016
6	0.104	0.011	0.000	38	-0.294	0.087	0.200	70	0.159	0.025	0.001
7	0.125	0.016	0.012	39	0.153	0.023	0.005	71	0.182	0.033	0.054
8	0.018	0.000	0.008	40	0.080	0.006	0.008	72	-0.051	0.003	0.000
9	0.106	0.011	0.043	41	-0.008	0.000	0.017	73	-0.065	0.004	0.017
10	0.313	0.098	0.023	42	0.121	0.015	0.037	74	0.066	0.004	0.002
11	0.162	0.026	0.115	43	-0.072	0.005	0.002	75	0.022	0.000	0.002
12	-0.178	0.032	0.057	44	-0.121	0.015	0.029	76	-0.023	0.001	0.004
13	0.061	0.004	0.022	45	0.050	0.003	0.145	77	-0.087	0.008	0.001
14	0.210	0.044	0.140	46	0.431	0.186	0.385	78	-0.059	0.004	0.002
15	0.584	0.341	0.893	47	-0.190	0.036	0.028	79	-0.108	0.012	0.048
16	-0.361	0.130	0.129	48	-0.024	0.001	0.006	80	0.111	0.012	0.005
17	-0.002	0.000	0.003	49	0.056	0.003	0.001	81	0.180	0.032	0.030
18	0.053	0.003	0.000	50	0.020	0.000	0.015	82	0.005	0.000	0.016
19	0.040	0.002	0.067	51	-0.101	0.010	0.005	83	0.133	0.018	0.009
20	-0.220	0.048	0.005	52	-0.173	0.030	0.055	84	0.037	0.001	0.006
21	-0.292	0.085	0.271	53	0.061	0.004	0.001	85	-0.040	0.002	0.002
22	0.228	0.052	0.235	54	0.084	0.007	0.011	86	-0.087	0.008	0.000
23	-0.257	0.066	0.129	55	0.191	0.037	0.020	87	-0.096	0.009	0.001
24	0.102	0.010	0.024	56	0.049	0.002	0.024	88	-0.072	0.005	0.008
25	0.257	0.066	0.012	57	-0.105	0.011	0.077	89	0.019	0.000	0.003
26	0.146	0.021	0.039	58	0.173	0.030	0.001	90	0.076	0.006	0.010
27	-0.053	0.003	0.110	59	0.150	0.023	0.036	91	-0.026	0.001	0.003
28	0.279	0.078	0.031	60	-0.038	0.001	0.046	92	-0.079	0.006	0.003
29	0.103	0.011	0.000	61	-0.252	0.064	0.052	93	-0.131	0.017	0.017
30	0.087	0.008	0.408	62	-0.025	0.001	0.007	94	-0.002	0.000	0.002
31	-0.551	0.304	0.009	63	0.061	0.004	0.008	95	0.041	0.002	0.002
32	-0.456	0.208	0.009	64	-0.029	0.001	0.002	96	0.001	0.000	
									Sums	2.948	4.797

Durbin Watson 1.627

$$DW = \frac{\sum_{t=2}^{N} (e_t - e_{t-1})^2}{\sum_{t=1}^{N} e_t^2}$$

Where e_t is the difference between the actual and the fitted value for the t^{th} observation of the dependent variable. A rule of thumb is that if no autocorrelation is present, the Durbin-Watson statistic should be between 1.5 and 2.5. For the sample size used in this example the result in Exhibit III is inconclusive. Therefore autocorrelation may or may not be present. To simplify the modelling process, no adjustment was made to the regression due to possible autocorrelation.

Once the regression coefficients have been estimated the model's expected values for Y for each forecast time period can be determined.

$$\hat{Y}_t = \hat{a} + \hat{b}\, t + \hat{c} \ln(t)$$

To simulate a value for a payout, the following algorithm is used:

1. Calculate $\hat{Y}$ (fitted Y) for the time period.

2. Generate a random variable, T, from a t distribution with N-k-1 degrees of freedom. N = the number of observations used to fit the parameters.

3. The simulated value for Y for year t is

$$S = \hat{Y} + \hat{\sigma} T$$

4. The percent paid in an interval is

$$\text{Percent Paid} = \frac{1}{\exp(\exp(S))} \cdot \text{length of interval}$$

5. Multiply the percent paid in the interval by the aggregate losses for the accident year to obtain the aggregate amount of losses paid.

INTEREST RATE RISK: THE RANDOM WALK MODEL

One of the most commonly used distributions to model the return on investments is the random walk model. In general, the random walk model assumes that the best guess for tomorrow's interest rate is today's interest rate. This model is denoted as follows:

$$\Delta i = a_t ,$$

$$\Delta i = i_t - i_{t-1}, \text{ or}$$

$$\Delta \ln(i) = \ln(i_t) - \ln(i_{t-1})$$

Where i_t is an interest rate or an investment return for a time period t. a_t is the residual term for the change in interest rates and is usually assumed to be normally distributed with mean zero and variance σ^2. The following assumptions are made when using this model:

1. The changes in returns are independent and identically distributed.

2. The variance of the Δi's is constant over time.

3. The series is level, i.e. it does not contain a trend.

4. The error term in the model is uncorrelated with the error terms from any prior period.

5. a_t is the residual term for the change in interest rates. Though it is frequently assumed to be normally distributed, normality is not a requirement of the model.

The random walk model is a subset of a more general class of time series models known as Autoregressive Integrated Moving Average (ARIMA) models. The notation for the more general model is:

W_t = the n^{th} difference of the interest rate series at time t

$$W_t = \phi_1 W_{t-1} + \phi_2 W_{t-2} + \ldots + \phi_N W_{t-N} - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_N e_{t-N} + a_t$$

Where:

ϕ_1 is a parameter relating the W at time t to the W at earlier times. It is also an "autoregressive" parameter.

θ_1 is a parameter relating the W at time t to the residual of earlier times. It is a "moving average" parameter.

e_t is the difference between the actual and the predicted values for W_t.

a_t is the same as previously defined.

The random walk model is a very simple model and has frequently been used with financial data in the past. However, the hypothesis that interest rates follow a random walk model cannot be assumed to be appropiate without further testing. The hypothesis was tested on a series of monthly yields on one month treasury bills (treasury bills with one month left to maturity) between January 1970 and December 1985.

The first step in assessing the reasonableness of the random walk model for interest rates is to examine the autocorrelation and partial autocorrelation functions. The autocorrelation for lag k is the correlation between the return at time t and the return at time k periods ahead of t. The sample estimate of the autocorrelation for the series is:[18]

$$r_k = \frac{\sum_{t=1}^{N-k} (i_t - \bar{i})(i_{t+k} - \bar{i})}{\sum_{t=1}^{N} (i_t - \bar{i})^2}$$

Where:

N is the number of observations

i_t is the interest rate for time t

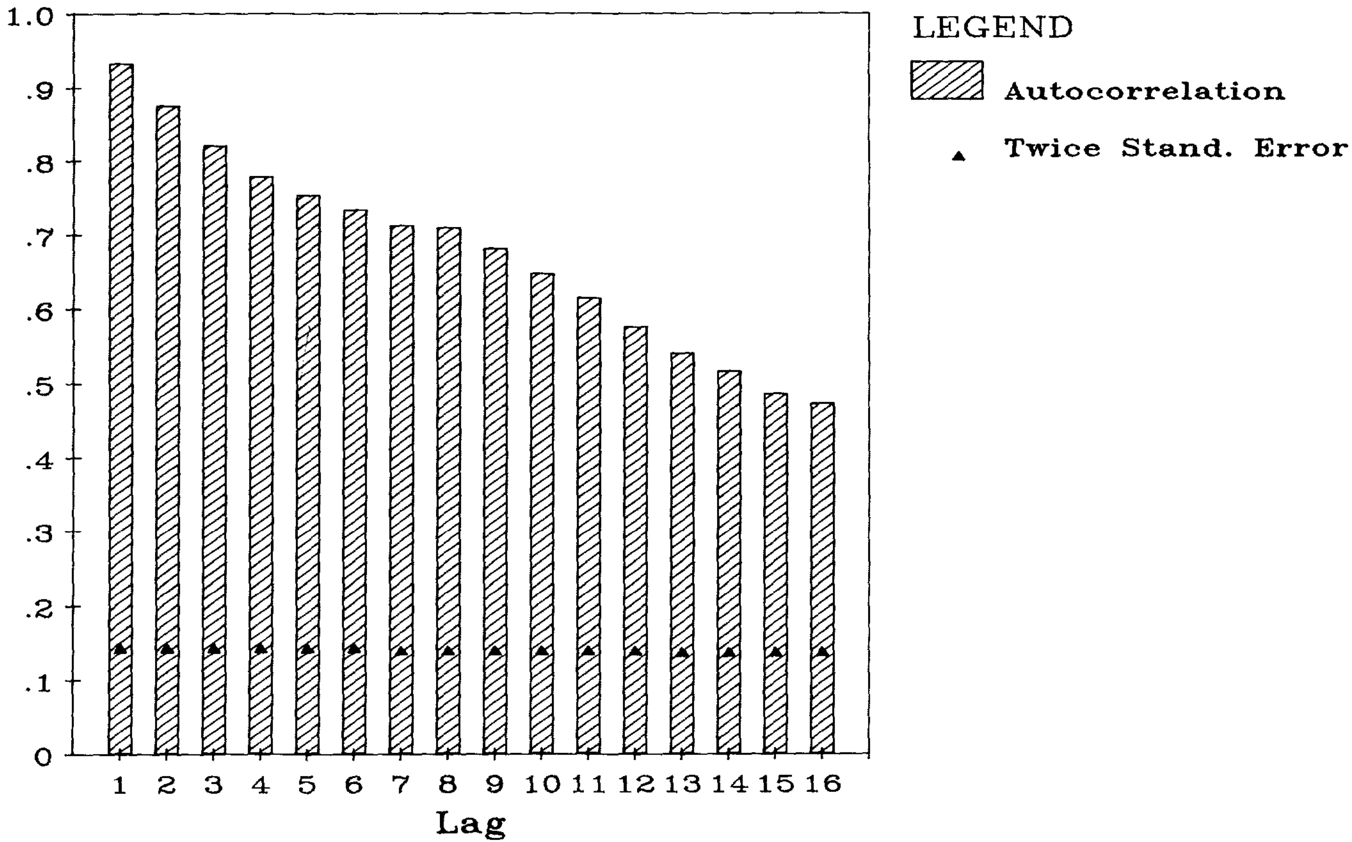
$\bar{i}$ is the average interest rate for the series

When returns are independent and identically distributed, i.e. no autocorrelation exists, this statistic is approximately normally distributed with a mean of zero and a variance of $\frac{1}{N}$. Thus a sample autocorrelation of greater than two times $\frac{1}{N}$ may indicate a significant autocorrelation at a particular lag. ($\frac{2}{N}$ represents approximately the 95th percent confidence level of the normal distribution.)

