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USING THE ODP BOOTSTRAP MODEL: A PRACTITIONER'S GUIDE

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CASUALTY ACTUARIAL SOCIETY

There are many papers that describe the over-dispersed Poisson (ODP) bootstrap model, but these papers are either limited to the basic calculations of the model or focus on the theoretical aspects of the model and always implicitly assume that the ODP bootstrap model is perfectly suited to the data being analyzed. In order to use the ODP bootstrap model on real data, the analyst must first test and review the assumptions of the model and may need to consider various modifications to the basic algorithm in order to put the ODP bootstrap model to practical use. This monograph starts by gathering the evolutionary changes from different papers into a complete ODP bootstrap modeling framework using a standard notation. Then it generalizes the basic model into a more flexible framework. Next it describes the adjustments or enhancements required for practical use and addresses the diagnostic testing of the model assumptions. While this monograph is focused on the ODP bootstrap model, we must recognize that it is a special subset of a larger framework of models and that there are a wide variety of other stochastic models that should also be considered. However, since no single model is perfect we also explore ways to combine or credibility weight the ODP bootstrap model results with various other models in order to arrive at a "best estimate" of the distribution, similar to how a deterministic best estimate is generally derived in practice. Finally, the monograph will also extend the model to illustrate the GLM Bootstrap and the model output to address other risk management issues and suggest areas for future research.

Keywords. Bootstrap, Over-Dispersed Poisson, Reserve Variability, Reserve Range, Distribution of Possible Outcomes, Generalized Linear Model, Best Estimate.

Availability of Excel workbooks. In lieu of technical appendices, several companion Excel workbooks are included that illustrate the calculations described in this monograph. The companion materials are summarized in the Supplementary Materials section and are available at https://www.casact.org/sites/default/files/2021-02/practitionerssuppl-shaplandmonograph04.zip. Other sources of ODP bootstrap modeling software that could be used for educational purposes would include working parties and other industry groups in North America and Europe, including but not limited to models freely available in the R statistical software package.

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Foreword

The concept of bootstrapping generally invokes the idea that once a process has been started, it can replicate without additional external input. Disciplines from biology and physics to business and statistics use bootstrapping to analyze numerous processes. For example, in statistics, bootstrapping involves starting with one sample and using it to derive many more subsamples drawn from the original sample. A specialized application within actuarial science involves derivation of a distribution of possible outcomes for each step in the loss development process.

Considerable literature has been developed over the past twenty-plus years regarding bootstrapping as it relates to actuarial science and the loss reserving process. In this work, Mr. Shapland collects the research from this vast literature base and frames it in one comprehensive presentation. The result is a complete over-dispersed Poisson (ODP) bootstrap model. At the same time, those who have worked with ODP bootstrapping know that these models have limitations when using real-world data. Mr. Shapland's work also proposes modifications and enhancements that allow more practical application of the ODP bootstrap model. In addition, he provides details on generalized linear models, of which the ODP bootstrap is one form.

With the knowledge that model risk is a real risk—no single model is perfect—Mr. Shapland further explores ways to combine the results of ODP bootstrapping with other types of models in an effort to determine a true "best estimate" of the distribution.

A set of illustrative Excel files, along with detailed instructions on how to use them, complements this monograph. With these files, the reader can follow through, step by step, the theory presented in monograph.

This monograph provides a one-stop shop for practical application of bootstrapping for the loss reserving process. The Monographs Editorial Board thanks the author for a valuable contribution to the casualty actuarial literature.

Leslie R. Marlo
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1. Introduction

The term "bootstrap" has a colorful history that dates back to German folk tales of the 18th century. It is aptly conveyed in the familiar cliché admonishing laggards to "pull oneself up by their own bootstraps." A physical paradox and virtual impossibility, the idea has nonetheless caught the imagination of scientists in a broad array of fields, including physics, biology and medical research, computer science, and statistics.

