Portfolio Based Pricing of Residual Basis Risk with Application to the S&P 500 Put Options

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ABSTRACT

Financial option pricing methodology since Black and Scholes (1973) defines option prices as the hedging cost to set up a riskless hedged portfolio. Financial options are treated as redundant contracts, since they can be replicated by trading the underlying assets. The so-called "relative valuation" method prices financial options in the world of the risk-neutral measure. On the actuarial side, there is no liquid secondary market for insurance contracts; thus, insurance and reinsurance contracts are viewed as non-redundant, primary contracts to complete the market. Actuarial risk models that price insurance liability contracts are not based on an assumption of hedging, instead considering the present value of future losses and the cost of allocated.

This paper is devoted to pricing "hybrid" (insurance and financial) risk products by combining the financial option pricing method with the actuarial pricing method. It suggests that the price of contingent claims containing both hedgeable risk and unhedgeable risk should reflect the average cost of hedging, plus a risk premium that compensates for the marginal residual risk the contract brings to the existing portfolio. To evaluate the residual risk, a portfolio-based pricing method is proposed to evaluate loss and systematically consider risk premium. The risk premium is charged to satisfy risk management and return on risk capital requirements. The proposed method is tested by pricing the Standard and Poor's 500 index (SPX) options in a simulated objective world.

Introduction to "Portfolio-Based Pricing of Residual Basis Risk with Application to the S&P 500 Put Options" by Sergei Esipov and Dajiang Guo

Donald Mango, FCAS, MAAA

American Re-Insurance

The authors of this paper, Sergei Esipov and Dajiang Guo, were colleagues of mine when I worked at Centre Solutions. These two gentlemen are not actuaries, but capital market quantitative analysts, with backgrounds in economics, finance and natural sciences. At Centre Solutions, they were involved in the evaluation of contracts that applied reinsurance techniques to risks with some capital market components. It was during discussions of these contracts that both they and their Centre Solutions actuarial colleagues saw just how divergent our respective approaches were. To Sergei's and Dajiang's credit, they reached out in earnest to learn the "actuarial" approach to evaluating risk. Once they embraced the theory and techniques, they applied them to one of the more difficult problems facing the capital market quantitative analysts: the **volatility smile** of option pricing.

This term may be foreign to most casualty actuaries, so some background on the issue would be helpful. To begin, recall the assumptions underlying the Black-Scholes formula (quoting the original source [1]):

- The short-term interest rate is known and is constant through time.
- The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The standard deviation rate of the return of the stock is constant.
- The stock pays no dividends or other distributions.
- The option is "European," that is, it can only be exercised at maturity.
- There are no transaction costs in buying or selling the stock or the option.
- It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.
- There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer at some future date by paying him an amount equal to the price of the security on that date. [1, p. 640]

Given these assumptions, a perfectly hedged portfolio is possible, one which exactly duplicates the payoffs of the option under all possible outcomes. By no-arbitrage pricing theory, the cost of this hedging portfolio must be the price of the option. That is the theory, anyway.

In the real market, however, transactions are not costless, short-term interest rates and volatilities vary over time, so dealers cannot achieve perfect hedges. The resulting Black-Scholes prices are therefore often different from the market prices. Quoting an excellent description of the situation from [2]:

> These [differences] have been documented with respect to the call option's exercise prices, its time to expiration, and the underlying common stock's volatility. Since there is a one-to-one relationship between volatility and option price through the Black-Scholes formula, the volatility is often used to quote the value of an option. An equivalent measure for the mispricing of Black-Scholes model is thus the implied or implicit volatility, i.e. the volatility which generates the corresponding option price. The Black-Scholes model imposes a flat term structure of volatility, i.e. the volatility is constant across both maturity and strike prices of options. If option prices in the market were confirmable with the Black-Scholes formula, all the Black-Scholes implied volatilities corresponding to various options written on the same asset would coincide with the volatility parameter σ of the underlying asset. In reality this is not the case, and the Black-Scholes implied volatility heavily depends on the calendar time, the time to maturity, and the moneyness of the options. The price distortions, well-known to practitioners, are usually documented in the empirical literature under the terminology of the **smile effect** [*emphasis mine*], referring to the U-shaped pattern of implied volatilities across different strike prices. [2, p. 23]

No-arbitrage pricing theory relies on the concept of the perfect hedge, which is not achievable in practice. With imperfect hedges, some residual basis risk remains. The question is how to address this basis risk in the pricing of derivatives.

Quoting from the third paragraph of their paper, Esipov and Guo suggest

> that the price of contingent claims containing both hedgeable risk and unhedgeable risk should reflect the average cost of hedging, plus a risk premium that compensates for the marginal risk the contract brings to the existing portfolio.