The partial autocorrelations for lag k measure the correlation between variables at different lags after removing the effect of the correlation of the variables at prior lags. Abraham and Ledolter provide a more complete explanation of the partial autocorrelation function and its estimation.[19] As was the case with the autocorrelation function, when the interest rates are independent, the partial autocorrelations are approximately normally distributed with a mean of zero and a variance of $\frac{1}{N}$.

Both autocorrelation and partial autcorrelation functions are part of the standard output of statistical packages which perform time series analysis. The SPSS statistical package was used to analyze the treasury bill data. The autocorrelations of the treasury bill data are shown in Chart I. These autocorrelations are large and do

AUTOCORRELATION OF TREASURY BILL RATES
CHART I



not die out quickly indicating that the series should be differenced, as is predicted by the random walk model. The autocorrelation and partial autocorrelation functions of the differenced series are shown in Charts II and III. The size of the autocorrelations and partial autocorrelations indicates that a more complicated model than the random walk model may be appropiate for this series.

A model of the following form (also known as an ARIMA(1,1,1) model) was found to fit the data:

$$i_t - i_{t-1} = \phi\,(i_{t-1} - i_{t-2}) - \theta\, e_{t-1} + a_t$$

An assumption of ARIMA models is that the residuals of each time period are uncorrelated with the residuals of prior time periods. A diagnostic check of the appropriateness of this assumption was performed using a Box-Ljung test. This statistic tests the joint null hypothesis that the autocorrelations at all lags are zero. The statistic used in the test is:[20]

$$Q = N\,(N+2) \sum_{i=1}^{k} (N-i)^{-1}\, r_i$$

Where: r_i is the i^{th} sample autocorrelation

N is the number of observations used to fit the model

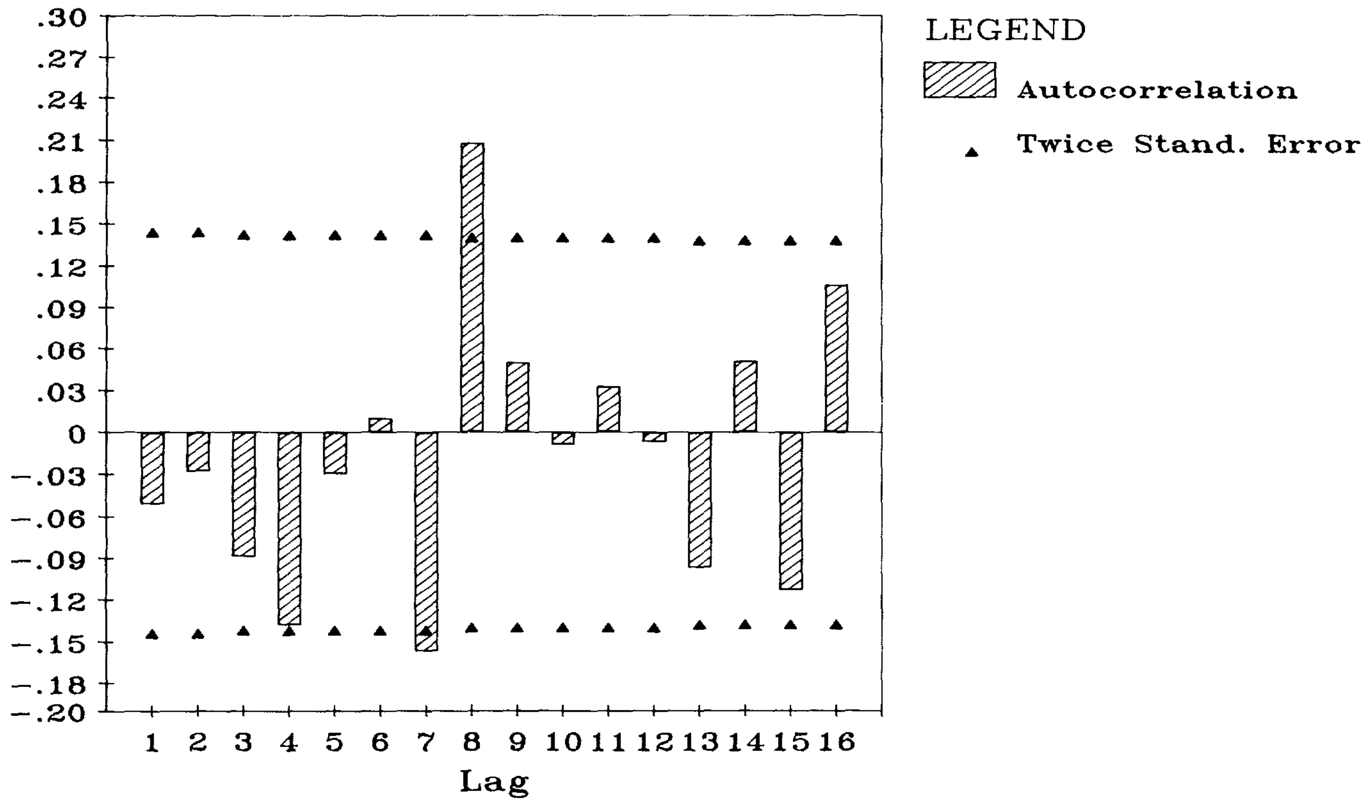
k denotes the number of autocorrelations tested for significance