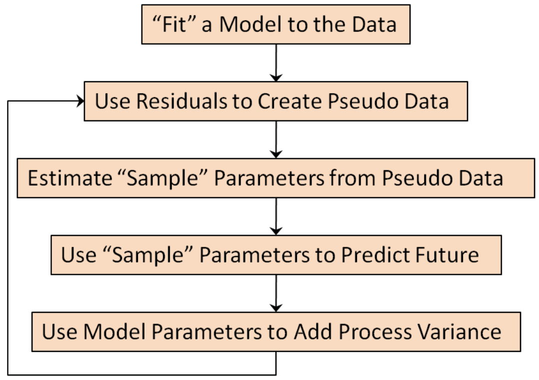
Bradley Efron (1979), Chairman of the Department of Statistics at Stanford University, is most often associated as the source of expanding bootstrapping into the realm of statistics, with his notion of taking one available sample and using it to arrive at many others through resampling.

In actuarial science, the concept of bootstrapping has become increasingly common in the process of loss reserving. The most commonly cited examples are England and Verrall (1999; 2002), Pinheiro, et al. (2003), and Kirschner, et al. (2008), who combine the bootstrap concept with a basic chain ladder model. These papers detail a form of the model where the incremental losses are modeled as over-dispersed Poisson random variables. In this monograph, it is called the over-dispersed Poisson bootstrap model, or the ODP bootstrap. The goal of the ODP bootstrap model is to generate a distribution of possible outcomes, rather than a point estimate, providing more information about the potential results.

At the present time, the vast majority of reserving actuaries in the U.S. are focused on deterministic point estimates. This is not surprising as the American Academy of Actuaries' primary standard of practice for reserving, ASOP 36, is focused on deterministic point estimates and the actuarial opinion required by regulators is also focused on deterministic estimates. However, actuaries are moving towards estimating an unpaid claim distribution, encouraged by the following factors:

- ASOP 43 defines "actuarial central estimate" in such a way that it could include either deterministic point estimates or a first moment estimate from a distribution;
- the SEC is looking for more reserving risk information in the 10-K reports filed by publicly traded companies;
- all of the major rating agencies have built or are building dynamic risk models to help with their insurance rating process and welcome the input of company actuaries regarding unpaid claim distributions;
- companies that use dynamic risk models to help their internal risk management processes need unpaid claim distributions;

Figure 1.1. Stochastic Model Diagram



- The Solvency II regime in Europe is moving many insurers towards unpaid claim distributions; and
- International Financial Accounting Standards, while still being discussed, shows actuaries that the future of insurance accounting may rely on unpaid claim distributions for booked reserves.

1.1. Objectives

One objective of this monograph is to provide more practical details on the Generalized Linear Model (GLM), of which the ODP bootstrap model[1] is a specific form. A GLM allows the user to "fit" the model to the data, as illustrated in Figure 1.1. The benefit of a GLM is that it can be specifically tailored to the statistical features found in the data under analysis. In contrast, consider algorithms that essentially force the data to be "fit" to a static method in order to predict the future as illustrated in Figure 1.2.[2]

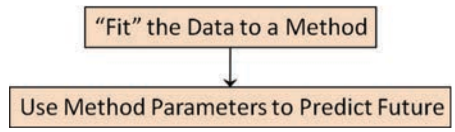
If a method does not use parameters or assumptions that fit the statistical features of the data then it may not project a reasonable point estimate. Similarly, if model assumptions and parameters do not fit the statistical features found in the data then the results of a simulation may not be a very good estimate of the distribution of possible outcomes. Thus, the modeling framework must be able to adapt to or "fit" the model to the data so this point will be elaborated on in later sections.

Another objective of this monograph is to show how the ODP bootstrap modeling framework can be used in practice, to help the wider adoption of unpaid claim distributions. Most of the papers describing stochastic models, including the ODP bootstrap model, tend to focus primarily on the theoretical aspects of the model while ignoring the data issues that commonly arise in practice. As a result the models can be quite elegantly implemented yet suffer from practical limitations such as only being useful

[1] Some authors define a model as having a defined structure and error distribution, so under this more restrictive definition bootstrapping would be considered to be a method or algorithm. However, using a less restrictive definition of a model as an algorithm that produces a distribution, bootstrapping would be defined as a model.

[2] For most deterministic reserving methods diagnostic tools can be used to test assumptions, adjust parameters and "fit" the method to the data, but not all assumptions can be adjusted and blindly applying a method is equivalent to a static method.

Figure 1.2. Static Method Diagram



for complete triangles or only for positive incremental values. Thus, while keeping as close to the theoretical foundation as possible, another objective is to illustrate how practical adjustments can be made to accommodate common data issues and allow the model to "fit" the data. As a practical matter, it is also possible that the model does not fit the data very well, or less well than other models, so the process of diagnosing the assumptions will inform the actuary's judgment when considering how much weight, if any, to give the model in relation to other models.