They take an actuarial approach to the problem, allocating capital to a new security based on its marginal impact upon the portfolio's Value-at-Risk (VaR) – the ruin threshold. Using this approach, they generate indicated option prices that closely resemble market prices – including the volatility smile.

This paper would be groundbreaking based on this achievement alone. From an actuarial point-of-view, however, it is even more revolutionary – even evolutionary. It represents validation of insurance pricing techniques in the capital markets community. There has been much discussion and interest lately regarding adoption of capital market techniques in insurance pricing. It is certainly encouraging to see this reciprocation.

As a bonus, the paper also provides an introduction to the simulation modeling of a capital market price process – the S&P 500 Index.

The focus of the 2000 Discussion Paper Program is "Insurance in the Next Century," a topic of great interest to the CAS membership. At first glance, this paper may not appear to be about what we might define today as *insurance*. However, it is about what insurance may evolve towards in the next century. More importantly, it is undeniably about *actuarial science* in the next century.

Donald Mango
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PORTFOLIO BASED PRICING OF RESIDUAL BASIS RISK WITH APPLICATION TO THE S&P 500 PUT OPTIONS[1]

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1. INTRODUCTION

In the convergence of capital and insurance markets, we increasingly observe a rapid merging of capital market products and insurance protection products, or so-called integrated financial and insurance risk management programs. Examples include the Catastrophe Linked Bond, Catastrophe Equity Put, Property-Catastrophe Services (PCS) options, Hybrid Corporate Risk Management Program, Collateralized Bond (Loan) Obligation (CBO/CLO), Structured Finance, etc. This presents great challenges to both financial and actuarial risk modeling. The financial models are challenged by the abundance of residual risk remaining after typical *sensitivity* hedging (by this we mean delta-, gamma-, vega-, etc. hedging), while the actuarial pricing models are unprepared to price products that are tradable in liquid markets.

Financial option pricing methodology since Black and Scholes (1973) defines option prices as the hedging cost to set up a riskless hedged portfolio. Financial options are treated as redundant contracts, since they can be replicated by trading the underlying assets. The so-called "relative valuation" method prices financial options in the world of the risk-neutral measure. On the actuarial side, there is no liquid secondary market for insurance contracts; thus, insurance and reinsurance contracts are viewed as non-redundant, primary contracts to complete the market. Actuarial risk models that price insurance liability contracts are not based on an assumption of hedging, instead considering the present value of future losses (loss theory) and the cost of allocated capital. The pricing is done in the world of the objective measure (Panjer and Willmot (1992)). In recent literature on risk pricing, actuaries and financial economists are struggling to connect various pieces of financial and insurance pricing theories into one unified risk pricing theory (Wang (1999)).

This paper is devoted to pricing "hybrid" (insurance and financial) risk products by combining the financial option pricing method with the actuarial pricing method. It suggests that the price of contingent claims containing both hedgeable risk and unhedgeable risk should reflect the average cost of hedging, plus a risk premium that compensates for the marginal risk the contract brings to the existing portfolio. In practice, when there is basis risk left after sensitivity hedging, it is important to incorporate risk management and capital allocation decisions into option underwriting. To evaluate the residual risk, a portfolio-based pricing method is proposed to evaluate loss and systematically consider risk premium. The risk premium is charged to satisfy risk management and return on risk capital requirements. The profit and loss distributions are priced based on a combination of Value-at-Risk and return on capital approaches. Its existing counterpart is the equilibrium capital asset pricing model (CAPM) developed by Sharpe (1964), Lintner (1965), and Mossin (1966).

In the derivatives markets, there are at least two stylized facts that puzzle the profession. One is that implied volatility is on average larger than realized volatility; the other is that implied volatility curves exhibit "smile" or "smirk" effects for out-of-the-money options with different strike levels. Fleming (1993), Jorion (1995), and Guo (1996) show that the implied volatility extracted from the stochastic volatility model of Hull and White (1987) is a dominant, but still a biased estimator in terms of *ex ante* forecasting power in the stock index and foreign currency option markets. There are at least three potential explanations for the puzzling empirical evidence that the implied volatility is a biased forecast of future volatility: the nonzero risk premium for distributed risk, transaction costs in dynamic hedging, and potential inefficiency of the option market. The hypothesis of the risk premium of volatility in the option prices is empirically supported by Guo (1998), who found that the market price of volatility risk is non-zero and time varying. The size of the estimated risk premium implies that the compensation for volatility risk is a significant component of the risk premium in option prices; therefore, a certain bias exists between implied volatility and realized volatility.