When the null hypothesis is true, this statistic has a

AUTOCORRELATION OF TREASURY BILL RATES

First Differences

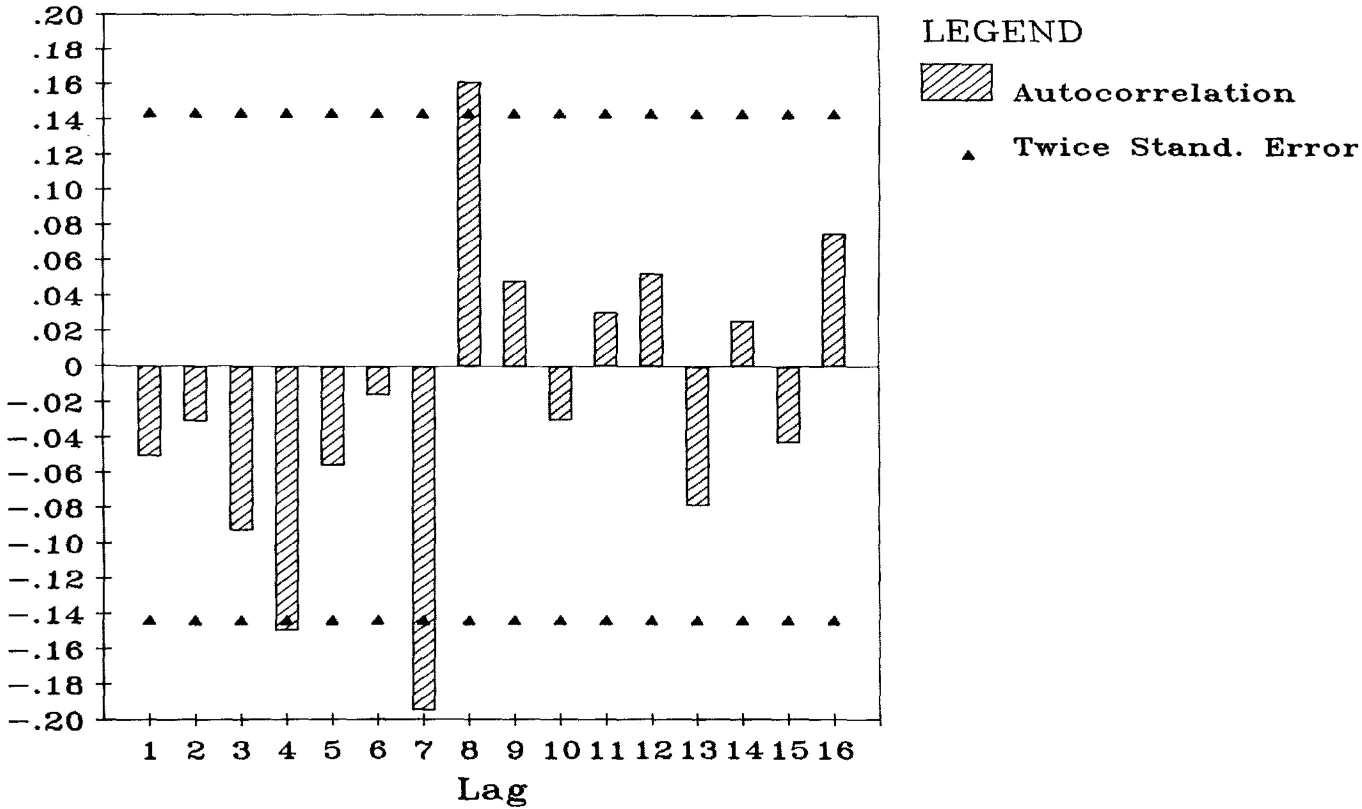
Chart II



PARTIAL AUTOCORRELATION OF TREASURY BILL RATES

First Differences

Chart III



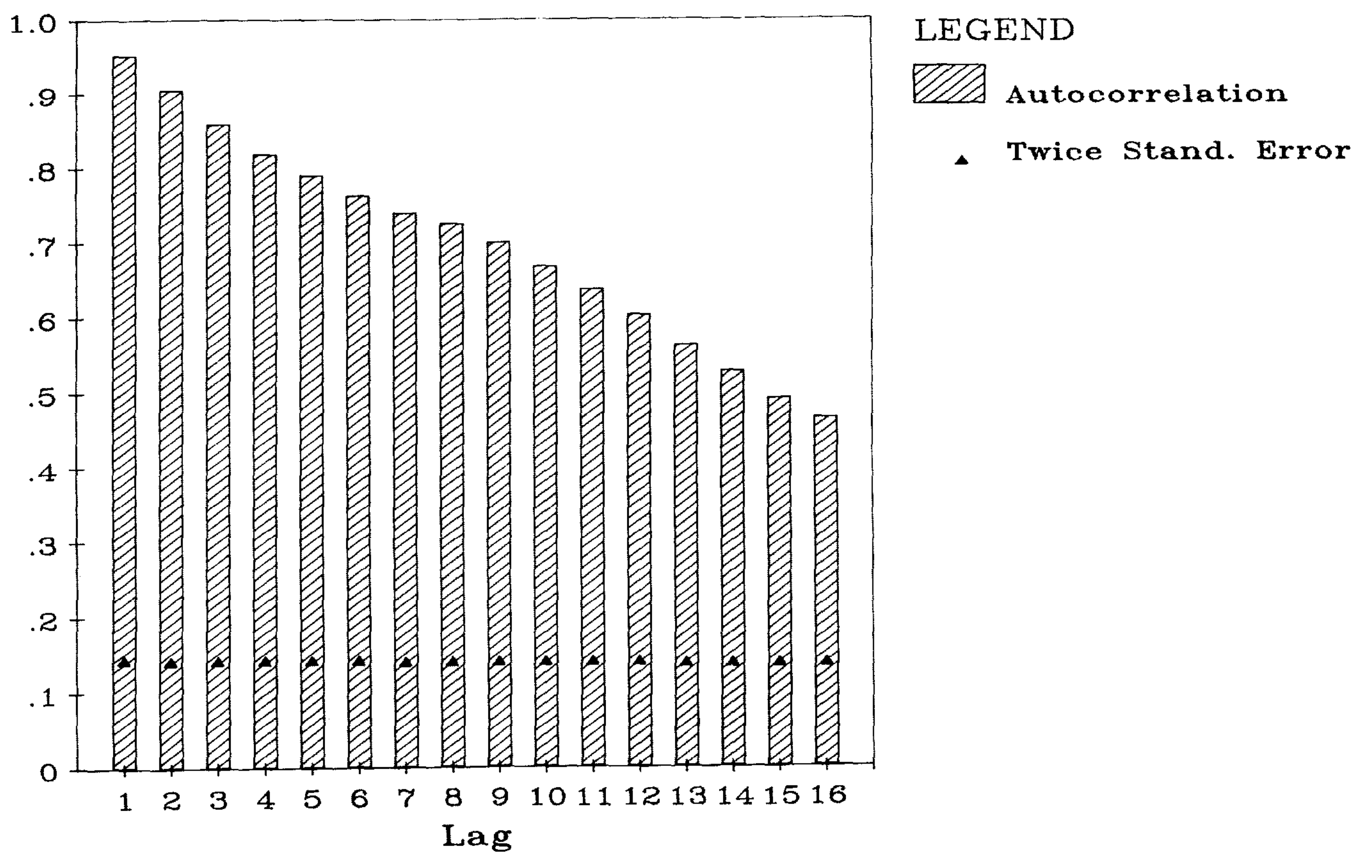
Chi-square distribution with k-m degrees of freedom, where m is the number of parameters in the model. The application of this test to the residuals of the fitted model indicated that the autocorrelations were not significant.

The procedures which were used to identify an appropriate model for the treasury bill rate data were also applied to the natural logarithms of the treasury bill rates. Chart IV displays the autocorrelation function for this data. The autocorrelations indicated that the series should be differenced. When the series was differenced no significant autocorrelations or partial autocorrelations appeared. These autocorrelations and partial autocorrelations are displayed in Charts V and VI. A Box-Ljung test was performed and this test found no significant autocorrelations in the residuals. Thus, for the logs of the interest rates, the need for a more complicated model than the random walk model was not indicated. Because it is a more parsimonious model and because it performed better than the ARIMA(1,1,1) model on statistical tests provided by the SPSS package, the random walk model for the logs of interest rates was chosen for use in simulation.

For the model selected only one parameter estimate, the variance of the distribution, is needed. This parameter is estimated by calculating the variance of the first difference of the data. This calculation is shown in Exhibit IV. In addition, the skewness and kurtosis for the model are shown. It should be noted that a negative skewness is seen in this

AUTOCORRELATION OF THE LOG OF TREASURY BILL RATES

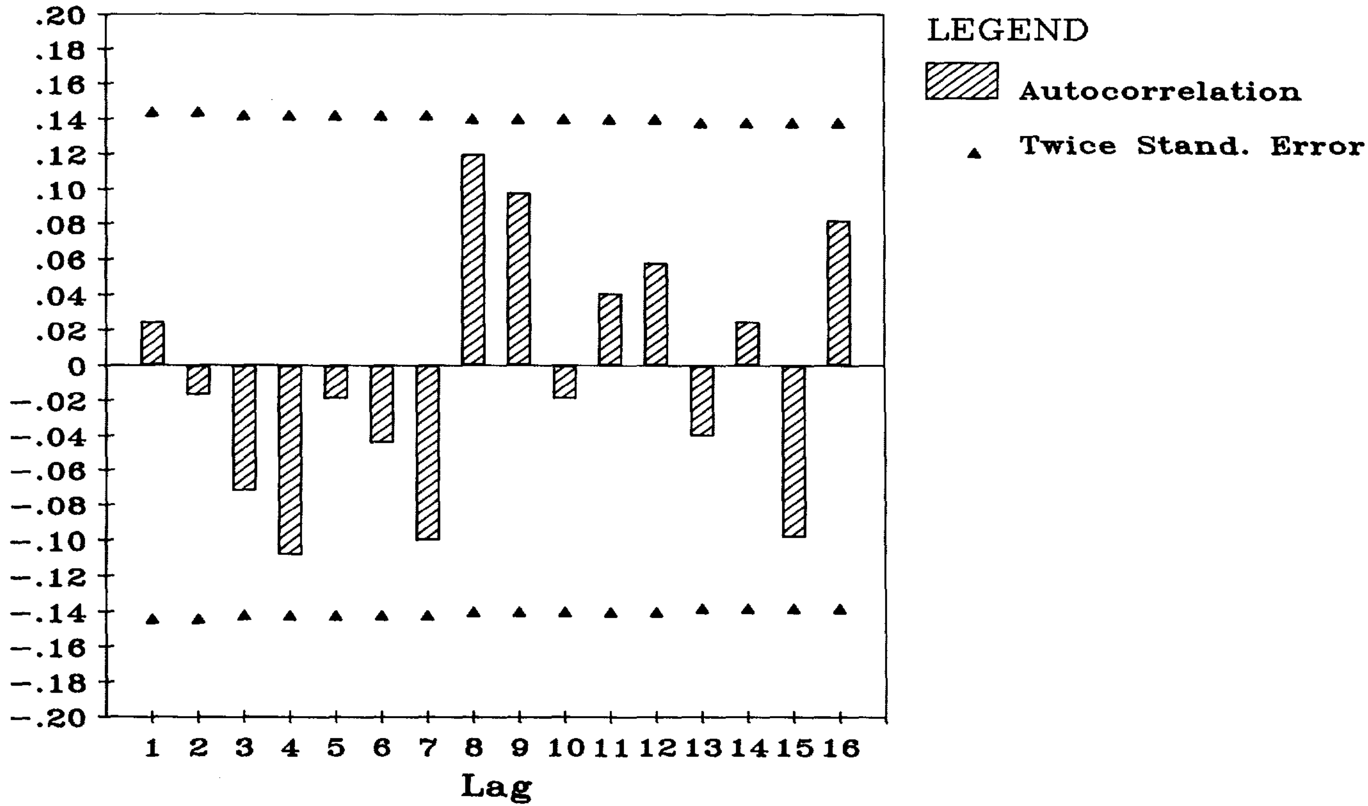
CHART IV



AUTOCORRELATION OF THE LOG OF TREASURY BILL RATES

First Differences

Chart V



PARTIAL AUTOCORRELATION OF THE LOG OF TREASURY BILL RATES
First Differences
Chart VI