Another potential roadblock seems to be the notion that actuaries are still searching for the perfect model to describe "the" distribution of unpaid claims, as if imperfections in a model remove it from all consideration since it can't be "the one." This notion can also manifest itself when an actuary settles for a model that seems to work the best or is the easiest to use, or with the idea that each model must be used in its entirety or not at all. Interestingly, this notion was dispelled long ago with respect to deterministic point estimates as actuaries commonly use many different methods, which range from easy to complex, and judgmentally weight the results to arrive at their best estimate.

Model risk—the risk that the model you have chosen is not the same as the one that generates future losses—is very real and weighting or combining multiple estimates is a very practical way of addressing model risk. Thus, another objective of this monograph is to show how stochastic reserving can be similar to deterministic reserving when it comes to analyzing and using the best parts of multiple models by illustrating how the results from an ODP bootstrap model can be weighted together with other models. More importantly, the monograph hopes to illustrate the advantage of using a more complete set of risk estimation tools (which can include both stochastic models and deterministic methods) to arrive at an actuarial best estimate of the distribution of possible outcomes, rather than to focus on deterministic methods to select the "mean" and then simply "add on" a simple approximation or use only a favorite model to turn that selected mean into a distribution.

2. Notation

The papers that describe the basic ODP bootstrap model use different notation, despite sharing common steps. Rather than pick the notation in one of the papers, the notation from the CAS Working Party on Quantifying Variability in Reserve Estimates Summary Report (CAS Working Party 2005) will be used since it is intended to serve as a basis for further research.

Many models visualize loss data as a two-dimensional array, (w, d) with accident period or policy period w, and development age d (think $w =$ "when" and $d =$ "delay"). For this discussion, we assume that the loss information available is an "upper triangular" subset for rows $w = 1, 2, \ldots, n$ and for development ages $d = 1, 2, \ldots, n - w + 1$. The "diagonal" for which $w + d$ equals the constant, k, represents the loss information for each accident period w as of accounting period k.[3]

For purposes of including tail factors, the development beyond the observed data for periods $d = n + 1, n + 2, \ldots, u$, where u is the ultimate time period for which any claim activity occurs—i.e., u is the period in which all claims are final and paid in full, must also be considered.

The monograph uses the following notation for certain important loss statistics:

$c(w, d)$: cumulative loss from accident[4] year w as of age d.

$q(w, d)$: incremental loss for accident year w from $d - 1$ to d.

$c(w, n) = U(w)$: total loss from accident year w when claims are at ultimate values at time n,[5] or

$c(w, u) = U(w)$: total loss from accident year w when claims are at ultimate values at time u.

$R(w)$: future development after age d for accident year w, i.e., $= U(w) - c(w, d)$.

$f(d)$: factor applied to $c(w, d)$ to estimate $q(w, d + 1)$ or can be used more generally to indicate any factor relating to age d.

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[3] For a more complete explanation of this two-dimensional view of the loss information, see the *Foundations of Casualty Actuarial Science* (2001), Chapter 5, particularly pages 210–226.

[4] The use of accident year is used for ease of discussion. All of the discussion and formulas that follow could also apply to underwriting year, policy year, report year, etc. Similarly, year could also be half-year, quarter or month.

[5] This would imply that claims reach their ultimate value without any tail factor. This is generalized by changing n to $n + t = u$, where t is the number of periods in the tail.

$F(d)$: factor applied to $c(w, d)$ to estimate $c(w, d+1)$ or $c(w, n)$ or can be used more generally to indicate any cumulative factor relating to age d.

$G(w)$: factor relating to accident year w—capitalized to designate ultimate loss level.

$h(k)$: factor relating to the diagonal k along which $w + d$ is constant.[6]

$e(w, d)$: a random fluctuation, or error, which occurs at the w, d cell.

$E(x)$: the expectation of the random variable x.

$Var(x)$: the variance of the random variable x.

x^*: a randomly sampled value of the variable x.

What are called factors here could also be summands, but if factors and summands are both used, some other notation for the additive terms would be needed. The notation does not distinguish paid vs. incurred, but if this is necessary, capitalized subscripts P and I could be used.