In the presence of residual basis risk, pricing is performed in two steps. The first step is to obtain the probability distribution of possible cumulative discounted profits and losses after hedging, the P&L distribution. The second step is to convert the P&L distribution into a single number - the suggested price. As for the first step, Esipov (1996) derived a general mixed backward-forward Kolmogorov-like equation for tracking the P&L distribution with arbitrary hedging strategy. (See Esipov and Vaysburd (1998), (1999) for full exposition and examples.) This equation allows one to bypass the Monte-Carlo simulation in a log-normal setting and obtain, most efficiently, the P&L distribution after hedging by numerical integration using finite differences. While we found this method to be excellent for generating realistic P&L distributions, it was also found that some practitioners are only comfortable with Monte-Carlo simulations, given that even a complex model of the dynamics of the underlying asset can be implemented in a straightforward manner. With this in mind we base this paper on an econometrically sound model of the underlying asset implemented by using the Monte-Carlo simulation. Given that the computer power is consumed by generating realistic details of the statistics of the underlying asset, we are no longer in a position to optimize the hedging strategy. Therefore, we *a priori* select a delta-hedge with implied volatility, which has been the usual choice for many trading desks for about two decades. Recently, we have heard of novel practices to hedge with historical volatility and, moreover, with forecasted volatility. We found such cost-effective solutions to bring little relief to the tails of the P&L-distribution of residual basis risk.

This paper also discusses in considerable detail how to convert the P&L-distribution into a price, and makes an application to index options. The question of what to do if the derivative security is already being traded, and how to take into account its existing market price is set aside for now. This question will be considered elsewhere.

Suppose a representative Dealer writes a put option and dynamically hedges her position by shorting a certain amount of the underlying assets, according to the Black-Scholes sensitivity. In a standard Black-Scholes world, if the underlying asset is log-normally distributed, and continuous frictionless hedging is possible, the full hedging (replication) cost would be a well-defined constant number, thus it is the value of the option contract. However, in reality, none of the Black-Scholes assumptions holds exactly: the underlying volatility could be time varying, the hedging invokes transaction costs, and can only be done discretely. Thus, the full hedging cost has a stochastic behavior characterized by a distribution, which means that the residual risk exists. This is where the present paper comes into play: what can the Dealer do while taking the residual basis risk to the portfolio consisting of assets and this derivative? By taking risk, the Dealer must (1) meet the internal risk management requirements, and (2) allocate scarce economic capital. (In return, the Dealer has to generate a certain minimal return. Many firms expect, at least, a 20% return on allocated economic capital.)

The solution of this problem is illustrated below by studying the P&L-distribution and subsequently by pricing the Standard and Poor's 500 index (SPX) options in a simulated objective world. While the complete list of risk factors is extensive, here we focus on three major concerns an option Dealer has to consider.

(1) Non-log-normal distribution of the underlying asset, as might be generated by stochastic volatility, jumps and memory. To generate a simulated world, we use a GJR-GARCH(1,1)/Jump-Diffusion model to approximate the underlying SPX volatility process. This is a minimal model which is capable of reproducing most of the empirically observed regularities of the volatility process (mean-reversion, persistence, and leverage effect) reasonably closely. Time series analyses reflected in the extensive literature on ARCH and GARCH models have already determined the statistical properties of the volatility process of the SPX index. The index exhibits time-varying volatility, nonzero skewness and leptokurtosis (Baillie and Bollerslev (1989), Engle and Lee (1997)). In addition, stock market downside jumps are added from a compound Poisson-Beta distribution with maximum loss of up to 25% in a day; thus, the 1987 stock market crash has a positive probability in the model. Smaller jumps are more frequent.

(2) We look at the transaction costs in dynamic hedging. In the Black-Scholes world with transactions costs, continuous rebalancing of a hedge position leads to infinite costs, while less frequent trading gives lower costs, but a less exact hedge.

(3) Finally, we examine, the risk premium for unhedgeable remaining basis risk.

With these complications, the final profit and loss (P&L) to the option Dealer is no longer certain. The Dealer has to develop a formal procedure to relate the P&L distribution at maturity to a (dollar) value of the derivative security. It is clear that charging the limit loss is overly expensive. Risk premiums for the residual basis risk calculated according to the method proposed in this paper exhibit the stylized fact of implied volatility biasedness. In general, in the simulated world, the implied volatility is about 10%-40% higher than the realized volatility. These numbers are consistent with the historical behavior of the S&P 500 options market. We can also generate the implied volatility "smile" or "smirk" curve, which closely matches the one-year SPX option's implied volatility curve. This suggests the volatility "smile" is a reflection of the risk (insurance) premium imbedded in the out-of-the-money options.

In practice, this approach can be extended to price long term stocks, currency, and, with some modifications, even commodity options. Long-term options form non-liquid, over-the-counter markets with a small number of market makers. The observed bid-ask spreads are much wider than for the short-term options. The demand of long-term options ("long volatility") is much larger than their supply ("short volatility"). Ideally, Dealers would hedge their positions by buying offset contracts from risk takers (like hedge funds, or insurance companies); however, the imbalance of demand and supply force them to use the underlying assets to hedge their positions and take residual risks.