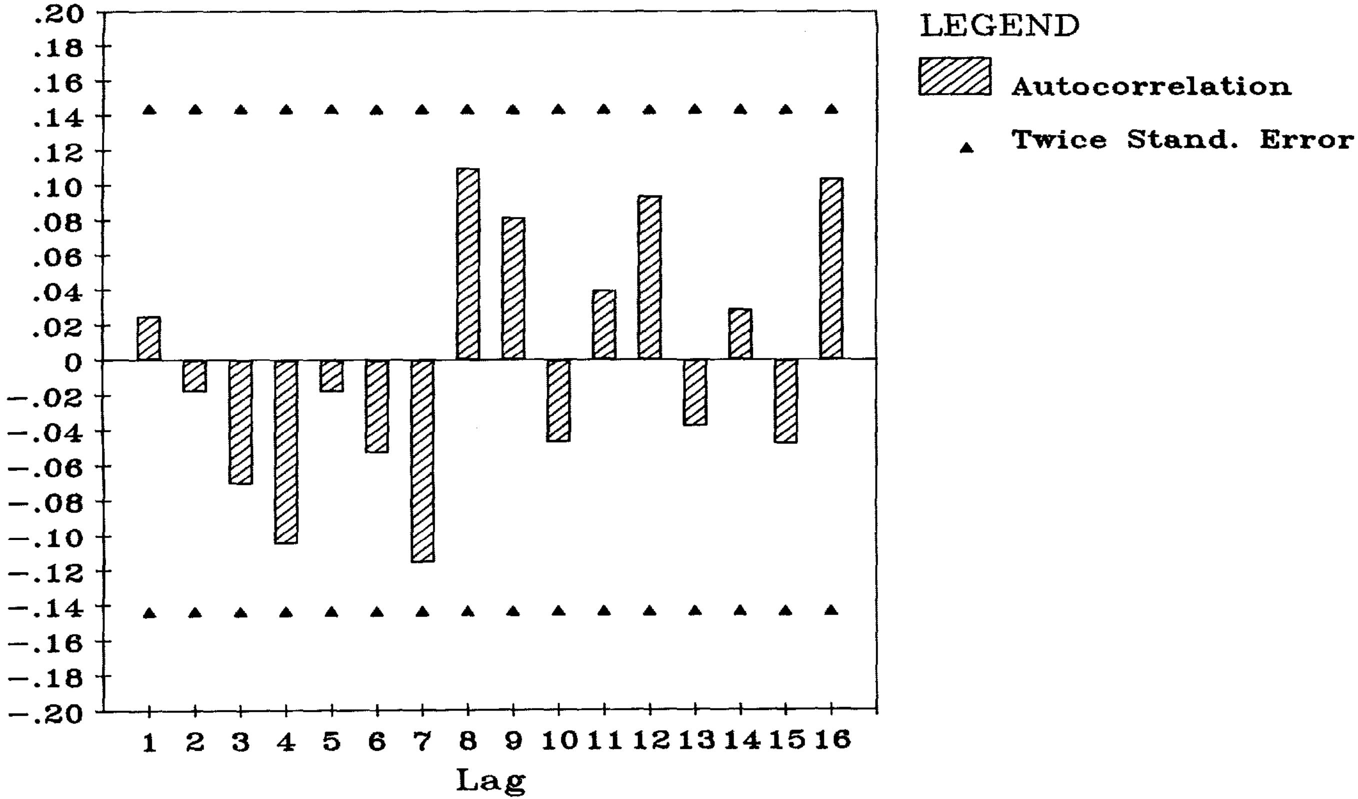


EXHIBIT IV

One Month Yield on Treasury Bills

(1) Year	(2) Month	(3) T-Bill Rate	(4) Ln(T-Bill Rate)	(5) First Difference Of Logs	(6) Squared First Difference	(7) Third Moment	(8) Fourth Moment
70	1	0.63%	-5.07				
70	2	0.53%	-5.23	-0.168	0.028	-0.005	0.001
70	3	0.55%	-5.21	0.023	0.001	0.000	0.000
70	4	0.54%	-5.22	-0.008	0.000	-0.000	0.000
70	5	0.53%	-5.24	-0.024	0.001	-0.000	0.000
70	6	0.50%	-5.31	-0.067	0.004	-0.000	0.000
70	7	0.50%	-5.30	0.009	0.000	0.000	0.000
70	8	0.53%	-5.25	0.050	0.002	0.000	0.000
70	9	0.48%	-5.33	-0.084	0.007	-0.001	0.000
70	10	0.44%	-5.42	-0.082	0.007	-0.001	0.000
70	11	0.38%	-5.58	-0.166	0.027	-0.005	0.001
70	12	0.36%	-5.64	-0.058	0.003	-0.000	0.000
71	1	0.34%	-5.69	-0.049	0.002	-0.000	0.000
71	2	0.27%	-5.90	-0.208	0.043	-0.009	0.002
71	3	0.28%	-5.87	0.030	0.001	0.000	0.000
71	4	0.30%	-5.79	0.072	0.005	0.000	0.000
71	5	0.35%	-5.65	0.142	0.020	0.003	0.000
71	6	0.41%	-5.51	0.146	0.021	0.003	0.000
71	7	0.42%	-5.47	0.041	0.002	0.000	0.000
71	8	0.37%	-5.59	-0.128	0.016	-0.002	0.000
71	9	0.38%	-5.58	0.011	0.000	0.000	0.000
71	10	0.34%	-5.69	-0.107	0.011	-0.001	0.000
71	11	0.33%	-5.71	-0.025	0.001	-0.000	0.000
71	12	0.28%	-5.88	-0.167	0.028	-0.005	0.001
72	1	0.26%	-5.94	-0.063	0.004	-0.000	0.000
72	2	0.27%	-5.91	0.032	0.001	0.000	0.000
72	3	0.28%	-5.87	0.046	0.002	0.000	0.000
72	4	0.28%	-5.88	-0.015	0.000	-0.000	0.000
72	5	0.29%	-5.85	0.030	0.001	0.000	0.000
72	6	0.29%	-5.84	0.015	0.000	0.000	0.000
72	7	0.29%	-5.84	0.000	0.000	0.000	0.000
72	8	0.36%	-5.63	0.209	0.044	0.009	0.002
72	9	0.37%	-5.59	0.035	0.001	0.000	0.000
72	10	0.38%	-5.58	0.011	0.000	0.000	0.000
72	11	0.42%	-5.47	0.117	0.014	0.002	0.000
72	12	0.40%	-5.52	-0.051	0.003	-0.000	0.000
73	1	0.45%	-5.40	0.119	0.014	0.002	0.000
73	2	0.47%	-5.37	0.028	0.001	0.000	0.000
73	3	0.51%	-5.28	0.088	0.008	0.001	0.000
73	4	0.50%	-5.30	-0.021	0.000	-0.000	0.000
73	5	0.54%	-5.22	0.086	0.007	0.001	0.000

EXHIBIT IV

One Month Yield on Treasury Bills

(1) Year	(2) Month	(3) T-Bill Rate	(4) Ln(T-Bill Rate)	(5) First Difference Of Logs	(6) Squared First Difference	(7) Third Moment	(8) Fourth Moment
73	6	0.62%	-5.08	0.139	0.019	0.003	0.000
73	7	0.69%	-4.98	0.103	0.011	0.001	0.000
73	8	0.71%	-4.95	0.030	0.001	0.000	0.000
73	9	0.60%	-5.12	-0.176	0.031	-0.005	0.001
73	10	0.58%	-5.15	-0.026	0.001	-0.000	0.000
73	11	0.63%	-5.07	0.074	0.006	0.000	0.000
73	12	0.61%	-5.09	-0.021	0.000	-0.000	0.000
74	1	0.62%	-5.09	0.008	0.000	0.000	0.000
74	2	0.62%	-5.08	0.003	0.000	0.000	0.000
74	3	0.71%	-4.94	0.141	0.020	0.003	0.000
74	4	0.72%	-4.93	0.008	0.000	0.000	0.000
74	5	0.63%	-5.07	-0.139	0.019	-0.003	0.000
74	6	0.64%	-5.05	0.021	0.000	0.000	0.000
74	7	0.61%	-5.09	-0.041	0.002	-0.000	0.000
74	8	0.75%	-4.89	0.203	0.041	0.008	0.002
74	9	0.51%	-5.28	-0.392	0.154	-0.060	0.024
74	10	0.57%	-5.16	0.118	0.014	0.002	0.000
74	11	0.61%	-5.09	0.071	0.005	0.000	0.000
74	12	0.55%	-5.20	-0.109	0.012	-0.001	0.000
75	1	0.45%	-5.39	-0.191	0.036	-0.007	0.001
75	2	0.38%	-5.57	-0.178	0.032	-0.006	0.001
75	3	0.44%	-5.44	0.136	0.018	0.002	0.000
75	4	0.41%	-5.51	-0.071	0.005	-0.000	0.000
75	5	0.41%	-5.49	0.015	0.000	0.000	0.000
75	6	0.45%	-5.40	0.094	0.009	0.001	0.000
75	7	0.50%	-5.30	0.098	0.010	0.001	0.000
75	8	0.47%	-5.37	-0.071	0.005	-0.000	0.000
75	9	0.52%	-5.27	0.104	0.011	0.001	0.000
75	10	0.44%	-5.43	-0.160	0.026	-0.004	0.001
75	11	0.41%	-5.49	-0.059	0.004	-0.000	0.000
75	12	0.43%	-5.44	0.045	0.002	0.000	0.000
76	1	0.37%	-5.61	-0.165	0.027	-0.004	0.001
76	2	0.39%	-5.55	0.057	0.003	0.000	0.000
76	3	0.39%	-5.55	-0.000	0.000	-0.000	0.000
76	4	0.40%	-5.53	0.021	0.000	0.000	0.000
76	5	0.43%	-5.45	0.082	0.007	0.001	0.000
76	6	0.44%	-5.43	0.019	0.000	0.000	0.000
76	7	0.42%	-5.47	-0.039	0.002	-0.000	0.000
76	8	0.41%	-5.49	-0.020	0.000	-0.000	0.000
76	9	0.42%	-5.47	0.020	0.000	0.000	0.000
76	10	0.39%	-5.54	-0.073	0.005	-0.000	0.000

EXHIBIT IV

One Month Yield on Treasury Bills

(1) Year	(2) Month	(3) T-Bill Rate	(4) Ln(T-Bill Rate)	(5) First Difference Of Logs	(6) Squared First Difference	(7) Third Moment	(8) Fourth Moment
76	11	0.36%	-5.64	-0.102	0.010	-0.001	0.000
76	12	0.35%	-5.66	-0.024	0.001	-0.000	0.000
77	1	0.38%	-5.57	0.093	0.009	0.001	0.000
77	2	0.37%	-5.59	-0.022	0.000	-0.000	0.000
77	3	0.37%	-5.59	-0.000	0.000	-0.000	0.000
77	4	0.36%	-5.62	-0.023	0.001	-0.000	0.000
77	5	0.39%	-5.54	0.073	0.005	0.000	0.000
77	6	0.40%	-5.51	0.032	0.001	0.000	0.000
77	7	0.43%	-5.44	0.071	0.005	0.000	0.000
77	8	0.44%	-5.43	0.015	0.000	0.000	0.000
77	9	0.49%	-5.32	0.109	0.012	0.001	0.000
77	10	0.50%	-5.30	0.017	0.000	0.000	0.000
77	11	0.47%	-5.37	-0.070	0.005	-0.000	0.000
77	12	0.45%	-5.40	-0.033	0.001	-0.000	0.000
78	1	0.50%	-5.30	0.103	0.011	0.001	0.000
78	2	0.50%	-5.29	0.009	0.000	0.000	0.000
78	3	0.56%	-5.18	0.107	0.012	0.001	0.000
78	4	0.52%	-5.27	-0.082	0.007	-0.001	0.000
78	5	0.53%	-5.23	0.032	0.001	0.000	0.000
78	6	0.55%	-5.20	0.032	0.001	0.000	0.000
78	7	0.54%	-5.22	-0.016	0.000	-0.000	0.000
78	8	0.62%	-5.08	0.139	0.019	0.003	0.000
78	9	0.65%	-5.03	0.047	0.002	0.000	0.000
78	10	0.73%	-4.92	0.111	0.012	0.001	0.000
78	11	0.74%	-4.90	0.017	0.000	0.000	0.000
78	12	0.72%	-4.93	-0.029	0.001	-0.000	0.000
79	1	0.79%	-4.84	0.096	0.009	0.001	0.000
79	2	0.78%	-4.85	-0.011	0.000	-0.000	0.000
79	3	0.80%	-4.83	0.017	0.000	0.000	0.000
79	4	0.78%	-4.85	-0.017	0.000	-0.000	0.000
79	5	0.81%	-4.82	0.032	0.001	0.000	0.000
79	6	0.74%	-4.91	-0.093	0.009	-0.001	0.000
79	7	0.75%	-4.89	0.022	0.001	0.000	0.000
79	8	0.81%	-4.81	0.076	0.006	0.000	0.000
79	9	0.82%	-4.81	0.005	0.000	0.000	0.000
79	10	0.95%	-4.65	0.154	0.024	0.004	0.001
79	11	0.85%	-4.77	-0.114	0.013	-0.001	0.000
79	12	0.81%	-4.82	-0.051	0.003	-0.000	0.000
80	1	0.92%	-4.68	0.132	0.018	0.002	0.000
80	2	1.12%	-4.49	0.193	0.037	0.007	0.001
80	3	1.27%	-4.37	0.122	0.015	0.002	0.000

EXHIBIT IV

One Month Yield on Treasury Bills

(1) Year	(2) Month	(3) T-Bill Rate	(4) Ln(T-Bill Rate)	(5) First Difference Of Logs	(6) Squared First Difference	(7) Third Moment	(8) Fourth Moment
80	4	0.72%	-4.93	-0.564	0.319	-0.180	0.101
80	5	0.57%	-5.16	-0.231	0.053	-0.012	0.003
80	6	0.51%	-5.28	-0.118	0.014	-0.002	0.000
80	7	0.59%	-5.13	0.154	0.024	0.004	0.001
80	8	0.75%	-4.90	0.230	0.053	0.012	0.003
80	9	0.91%	-4.70	0.201	0.040	0.008	0.002
80	10	0.90%	-4.71	-0.010	0.000	-0.000	0.000
80	11	1.24%	-4.39	0.315	0.099	0.031	0.010
80	12	1.00%	-4.61	-0.216	0.047	-0.010	0.002
81	1	1.16%	-4.45	0.154	0.024	0.004	0.001
81	2	1.16%	-4.45	0.001	0.000	0.000	0.000
81	3	1.09%	-4.52	-0.067	0.005	-0.000	0.000
81	4	1.12%	-4.49	0.027	0.001	0.000	0.000
81	5	1.31%	-4.34	0.155	0.024	0.004	0.001
81	6	1.20%	-4.42	-0.081	0.007	-0.001	0.000
81	7	1.24%	-4.39	0.029	0.001	0.000	0.000
81	8	1.25%	-4.38	0.006	0.000	0.000	0.000
81	9	1.12%	-4.49	-0.105	0.011	-0.001	0.000
81	10	1.06%	-4.55	-0.058	0.003	-0.000	0.000
81	11	0.86%	-4.75	-0.204	0.042	-0.009	0.002
81	12	0.78%	-4.86	-0.108	0.012	-0.001	0.000
82	1	1.00%	-4.60	0.257	0.066	0.017	0.004
82	2	1.00%	-4.61	-0.007	0.000	-0.000	0.000
82	3	1.12%	-4.49	0.118	0.014	0.002	0.000
82	4	1.03%	-4.57	-0.083	0.007	-0.001	0.000
82	5	0.96%	-4.64	-0.070	0.005	-0.000	0.000
82	6	0.95%	-4.65	-0.010	0.000	-0.000	0.000
82	7	0.74%	-4.90	-0.247	0.061	-0.015	0.004
82	8	0.52%	-5.25	-0.353	0.125	-0.044	0.016
82	9	0.59%	-5.14	0.115	0.013	0.002	0.000
82	10	0.63%	-5.07	0.070	0.005	0.000	0.000
82	11	0.64%	-5.06	0.013	0.000	0.000	0.000
82	12	0.68%	-4.99	0.066	0.004	0.000	0.000
83	1	0.66%	-5.02	-0.027	0.001	-0.000	0.000
83	2	0.65%	-5.03	-0.019	0.000	-0.000	0.000
83	3	0.71%	-4.95	0.087	0.007	0.001	0.000
83	4	0.66%	-5.02	-0.075	0.006	-0.000	0.000
83	5	0.68%	-5.00	0.026	0.001	0.000	0.000
83	6	0.72%	-4.93	0.068	0.005	0.000	0.000
83	7	0.74%	-4.91	0.023	0.001	0.000	0.000
83	8	0.71%	-4.95	-0.048	0.002	-0.000	0.000

EXHIBIT IV

One Month Yield on Treasury Bills

(1) Year	(2) Month	(3) T-Bill Rate	(4) Ln(T-Bill Rate)	(5) First Difference Of Logs	(6) Squared First Difference	(7) Third Moment	(8) Fourth Moment
83	9	0.72%	-4.93	0.024	0.001	0.000	0.000
83	10	0.71%	-4.95	-0.023	0.001	-0.000	0.000
83	11	0.71%	-4.95	-0.000	0.000	-0.000	0.000
83	12	0.71%	-4.95	0.003	0.000	0.000	0.000
84	1	0.74%	-4.91	0.046	0.002	0.000	0.000
84	2	0.69%	-4.98	-0.076	0.006	-0.000	0.000
84	3	0.77%	-4.87	0.117	0.014	0.002	0.000
84	4	0.76%	-4.87	-0.008	0.000	-0.000	0.000
84	5	0.69%	-4.97	-0.096	0.009	-0.001	0.000
84	6	0.77%	-4.86	0.105	0.011	0.001	0.000
84	7	0.80%	-4.83	0.038	0.001	0.000	0.000
84	8	0.79%	-4.84	-0.017	0.000	-0.000	0.000
84	9	0.86%	-4.75	0.089	0.008	0.001	0.000
84	10	0.69%	-4.98	-0.224	0.050	-0.011	0.003
84	11	0.62%	-5.09	-0.111	0.012	-0.001	0.000
84	12	0.64%	-5.05	0.038	0.001	0.000	0.000
85	1	0.62%	-5.08	-0.031	0.001	-0.000	0.000
85	2	0.60%	-5.11	-0.028	0.001	-0.000	0.000
85	3	0.67%	-5.00	0.111	0.012	0.001	0.000
85	4	0.61%	-5.10	-0.100	0.010	-0.001	0.000
85	5	0.51%	-5.27	-0.175	0.030	-0.005	0.001
85	6	0.55%	-5.19	0.080	0.006	0.001	0.000
85	7	0.58%	-5.14	0.052	0.003	0.000	0.000
85	8	0.59%	-5.13	0.008	0.000	0.000	0.000
85	9	0.58%	-5.15	-0.015	0.000	-0.000	0.000
85	10	0.59%	-5.13	0.020	0.000	0.000	0.000
85	11	0.55%	-5.20	-0.073	0.005	-0.000	0.000
85	12	0.49%	-5.32	-0.114	0.013	-0.001	0.000
N =	192		Avg =	-0.00131	0.0127	-0.0014	0.0010
			S.D. =	0.113			
			Skewness =	-0.965			
			Kurtosis =	6.425			

data and the kurtosis of greater than six is substantially higher than the theoretical kurtosis for a normally distributed variable of three. A further test of the distributional assumptions was performed. The hypothesis that the model's residual had a normal$(0,\sigma^2)$ distribution was tested using a Kolmogorov-Smirnov statistic and rejected at the 95th percent confidence level. Thus the statistical data indicated that the normal distribution is not an appropiate model for generating the disturbance terms for the interest rate model.