The result of the present study can be subsequently used by academia to improve on option pricing models, and by practitioners to structure over-the-counter derivative products and perform firm-wide risk management, considering the entire portfolio as an "option".

The paper proceeds as follows. Section 2 is devoted to a portfolio-based pricing method for risk evaluation. Section 3 shows how to create a simulated world of the SPX index. Section 4 describes the price generating process of short term options and compares with the real world observations. Section 5 extends the result to price long term put options, and Section 6 concludes.

2. PORTFOLIO-BASED RISK PRICING

The pricing technique presented in this section is based on historical probabilities, and correlations with the existing portfolio. This technique combines actuarial methods for computing the risk premiums with the Value-at-Risk approach. The calibration procedure used in the conventional risk-neutral approach is fully replaced here by a set of constraints on the values of the risk premiums expressed though a return on allocated capital.

An important characteristic of a portfolio is its Value-at-Risk, VaR. Notwithstanding the growing popularity of this term, its definition is far from unique. We adopt the narrow definition where VaR is the 99%-ile (or 1% left limit) of the P&L distribution of the net cash flows associated with a given time horizon. (Really, full VaR is a set of numbers - the inverted cumulative P&L-distribution.) The conventional approach to determine VaR is a Monte-Carlo simulation of all the contracts under management. We are going to show how to bypass this complex step in a simple market-wide setting. The situation simplifies greatly when one focuses on a single contract whose cash flows are small relative to the scale of the whole portfolio. This is usually the case for the so-called "Benchmark Market Portfolio", i.e. the collection of contracts traded on a market or held by major players. The same assumption holds true for large financial institutions. Under this assumption, the change of VaR caused by adding a single contract is much smaller than the current VaR itself. Because the new contract is small relative to the portfolio, the detailed shape of its P&L distribution is irrelevant.

Below, the P&L distribution of a Market Portfolio is assumed to be normal, $N(E(X),\Xi)$, although the same technique is applicable to an arbitrary distribution. Adding a contract with probability distribution *p(x)* to this portfolio shifts the VaR (99%-ile) by

$$\Delta\text{VaR} = E(X+x) + N^{-1}(0.01)\sqrt{E[(X+x-E(X)-E(x))]^2]} - E(X) - N^{-1}(0.01)\sqrt{E[(X-E(X))]^2} \approx$$

$$\approx E(x) - \zeta\left(\frac{\sigma_x^2}{2\Xi} + \rho\sigma_x\right) \approx E(x) - \zeta\rho\sigma_x \qquad (1)$$

where $E(x)$ is the (objective) expected value under the added P&L distribution, σ_x is the corresponding standard deviation, ρ is the correlation between the P&L of the Market Portfolio and that of the contract, $\Xi = \sqrt{E[(X-E(X))^2]} >> \sigma_x$ is the standard deviation of the Benchmark Market Portfolio. The shift of VaR is usually a negative number, indicating a loss to the portfolio, $\zeta = -N^{-1}(0.01) \approx 2.326$ is a constant. Note, that the above-given formula can be easily generalized to account for a non-normal Market Portfolio. In this case the second term is simply proportional to the log-derivative of the P&L-distribution of the Market Portfolio.

In most markets, the market premiums charged for contracts are smaller than the change ofVaR, so additional risk capital has to be allocated and invested. If the actual premium is Pr, one has to allocate an additional amount $-\Delta\text{VaR} - \text{Pr}$, i.e. $\max(-\Delta\text{VaR},0)$ altogether to keep the VaR limit unchanged (clearly, if ΔVaR is positive there is no need to allocate capital at all; such a contract has infinite return within our assumptions, and seems to be very attractive). In what follows we assume that ΔVaR is negative. The formal expected continuous return on such investment μ is simply

$$e^{\mu T} = \frac{[-\Delta\text{VaR} - \text{Pr} + E(x)]e^{rT}}{-\Delta\text{VaR} - \text{Pr}} \qquad (2)$$

Indeed, the numerator is the future amount of allocated capital plus the expected profit on the deal, while the denominator is the allocated capital. Substituting the explicit expression for ΔVaR from (1), one finds the expected continuous return

$$\mu = r + \frac{1}{T}\ln\left[1 + \frac{\text{Pr} + E(x)}{-E(x) + \zeta\rho\sigma_x - \text{Pr}}\right] \qquad (3)$$

where r is the risk-free rate. In the market, where participants expect to earn a certain return on investment, the value of $\mu = \bar{\mu}$ is roughly fixed by the majority of participants. Solving Eq(3) for Pr gives

$$\text{Pr} = -E(x) + \zeta\rho\sigma_x[1 - e^{-(\bar{\mu}-r)T}] = -E(x) + \kappa\sigma_x, \qquad (4)$$

where, again, $\mu = \bar{\mu}$ is the market-*required* return on risk capital, and the constant κ helps to indicate that, effectively, we "charge by standard deviation", which is one of the traditional actuarial risk loading principles. The latter formula is surprisingly robust. Note that it is *not* consistent with any choice of a utility function on a stand-alone basis. Formally, it is consistent with the following utility "function" on the market level:

$$u(X + x) = x - \frac{\kappa}{\rho\Xi}(X - E(X))(x - E(x)). \qquad (5)$$

This function has to be integrated by using the joint probability distribution of the market and the given deal. However, a second glance at this latter formula reveals that we are dealing with a *functional*, which is not even reducible to a cumulant expansion.

In summary, the minimum premium earned must ensure that: (1) the new portfolio (the contract plus the original portfolio) will meet the risk management requirement, that is, the VaR is kept at the same threshold as the original portfolio; and (2) the return on capital requirement is met for the risk capital allocated to this new contract. The premium formula given above is far from being universal. It simply reflects some wide-spread capital allocation guidelines. It is worth noting here that at present a single universal formula can hardly exist for capital allocation, and attempts to formalize this issue usually spark a debate.

3. THE DATA GENERATING PROCESS IN A SIMULATED WORLD

This section establishes a simulated SPX world which closely replicates the empirical distributional properties of the SPX index during the period 1988-1997. The SPX index volatility has well documented time-series properties of mean-reversion, persistence, and the leverage effect. Mean-reversion means that after experiencing positive or negative shocks, the volatility itself will converge back to its long term average value. Persistence means that a large movement of stock volatility tends to be followed by a large movement, while small volatility changes tend to be followed by subsequent small volatility changes. The leverage effect means that negative moves of the underlying have a greater impact on volatility than positive moves of equal amount. We use the GJR-GARCH(1,1) model of Glosten, Jagannathan, and Runkle (GJR) (1993), modified by using an additional jump component to capture conservatively the large downside movement of a stock index, especially large market corrections and crashes. While the adopted model generates rather realistic time series, and is similar to other minimal models in its class, it is not necessarily unique. It would be of interest to test the expected stability of our results for a range of different models for the underlying index.

The diffusion process for the stock index process is assumed to be a standard discrete Geometric Brownian Motion with jumps,

$$S_t - S_{t-1} = S_{t-1}(m + \sigma_t + q_t) \qquad (6)$$

where m is the log-drift per time step, σ_t is the volatility of the process, which has zero mean and its variance h_t is given below. q_t is the jump component. The GJR-GARCH(1,1) volatility process is

$$h_t = \bar{\sigma}^2\left(1 - \alpha - \beta - \frac{\gamma}{2}\right) + (\alpha + \gamma D_{t-1})\sigma_{t-1}^2 + \beta h_{t-1}, \qquad (7)$$

where $\bar{\sigma}$ is the long-run average volatility per time step (volatility scales as the square root of number of the time steps). D_t reflects the shock momentum with

$$D_t = \begin{cases} 1, & \sigma_t \le 0 \\ 0, & \sigma_t > 0 \end{cases} \qquad (8)$$

A positive parameter γ reflects the "leverage effect". β is the measure of memory which decays as $(1-\beta)^n$ after n time steps. α is the amplitude of the stochastic component from the previous time step. During the period from September 1, 1988 to July 1, 1997, the key descriptive statistics for the daily SPX index return are

Annualized SPX mean return	13.86%
Annualized SPX return volatility	11.95%

Daily Probability of Jump	0.99%
Daily Average Jump Size	2.70%

The corresponding parameters of the GJR-GARCH(1,1) model for the S&P 500 index return (using the estimation techniques in Engle and Lee (1996)) are: $\alpha = 0.0332, \beta = 0.9122, \gamma = 0.0925$. The time step here is one business day, and all units are adjusted correspondingly.

Unfortunately, the GARCH-class models are not satisfactory in generating large price deviations. In view of this, it is important to add the jump component. We didn't modify the GJR parameters while adding the q_t term, which is drawn from a compound Poisson-beta distribution. As a result our model is somewhat more conservative than the market historical data. In the compound distribution smaller jumps are more frequent, and daily positive and negative jumps arrive at an average arrival rate of 0.99% (each as a separate Poisson process). The jump size is drawn from a beta distribution, with a mean of 2.7%, and a minimum of 2%. In particular, we allow a maximum loss of up to 25% in a day; thus, the 1987 stock market crash could have a positive probability in the model.

4. FROM HEDGING COST TO OPTION PRICES

From Distributions to Prices

In the simulated world generated above, assume a representative Dealer writes a one year at-the-money put option, then dynamically hedges her position on a daily basis by buying or selling a certain amount of stocks, as required by the Black-Scholes delta-hedge.