Taylor[21] and Burn[22] have observed that financial time series frequently are nonnormal and they suggest procedures for modelling time series when normality is not assumed. For the simulation described in this paper a Cornish Fisher expansion was selected to model the random shock terms of the series. This approximation was selected because it does not significantly increase the difficulty of fitting models to the data or programming the simulation. It also provides a correction for the departure from normality exhibited by the skewness and kurtosis of the interest rate data. The Cornish Fisher expansion is discussed in Appendix II.

In order to simulate a return from the random walk model, the following steps are performed:

1. Generate a random shock term a_t using the Cornish Fisher expansion.

2. Since $\ln(X_t) - \ln(X_{t-1}) = a_t$

$$\text{or } \ln\left(\frac{X_t}{X_{t-1}}\right) = a_t$$

$$\frac{X_t}{X_{t-1}} = \exp(a_t)$$

$$X_t = X_{t-1} \cdot \exp(a_t)$$

SUMMARY

A complete model has now been described which can be used to generate random present values of losses. Losses paid in a particular period are assumed to earn interest from the inception of the policy period until the losses are paid out. Thus, the simulation works by first generating a random aggregate loss amount. Then a random payment pattern is generated for the aggregate loss amount. Finally, random interest rates are generated for present valuing the payouts.

APPLICATION OF MODEL

The first example presents results of the application of the simulation model to the determination of the capital requirements of a captive insuring automobile liability losses.

An expected claim count of 200 was selected for this example. A negative binomial distribution with a variance equal to twice the mean was used. The parameters of this distribution were not derived from actual data, but are typical of claim count distributions observed in practice.

The payout pattern, as displayed in Exhibit I, has been derived from actual automobile liability data, as have the severity distribution parameters. These parameters are shown on the bottom of Exhibit V. A rate of 7.54% was used as the initial interest rate in the model.

In this example, the capital requirements for the captive are determined from risk theoretic considerations. A criteria for selecting a capital requirement is that the probability that the company's actual loss will exceed premiums collected plus investment income on the premiums plus the capital will be very small. Typically, this probability is selected to be 1% or 0.1%. If the difference between the 99^{th} percentile or the 99.9^{th} percentile of the present value distribution and the expected present value is used as the capital requirement, this criteria is satisfied.

Exhibits V and VI present the results of the simulation model for the first example. Exhibit V displays the selected percentiles of the undiscounted aggregate loss distribution and the distribution of the present value of losses.

An average discount factor equal to the average of the present value distribution divided by the average undiscounted losses was determined. (This discount factor was tested for reasonableness by comparing it to a discount factor derived from the original payment pattern data and a 7.54% rate of interest.) This discount factor was applied to

EXHIBIT V

Aggregate Loss Simulation
Automobile Liability
Retention = $500,000

(1)	(2)	(3)
Percentile	Aggregate Loss Amount	Present Value Loss
5%	$1,715,060	$1,236,175
10%	$1,898,247	$1,418,835
15%	$2,028,626	$1,553,496
20%	$2,140,948	$1,667,683
25%	$2,238,017	$1,767,912
30%	$2,324,570	$1,855,017
35%	$2,410,222	$1,937,643
40%	$2,500,990	$2,022,185
45%	$2,578,671	$2,103,421
50%	$2,657,362	$2,178,678
55%	$2,742,001	$2,264,754
60%	$2,827,389	$2,347,769
65%	$2,925,761	$2,435,732
70%	$3,027,118	$2,521,824
75%	$3,139,571	$2,629,880
80%	$3,275,600	$2,755,147
85%	$3,421,406	$2,895,081
90%	$3,623,862	$3,081,293
95%	$3,913,000	$3,349,684
99%	$4,501,768	$3,897,830
99.5%	$4,733,175	$4,123,510
99.9%	$5,313,894	$4,619,050
Avg	$2,719,628	$2,223,868

Simulation Parameters

Claim Count Mean:	200
Claim Variance/Mean:	2
Log t Mu:	7.9
Log t Sigma:	1.74
Log t v:	12
Number of trials	10,000

Aggregate Loss Simulation
Automobile Liability
Retention = $500,000

Discount Factor = 81.8%

(1) Percentile	(2) PV of Aggregate Loss Amount	(3) Simulated Present Value Loss	(4) Percent Difference [(3)/(2)]-1	(5) Simulation Discount Factor [(3)/ Exh V (2)]
5%	$1,402,919	$1,236,175	-11.9%	72.1%
10%	$1,552,766	$1,418,835	-8.6%	74.7%
15%	$1,659,416	$1,553,496	-6.4%	76.6%
20%	$1,751,295	$1,667,683	-4.8%	77.9%
25%	$1,830,697	$1,767,912	-3.4%	79.0%
30%	$1,901,498	$1,855,017	-2.4%	79.8%
35%	$1,971,561	$1,937,643	-1.7%	80.4%
40%	$2,045,809	$2,022,185	-1.2%	80.9%
45%	$2,109,353	$2,103,421	-0.3%	81.6%
50%	$2,173,722	$2,178,678	0.2%	82.0%
55%	$2,242,957	$2,264,754	1.0%	82.6%
60%	$2,312,804	$2,347,769	1.5%	83.0%
65%	$2,393,272	$2,435,732	1.8%	83.3%
70%	$2,476,182	$2,521,824	1.8%	83.3%
75%	$2,568,169	$2,629,880	2.4%	83.8%
80%	$2,679,441	$2,755,147	2.8%	84.1%
85%	$2,798,710	$2,895,081	3.4%	84.6%
90%	$2,964,319	$3,081,293	3.9%	85.0%
95%	$3,200,834	$3,349,684	4.7%	85.6%
99%	$3,682,446	$3,897,830	5.8%	86.6%
99.5%	$3,871,737	$4,123,510	6.5%	87.1%
99.9%	$4,346,765	$4,619,050	6.3%	86.9%

Capital Requirement			Difference
@ 99%ile	$1,458,578	$1,673,962	$215,384
@ 99.9%ile	$2,122,897	$2,395,182	$272,284

the distribution of the undiscounted aggregate losses shown on Exhibit V. Thus, the percentiles of the aggregate loss distribution were present valued using a selected constant discount factor. In Exhibit VI these discounted losses are compared to the percentiles of the present value loss distribution.

A conclusion which can be drawn based on Exhibit VI is that the percentiles of the present value loss distribution are not equal to the present value of the percentiles of the aggregate loss distribution. While these two distributions are approximately equal at intermediate percentiles, at the extremes of the distribution, there is a signficant difference between the values of the two distributions. The 99^{th} percentile of the present value distribution is higher than the present value of the 99^{th} percentile of the aggregate loss distribution. The fifth percentile of the present value distribution is lower that the fifth percentile of the present valued loss distribution.

The capital requirement derived from the model described in this paper is higher at both the 99^{th} percentile and the 99.9^{th} percentile than the capital requirement derived by present valuing the 99^{th} and 99.9^{th} percentiles of the aggregate loss distribution.

Because automobile liability is a short tail line, payouts are more predictable and investment income earned on premiums is a smaller component of loss funding requirements than it is for a longer tail line. Thus, the difference in capital requirements derived using the two different methods of evaluating risk is not as dramatic as it would be for a longer tail line.

To illustrate the significant impact that timing and interest rate risk can have on a company's financial well-being, it is necessary to apply the method described in this paper to a line of business where payments are made more slowly. This is illustrated in the second example which simulates the experience of a captive insuring medical malpractice losses.

A payout pattern curve was fit to industry medical malpractice data compiled by A.M. Best. The parameters of the frequency and the severity distribution were selected to be representative of this line of business.

A model of the following form was fit to the medical malpractice payment data.

$$f(t) = e^{-ab^{t}c^{t^{1.2}}}$$

The results of the simulation are presented in Exhibits VII and VIII. As suggested earlier, the difference between the present value of the aggregate loss distribution and the distribution of the present value of losses is much more significant for this example.