The table below describes the key parameters for a one year option,

Forward SPX	106.18%
SPX Spot	100.00%
SPX Strike	106.18%
Short-term Risk-free Rate (per annum)	6.00%
Selected Volatility for delta-hedging (per annum)	11.95%
Time to Maturity (yrs)	1
Proportional Transaction Costs	0.05%

Note that all SPX-related (dollar) values are measured in SPX spot units. For simplicity, the hedge volatility of 11.95% is an indicative single-number estimate for the range of volatilities observed during the day when hedging was performed. In the real world, Dealers may use the observed implied volatility from that day to compute the hedge-delta; however, theimplied volatility changes over time as the underlying index moves and derivatives are traded.

In an ideal Black-Scholes world, where the underlying index follows a Geometric Brownian Motion, and hedging can be performed continuously, the full accumulated hedging cost has a fixed value, which should be equal to the theoretical Black-Scholes option price. However, when the underlying spot process does not follow a log-normal distribution, and hedging is discrete, the hedging cost is a random variable depending on the realization of each pass in the simulated world. Usage of the sensitivity, however advanced, at best generates an over-hedge for many scenarios, but cannot fully cover the risk.

In this simulated world, with the presence of stochastic volatility, jumps, and discrete hedging, the final profit and loss (P&L) to the option Dealer is no longer certain. The Dealer has to develop a formal procedure which would allow her to relate the P&L distribution at maturity to a (dollar) value of the derivative security. The portfolio-based risk pricing can be applied to determine the price of the put contract along the lines described in Section 2.

As before, let σ_x denote the standard deviation of P&L associated with the position of writing the option and hedging, $E(x)$ refer to the expected cost, and ρ refer to the correlation of the "option + hedge" position with the Benchmark Market Portfolio. In actuarial modeling, ρ is used in the calculation of the risk loading. Its implicit value can be reverse-engineered from the market prices. For the purpose of comparison, we have listed *four* different versions of the price formula, as we have seen or heard of their existence in practice (Pr_1 is the one we derived in Section 2 above):

(1) Annual return on allocated capital is $\bar{\mu}$

$$\mathrm{Pr}_1 = -E(x) + \zeta\rho\sigma_x[1 - e^{-(\bar{\mu}-r)T}], \qquad (5')$$

(2) Return is infinite or no capital is allocated,

$$\mathrm{Pr}_2 = -E(x) + \zeta\rho\sigma_x \qquad (9)$$

(3) Maximum Observed Cost ($\mathrm{Pr}_3 = \mathrm{MOC}$) is used, which is the 100^{th} percentile of the hedging cost distribution.

(4) Additional 1% of the Maximum Observed Cost is charged on top of the value in (5')

$$\mathrm{Pr}_4 = -E(x) + \zeta\rho\sigma_x[1 - e^{-(\bar{\mu}-r)T}] + \frac{\mathrm{MOC}}{100}. \qquad (10)$$

One year at-the-money put option price

Figure 1 shows, in the simulated world, the cumulative probability distribution function of the costs of writing a one year put option and performing the Black-Scholes delta-hedge on a daily basis.

This strategy is insufficient for loss immunization. Moreover, the tails of the distribution of replication costs are very weakly sensitive to whether historical or implied volatility is used as is the standard deviation (since we are dealing with a parabolic minimum). The probability distribution of the hedging P&L has been studied analytically by using the technique similar to what can be found in Esipov and Vaysburd (1998, 1999). Evolution of this distribution obeys a certain integro-differential equation (not presented here). In the ideal Black-Scholes world this distribution becomes a delta-function centered at the Black-Scholes price.

As an example, we choose $\rho = 50\%$, and annual return on allocated capital $\bar{\mu} = 30\%$. The following prices were generated using the four pricing methods (1-4) at different strike levels.

Strike/Forward	Pr_1	Pr_2	Pr_3	Pr_4
0.75	0.0035	0.0096	0.1165	0.0046
1.00	0.0568	0.0690	0.1426	0.0580
1.25	0.2538	0.2588	0.3106	0.2569

Volatility "Bias" and "Smile"

In the derivatives markets, there are at least two stylized facts that puzzle the profession: one is that implied volatility is on average larger than realized volatility, the other is that implied volatility curve exhibits "smile" or "smirk" effects for out-of-the-money options with different strike levels. These empirical stylized facts can be reproduced by the model.

Implied Volatility vs. Realized Volatility

The empirical study on SPX daily returns during the period from September 1, 1988 to July 1, 1997 suggests a three-month moving average annualized volatility of 11.95%, with a maximum of 34.14% and a minimum of 3.74%; and volatility of volatility is around 39%. During the same period, the averaged annual implied volatility of the SPX one year put is 16.73%, which is 40% higher than the actual realized volatility.

If we use the Black-Scholes formula to reverse-engineer the so-called "implied" volatility from the at-the-money price suggested by Pr_1 above, we find that the "implied" volatility is 14.23%. This is much higher than the objective volatility of 11.95% that is used to generate the simulated SPX world (or 13.43% after adjustment of time varying volatility and jumps). Indeed, this price generating process can reproduce (qualitatively, at this level) the observed difference between implied volatility and realized volatility. One can further tailor the model by calibrating the risk loading parameter ρ to match the observed difference of SPX implied volatility and realized volatility.

Volatility Smile

For out-of-the-money SPX put options, the implied volatilities exhibit a "smile" or "smirk" effect for different Strike/Forward ratios. Conventionally, these volatilities are reverse-engineered out from the observed market prices using the Black-Scholes formula. Similarly, "implied" volatilities can be inferred from the option prices generated by the model. Among the four pricing methods described above, the first and fourth formulas provide the best fit of the observed implied volatility "facial expression" of SPX put options.

Strike/Forward	Vol from Pr_1	Vol from Pr_2	Vol from Pr_3	Vol from Pr_4
0.75	0.1829	0.2314	0.6434	0.1928
1.00	0.1423	0.1722	0.3582	0.1458
1.25	0.1436	0.1757	0.3479	0.1600

Figure 2 shows the implied volatility "smile" curve for the different pricing formulas. The method suggests that the origin of the smile is the relative growth of the risk premium for the away-from-money options.

5. VALUATION OF STRUCTURED DERIVATIVE PRODUCTS

In practice, this approach can be extended to value structured derivative products in the non-liquid, over-the counter market. Structured derivative products can be valued in two steps. First, calibrate the model towards the liquid options market (usually short term options). Second, use the calibrated model to price other non-liquid instruments, like long term out-of-the-money stocks, currency, and commodity options . Option prices observed from liquid option markets contain rich information about the market's expectation of the future distribution of the underlying asset, and the risk appetite towards profit and loss distributions. This information can be fully utilized by calibrating the model (for example, the risk loading parameter ρ , and the future objective volatility parameter) to make it exactly fit the implied volatility curves of the liquid options, in term of slope ("smile" or "smirk") and values. To get an exact fit, we can first calibrate ρ from recent historical implied and realized volatility, then use the "current" market implied volatility term structure to calibrate out the term structure of market expectation of future objective volatility (not implied volatility).

Calibration to liquid option market prices

On the day (November 17,1998) when the test described here was performed, the price for a one year at-the-money (Strike/Forward = 1) European-style put was 8.52%, or the annual market implied volatility was approximately 21.4%. For illustrative purposes, we keep the correlation parameter ρ unchanged at 50%. In the calibration, when we choose the objective volatility parameter to be 17%, the observed market price of the option can be reproduced by Pr_1. Thus, when the "future" objective volatility of the underlying SPX index is assumed to be 17.00%, the observed one year market implied volatility can be reproduced by the "calibrated" model. In practice, this exercise can be extended to calibrate out a term structure of objective volatility. In risk management, this objective volatility curve contains very useful information in predicting future volatility and conducting stress tests. After the model is "calibrated" successfully to capture the price generating process of the liquid options, these "calibrated" parameters are used to generate the price of long term options or exotic options in the non-liquid, over-the counter market.

Pricing a five year SPX put option

Option Price

Now let's price a five year SPX European-style put option that is mostly traded on the over-the-counter market. Using the flat term structure of objective volatility, we adopt the same parameters to generate a simulated SPX world for five years with the exception of volatility magnitude of the process. It is now assumed to be larger, 17%. Note that in a Black-Scholes hedged world, the expected return will not affect the value of the options.

Figure 3 shows, in the simulated world, the cumulative probability distribution function of the costs of writing a five year put option and performing the Black-Scholes delta-hedge on a daily basis. If we use Pr_1 then the suggested price for the put option is 17.69% (or "implied" volatility is 21.24%). For comparison, suppose the "market-implied" volatility curve is flat, that is, the five year "market-implied" volatility equals the one-year "market-implied" volatility of 21.4%, then the Black-Scholes price for a five year at-the-money European-style put is 17.82%. The following prices are generated by the different pricing methods at different Strike/Forward ratios.

Strike/Forward	Pr_1	Pr_2	Pr_3	Pr_4
0.75	0.0709	0.0805	0.3117	0.0741
1.00	0.1769	0.1886	0.4972	0.1819
1.25	0.3371	0.3508	0.8334	0.3454

Volatility Smile

Figure 4 shows the implied volatility "smirk" curves. It is important to note that the "smirk" is less friendly as compared to Figure 2. However, it is consistent with the shape observed in the option market for long term SPX put options. The model reproduces (qualitatively, at this level) the well-known property of the volatility term-structure: the "smile" weakens at longer maturities. In the model results, this is a direct consequence of the portfolio based or VaR-based risk premium loading. The following table summarizes the "implied volatilities" generated from different pricing methods for different Strike/Forward ratios.