The discount factor shown in Exhibit VIII was derived as described above for automobile liability. This average

EXHIBIT VII

Aggregate Loss Simulation
Medical Malpractice
Retention = $1 Million

(1)	(2)	(3)
Percentile	Aggregate Loss Amount	Present Value Loss
5%	$8,657,779	$4,663,561
10%	$9,631,627	$5,427,865
15%	$10,274,340	$5,933,729
20%	$10,788,455	$6,443,450
25%	$11,276,025	$6,827,288
30%	$11,729,180	$7,245,040
35%	$12,156,905	$7,617,591
40%	$12,544,930	$7,983,861
45%	$12,896,100	$8,297,990
50%	$13,276,955	$8,625,623
55%	$13,644,115	$8,984,258
60%	$14,026,925	$9,320,487
65%	$14,442,800	$9,653,176
70%	$14,886,320	$10,007,271
75%	$15,361,065	$10,454,555
80%	$15,844,000	$10,910,135
85%	$16,449,865	$11,518,205
90%	$17,325,780	$12,318,570
95%	$18,733,620	$13,535,480
99%	$21,264,935	$15,670,860
99.5%	$22,108,500	$16,516,385
99.9%	$24,184,735	$18,420,840
Avg	$13,414,055	$8,771,733

Simulation Parameters

Claim Count Mean:	100
Claim Variance/Mean:	2
Log t Mu:	10.5
Log t Sigma:	1.8
Log t v:	10
Number of trials	10,000

Aggregate Loss Simulation
Medical Malpractice
Retention = $1 Million

Discount Factor = 65.4%

(1) Percentile	(2) PV of Aggregate Loss Amount	(3) Simulated Present Value Loss	(4) Percent Difference [(3)/(2)]-1	(5) Simulation Discount Factor [(3) / Exh VII (2)]
5%	$5,661,504	$4,663,561	-17.6%	53.9%
10%	$6,298,323	$5,427,865	-13.8%	56.4%
15%	$6,718,607	$5,933,729	-11.7%	57.8%
20%	$7,054,797	$6,443,450	-8.7%	59.7%
25%	$7,373,630	$6,827,288	-7.4%	60.5%
30%	$7,669,957	$7,245,040	-5.5%	61.8%
35%	$7,949,656	$7,617,591	-4.2%	62.7%
40%	$8,203,393	$7,983,861	-2.7%	63.6%
45%	$8,433,031	$8,297,990	-1.6%	64.3%
50%	$8,682,080	$8,625,623	-0.7%	65.0%
55%	$8,922,174	$8,984,258	0.7%	65.8%
60%	$9,172,501	$9,320,487	1.6%	66.4%
65%	$9,444,450	$9,653,176	2.2%	66.8%
70%	$9,734,478	$10,007,271	2.8%	67.2%
75%	$10,044,923	$10,454,555	4.1%	68.1%
80%	$10,360,725	$10,910,135	5.3%	68.9%
85%	$10,756,912	$11,518,205	7.1%	70.0%
90%	$11,329,692	$12,318,570	8.7%	71.1%
95%	$12,250,308	$13,535,480	10.5%	72.3%
99%	$13,905,588	$15,670,860	12.7%	73.7%
99.5%	$14,457,213	$16,516,385	14.2%	74.7%
99.9%	$15,814,907	$18,420,840	16.5%	76.2%

Capital Requirement			Difference
@ 99%ile	$5,133,855	$6,899,128	$1,765,272
@ 99.9%ile	$7,043,174	$9,649,108	$2,605,933

discount factor of 65.4% can be compared to the discount factors shown in Exhibit VIII for the present value distribution. At the more extreme percentiles there are approximately ten percentage points difference in discount factors. There is also a $1,765,272 difference at the 99th percentile and a $2,605,933 difference at the 99.9th percentile between the capital requirements derived using the distribution of present values, as opposed to a discounted aggregate loss distribution. Thus, by ignoring timing and interest rate risk, the amount of cushion against adverse experience that is provided by the required capital is considerably less than what it is believed to be.

PRACTICAL CONSIDERATIONS WHEN APPLYING THE MODEL: INTEREST RATE SELECTION

Much of the actuarial literature on the selection of interest rates for use in discounting liabilities suggests that a risk free rate of return be used.[23,24] The return on United States treasury bills and treasury bonds are frequently used as a proxy for the risk free rate of return. However, as the yields on United States treasury obligations vary with the maturity of the note or bond, a number of possible rates can be selected.

The short term rates used in this analysis were part of a data base provided by the SPSS time series software package. These rates were used only as an illustration of a method of analysis, not because they are believed to be the most appropriate rate for discounting losses. These rates may appear to be unnecessarily conservative. Because companies hold investments of varying maturities, they can, on average, realize a return which exceeds the return on the one month treasury bills.

In a recent analysis performed by Bryan and Linke[25], a number of different portfolios were tested for their ability to offset the effect of inflation on the liabilities of a personal injury settlement. The portfolios examined included one year, five year, ten year, fifteen year, and twenty year United States treasury obligations. Their research indicated that the one year treasury bill portfolio was better than the longer maturity portfolios at offsetting inflation and also was the lowest risk portfolio. Bryan and Linke concluded that "the least cost means of providing for a future payment stream of uncertain size is to construct a dedicated portfolio of very short term securities."[26]

However, the conclusions reached by Bryan and Linke may not apply to large companies in the same manner as to individuals, since the companies can use sophisticated techniques such as asset liability matching to reduce their risk. Nevertheless, for a relatively small trust fund or captive, it may be prudent to use short term rates when discounting liabilities.

An alternative to the use of the rate of return on investments with a single maturity is suggested by the work of Bradley.[27] Bradley proposed that instead of discounting assets and liabilities separately, the analyst should discount only the mismatch in cash flow between the assets and the liabilities. This implies that a portfolio of securities could be constructed with cash flows equal to the expected payout of the company's losses. The portfolio would have a present value equal to the purchase price of the bonds on the day the transaction is made. That is, the interest rate applied to each expected payment is the yield on zero coupon bonds maturing in the period when the payment is made. The expected losses, then, would be discounted at the same interest rates as the bonds which could, but need not actually be purchased to fund the losses. Actual payments, however will vary from their expected value and it is this variability which can be modeled in a simulation.

To model the asset liability mismatch via simulation, an aggregate loss amount and a payout pattern are generated. The difference between each payment and the cash flow provided by a bond maturing in the period the payment is made is calculated. This difference, which is the asset liability mismatch for the period, is present valued using a random interest rate. The present value of the asset liability mismatch is added to the present value of the bond portfolio to determine the present value of losses.

PRACTICAL CONSIDERATIONS WHEN APPLYING THE MODEL: SELECTION OF PAYMENT PATTERN MODEL

A variety of curves can be used to model payout pattern data. Kolb[28] describes over twenty simple curves, a number of which have been successfully applied to model development data.

A problem which occurs when regressions are used to model payment patterns is that the payment pattern derived from the model may not sum to one. Although the simulation program can be constructed to force the sum of all payout percentages to equal to 100%, the payments derived from the fitted model may be significantly faster or slower than the actual payments the curve was fit to. This can happen even for a curve which is a "good" fit, as determined by the R^2 and F statistics.

Statistics such as average payment lag, average time for 100% of losses to pay out and average discount factor should be tabulated by the simulation. These statistics should be substantially similar to those derived from the data the regression was fit to, otherwise another model should be used.

When using a curve of the following form,

$$f(t) = e^{-ab^{t}c^{t^{p}}}$$

the speed of payments can be adjusted by changing the exponent p. Increasing p will cause the model's payments to be made slower and decreasing p will cause the model's payments to be made faster.

CONCLUSION

In this paper, a model has been presented which can be used to incorporate timing and investment risk into the determination of solvency or risk margins. It has been shown that, when determining risk margins, discount factors derived from deterministic payout patterns and constant interest rates are too low relative to discount factors derived from a random distribution of present values.

The model described in this paper has other applications, such as the determination of risk margins for reserves and the estimation of the present value cost of aggregate excess reinsurance. In addition, the timing, interest rate, and aggregate loss distributions could be incorporated into a more complete stochastic model of insurance company operations.

APPENDIX I: NEGATIVE BINOMIAL DISTRIBUTION

The probability density function for the negative binomial is:

$$P(X = x) = \binom{x + r - 1}{x} p^r (1 - p)^x \qquad x = 0,1,\ldots,r$$

The mean and variance of the negative binomial distribution are as follows:

$$\mu = \frac{r(1-p)}{p}$$

$$\sigma^2 = \frac{r(1-p)}{p^2}$$

A straightforward method of estimating the parameters of a negative binomial is the method of moments. The parameter p is equal to the mean of a number of years of claims adjusted to a common exposure level divided by the variance of claims. Once p has been determined r can be solved for by substituting p into one of the two moment formulas above.

When the parameter r of the negative binomial distribution is less than 40, claim counts can be simulated using an exact formula:

$$P(X = 0) = p^r$$

$$\text{For } X > 0,\ P(X = x) = \left(\frac{x + r - 1}{x} \right) (1 - p)\ P(X = x-1)$$

For large values of r, a Cornish Fisher expansion can be used to approximate a negative binomial random variable. For this approximation, first simulate a standard normal variable Z.

Then:

$$Y = Z + \frac{k_3}{6}(Z^2 - 1) + \frac{k_4}{24}(Z^3 - 3Z) - \frac{k_3^2}{36}(2Z^3 - 5Z)$$

$$k_3 = \text{Coefficient of skewness} = \frac{2+p}{\sqrt{r\,(1-p)}}$$

$$k_4 = \text{Coefficient of kurtosis} = 3 + \frac{6}{r} + \frac{p^2}{r\,(1-p)}$$

A negative binomial random variable X is obtained from $\mu + Y\sigma$. Where μ and σ are the mean and standard deviation of the negative binomial distribution.

APPENDIX II: THE CORNISH FISHER EXPANSION

A number of methods have been developed by statisticians which attempt to approximate the percentiles of a probability distribution with percentiles of another distribution. A purpose for using such a procedure is that the distribution used for the approximation is more convenient computationally than the original distribution.

One of the most common approximations is the normal approximation. Because the sum of a large number of independently and identically distributed random variables is asymptotically normal, the normal distribution is frequently used to approximate variables which are the sums of nonnormal random variables. However the distribution being approximated is frequently more skewed and has a thicker tail than its normal distribution approximation would indicate.