Strike/Forward	Vol from Pr_1	Vol from Pr_2	Vol from Pr_3	Vol from Pr_4
0.75	0.2229	0.2374	0.5982	0.2279
1.00	0.2124	0.2261	0.6190	0.2182

1.25	0.2117	0.2272	0.8310	0.2212

6. CONCLUSION

Large financial institutions, which are permanently involved in selling and trading derivative securities, should pay more attention to estimating the profit and loss distributions associated with these contracts. While such a general statement may be regarded as self-obvious in actuarial communities, its implications for financial quantitative analysis are far-reaching. On the other side, the market price provides an ultimate constraint on the actuarial approach to premium. The message which emerges from this work is that one has to be prepared to deal with basis risk in all circumstances. We have shown that the profit and loss distributions may lead to consistent pricing which can potentially reproduce complex market effects such as volatility "smile" and its term structure. The merger of financial and actuarial worlds is taking place today, and its complete quantitative description has a long-term market value.

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Figure 1 . The Cost of Writing a One Year Put Option and Hedging X / F=1

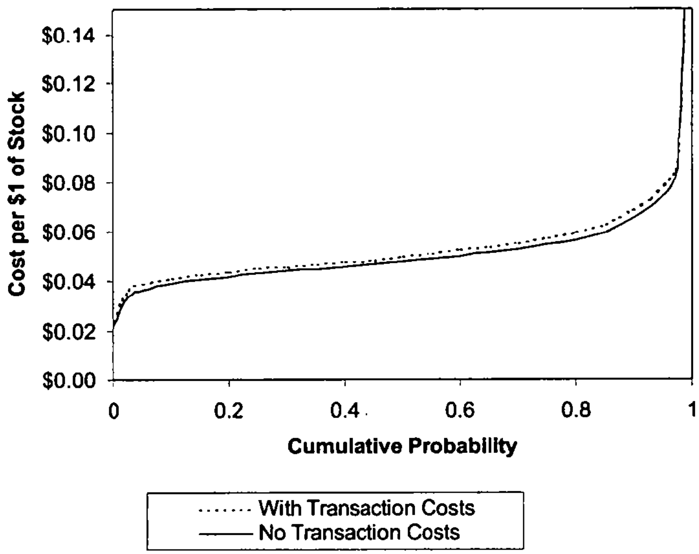

Figure 2. Implied Volatility Smile from Simulator, One Year Put

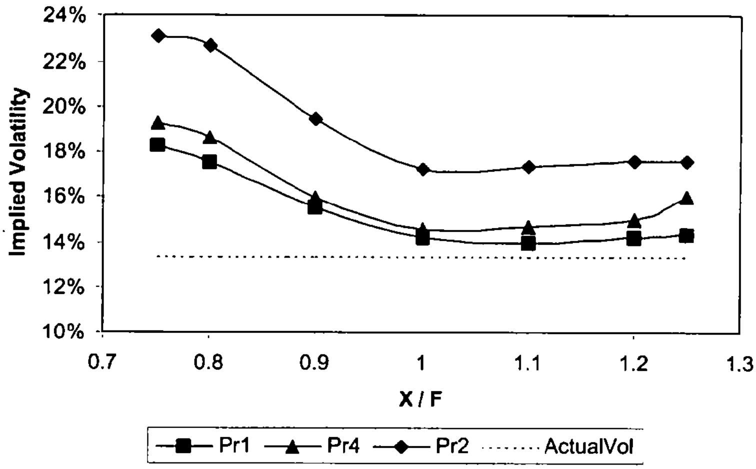

Figure 3 . The Cost of Writing a Five Year Put Option and Hedging It
X / F=1

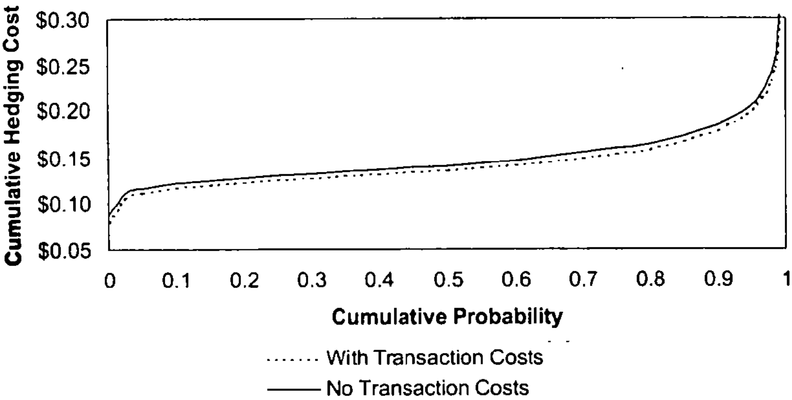

Figure 4, Implied Volatility "Smile", Five Year Put