Cornish and Fisher developed an approximation which uses the normal distribution but has the same moments as the distribution being approximated. The details of their method of derivation are provided by Beard, Pentikäinen, and Pesonen, and Johnson and Kotz.[29,30] The first twelve terms of the approximation they derived are:

$$
\begin{aligned}
Y_\alpha = {} & Z_\alpha + \frac{1}{6}\left(Z_\alpha^2 - 1\right)k_3 \\
& + \frac{1}{24}\left(Z_\alpha^3 - 3Z_\alpha\right)k_4 - \frac{1}{36}\left(2Z_\alpha^3 - 5Z_\alpha\right)k_3^2 \\
& + \frac{1}{120}\left(Z_\alpha^4 - 6Z_\alpha^2 + 3\right)k_5 - \frac{1}{24}\left(Z_\alpha^4 - 5Z_\alpha^2 + 2\right)k_3 k_4
\end{aligned}
$$

$$+ \frac{1}{324}\left(12Z_\alpha^4 - 53Z_\alpha^2 + 17\right)k_3^3$$

$$+ \frac{1}{720}\left(Z_\alpha^5 - 10Z_\alpha^3 + 15Z\right)k_6$$

$$- \frac{1}{180}\left(2Z_\alpha^5 - 17Z_\alpha^3 + 21Z_\alpha\right)k_3k_5$$

$$- \frac{1}{384}\left(3Z_\alpha^5 - 24Z_\alpha^3 + 29Z_\alpha\right)k_4^2$$

$$+ \frac{1}{288}\left(14Z_\alpha^5 - 103Z_\alpha^3 + 107Z_\alpha\right)k_3^2k_4$$

$$- \frac{1}{7776}\left(252Z_\alpha^5 - 1688Z_\alpha^3 + 1511Z_\alpha\right)k_3^4$$

$$+ \ldots$$

Z_α = the α^{th} percentile of the standard normal distribution

k_i = the i^{th} moment of the distribution of Y

$Y_\alpha = \frac{X_\alpha - \mu}{\sigma}$ the standardized variable of the distribution approximated

X_α = the α^{th} percentile of the distribution approximated

Venter[31] notes that since this series is an alternating and sometimes nonconvergent series, adding more terms may not improve its accuracy. The simulation described in the paper used the first four terms of the series.

In insurance, the first two terms of the series (also known as the normal power approximation) are frequently used to model aggregate loss distributions. The approximation has been found to be accurate for moderately skewed distributions. The accuracy of the approximation which incorporates the first four moments of the original distribution (and first four terms of the series) has not been thoroughly studied.

APPENDIX III: COMPUTATION OF THE VARIANCE OF THE FORECAST FOR A TWO VARIABLE REGRESSION

The purpose of this appendix is to provide details of the calculation of the variance of the forecast for actuaries who program in a language that does not have matrix operations.

Let Y be the dependent variable; X be an array of observations for the first independent variable; Z be the observations of the second independent variable; X_t, Z_t are actual values of the independent variables for one forecast period; $\hat{Y}_t$ is the forecast for Y at time t.

$$MSE = \frac{\sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2}{N-k-1}$$

Then $\hat{\sigma}^2$ = Estimate of the variance of the forecast

$$= \left(\begin{bmatrix} 1 & X_t & Z_t \end{bmatrix} \begin{bmatrix} N & \sum_{i=1}^{N} X_i & \sum_{i=1}^{N} Z_i \\ \sum_{i=1}^{N} X_i & \sum_{i=1}^{N} X_i^2 & \sum_{i=1}^{N} X_i Z_i \\ \sum_{i=1}^{N} Z_i & \sum_{i=1}^{N} X_i Z_i & \sum_{i=1}^{N} Z_i^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ X_t \\ Z_t \end{bmatrix} \right) \cdot MSE$$

Where:

$$\begin{bmatrix} N & \sum_{i=1}^{N} X_i & \sum_{i=1}^{N} Z_i \\ \sum_{i=1}^{N} X_i & \sum_{i=1}^{N} X_i^2 & \sum_{i=1}^{N} X_i Z_i \\ \sum_{i=1}^{N} Z_i & \sum_{i=1}^{N} X_i Z_i & \sum_{i=1}^{N} Z_i^2 \end{bmatrix}^{-1} = \frac{\text{INV}}{\text{DET}}$$

$$\text{INV} = \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}$$

$$A = \sum_{i=1}^{N} X_i^2 \sum_{i=1}^{N} Z_i^2 - \left(\sum_{i=1}^{N} X_i Z_i \right)^2$$

$$B = \sum_{i=1}^{N} Z_i \sum_{i=1}^{N} X_i Z_i - \sum_{i=1}^{N} X_i \sum_{i=1}^{N} Z_i^2$$

$$C = \sum_{i=1}^{N} X_i \sum_{i=1}^{N} X_i Z_i - \sum_{i=1}^{N} Z_i \sum_{i=1}^{N} X_i^2$$

$$D = N \sum_{i=1}^{N} Z_i^2 - \left(\sum_{i=1}^{N} Z_i \right)^2$$

$$E = \sum_{i=1}^{N} X_i \sum_{i=1}^{N} Z_i - N \sum_{i=1}^{N} X_i Z_i$$

$$F = N \sum_{i=1}^{N} X_i^2 - \left(\sum_{i=1}^{N} X_i \right)^2$$

DET = determinant of the matrix

$$= N\sum_{i=1}^{N} X_i^2 \sum_{i=1}^{N} Z_i^2 + 2\sum_{i=1}^{N} X_i \sum_{i=1}^{N} Z_i \sum_{i=1}^{N} X_i Z_i - \sum_{i=1}^{N} X_i^2 \left(\sum_{i=1}^{N} Z_i \right)^2$$

$$- N\left(\sum_{i=1}^{N} X_i Z_i \right)^2 - \sum_{i=1}^{N} Z_i^2 \left(\sum_{i=1}^{N} X_i \right)^2$$

An approximation to $\hat{\sigma}^2$ of $\left(1 + \frac{1}{N} \right) \cdot$ MSE was suggested in the paper. This approximation may not be very accurate if $|X_t - \bar{X}|$, or $|Z_t - \bar{Z}|$ is large. However, for the regression described in the paper, this tends to occur at time periods when the payout percentage is very small, and therefore the overall effect on simulation results is frequently not substantial.

APPENDIX IV: SIMULATION OF NORMAL, LOGNORMAL, AND T DISTRIBUTION RANDOM VARIABLES

One of the most commonly used procedures for simulating losses from a normal distribution is the Box Muller transformation, which is described as follows.

Let RND_1 and RND_2 equal two random numbers from a uniform distribution on the interval between zero and one.

$$\text{Then } Z_1 = \sqrt{-2 \ln (RND_1)} \cos (2\pi RND_2)$$

$$\text{and } Z_2 = \sqrt{-2 \ln (RND_1)} \sin (2\pi RND_2)$$

will be independent variables from a standard normal distribution. The quantity $X = \mu + Z_1\sigma$ will be distributed normal(μ,σ^2) and $e^{\mu+Z_1\sigma}$ will have a lognormal distribution. A variable from a t distribution can be obtained by using Z_1 or Z_2 in the following approximation[32]:

$$t_v = Z_1 + \frac{1}{4v} Z_1(Z_1^2 + 1) + \frac{1}{96v^2} Z_1(5Z_1^4 + 16Z_1^2 + 3)$$

$$+ \frac{1}{384v^3} Z_1(3Z_1^6 + 19Z_1^4 + 17Z_1^2 - 15)$$

Where t_v is a random variable from a t distribution with v degrees of freedom.

This approximation is typically accurate to three decimal places except when the degrees of freedom are less than ten.

APPENDIX V: THE LOG STUDENT'S T DISTRIBUTION

The density function of the log Student's t distribution is as follows:

$$f(x) = \frac{1}{x\sigma\sqrt{v}\ \beta\left(\frac{1}{2},\frac{v}{2}\right)} \left(1 + \frac{\left(\frac{\ln(x) - \mu}{\sigma}\right)^2}{v} \right)^{-\frac{(v+1)}{2}} \qquad x \geq 0$$

Where x denotes a severity random variable and β is the beta function.

This distribution, which is based upon the assumption that the standardized logarithms of losses follow a t distribution, may be appropriate for severities which are believed to come from a mixture of lognormal distributions. Thus the conditional distribution of data is lognormal, with parameters μ and τ. The parameter τ is itself a random variable which has an inverted gamma distribution[33]:

$$f(\tau) = \frac{2}{\Gamma(v/2)} \left(\frac{v\sigma^2}{2} \right)^{v/2} \tau^{-(v+1)}\ e^{-v\sigma^2/2\tau^2}$$

Then the unconditional distribution of the logarithms of the losses is the Student's t distribution with mean μ, variance σ^2, and v degrees of freedom. The parameters of this distribution can be fit from the mean, variance, and the coefficient of kurtosis of the logged data.

$$\hat{\mu} = \frac{\sum_{i=1}^{N} \ln(x_i)}{N}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{N} \left(\ln(x_i) - \hat{\mu} \right)^2}{N - 1}$$

k_4 = coefficient of kurtosis

$$= \frac{\sum_{i=1}^{N} \left(\ln(x_i) - \hat{\mu} \right)^4}{(N - 1)\ \hat{\sigma}^4}$$

For the t distribution, the coefficient of kurtosis, k_4, is as follows.

$$k_4 = 3 + \frac{6}{v-4}, \quad v > 4$$

Therefore,

$$\hat{v} = \frac{6}{k_4-3} + 4$$

The log Student's t distribution is appropriate if the kurtosis of the empirical logged severities exceeds the kurtosis for the normal distribution of three. This distribution is more dispersed than its lognormal analog.

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