[6] Some authors define $d = 0, 1, \ldots, n-1$ which intuitively allows $k = w$ along the diagonals, but in this case the triangle size is $n \times n-1$ which is not intuitive. With $d = 1, 2, \ldots, n$ defined as in this monograph, the triangle size $n \times n$ is intuitive, but then $k = w + 1$ along the diagonals is not as intuitive. A way to think about this which helps tie everything together is to assume the w variables are the beginning of the accident periods and the d variables are at the end of the development periods. Thus, if we are using years then cell $c(n, 1)$ represents accident year n evaluated at $12/31/n$, or essentially $1/1/n+1$.

3. The Bootstrap Model

Although many variations of a bootstrap model framework are possible, this monograph will focus on the most common example which reproduces the basic chain ladder method—the ODP bootstrap model. Let's briefly review the assumptions of the basic chain ladder method, because these assumptions are important in understanding the distribution created by the ODP bootstrap model.

Start with a triangle array of cumulative data:

		d					
		1	**2**	**3**	. . .	**n–1**	**n**
w	**1**	c(1, 1)	c(1, 2)	c(1, 3)	. . .	c(1, n–1)	c(1, n)
	2	c(2, 1)	c(2, 2)	c(2, 3)	. . .	c(2, n–1)	
	3	c(3, 1)	c(3, 2)	c(3, 3)	. . .		
	. . .	. . .	. . .				
	n–1	c(n–1, 1)	c(n–1, 2)				
	n	c(n, 1)					

A typical deterministic analysis of this data will start with an array of development ratios or development factors:

$$F(w,d)=\frac{c(w,d)}{c(w,d-1)}. \tag{3.1}$$

Then two key assumptions are made in order to make a projection of the known elements to their respective ultimate values. First, it is assumed that each accident year has the same development factor. Equivalently, for each $w = 1, 2, \ldots, n$:

$$F(w,d)=F(d).$$

Under this first assumption, one of the more popular estimators for the development factor is the weighted average:

$$\hat{F}(d)=\frac{\sum_{w=1}^{n-d+1}c(w,d)}{\sum_{w=1}^{n-d+1}c(w,d-1)}. \tag{3.2}$$

Certainly there are other popular estimators in use, but they are beyond our scope at this stage yet most are still consistent with our first assumption that each accident year has the same factor. Projections of the ultimate values, or $\hat{c}(w, n)$ for $w = 1, 2, \ldots, n$ are then computed using:

$$\hat{c}(w,n) = c(w,d)\prod_{i=d+1}^{n}\hat{F}(i), \text{ for all } d = n - w + 1. \tag{3.3}$$

This part of the claim projection algorithm relies explicitly on the second assumption, namely that each accident year has a parameter representing its relative level. These level parameters are the current cumulative values for each accident year, or $c(w, n - w + 1)$. Of course variations on this second assumption are also common, but the point is that every model has explicit assumptions that are an integral part of understanding the quality of that model.

One variation on the second assumption is to assume that the accident years are completely homogeneous.[7] In this case we would estimate the level parameter of the accident years using:

$$\frac{\sum_{w=1}^{n-d+1} c(w,d)}{n-d+1}. \tag{3.4}$$

Complete homogeneity implies that the observations $c(1, d)$, $c(2, d)$, . . . , $c(n - d + 1, d)$ are generated by the same mechanism. Thus, the column averages from (3.4) would replace the last actual values along the diagonal to calculate an estimate assuming homogeneity of accident years.

Interestingly, the basic chain ladder algorithm treats the processes generating the observations as NOT homogeneous[8] and effectively that "pooling" of the data does not provide any increased efficiency.[9] In contrast, it could be argued that the Bornhuetter-Ferguson (1972) and Cape Cod methods are a "blend" of these two extremes as the homogeneity of the future expected result depends on the consistency of the *a priori* loss ratios and decay rate, respectively.

3.1. Origins of Bootstrapping

Possibly the earliest development of a stochastic model for the actuarial array of cumulative development data is attributed to Kremer (1982) and the earliest discussion of bootstrapping is in Ashe (1986). The basic model used by Kremer is described by England and Verrall (1999) and Zehnwirth (1989), so there will be no further elaboration here. It should be noted, however, that this model can be extended by considering alternatives which are discussed in Barnett and Zehnwirth (2000) and Zehnwirth (1994), Renshaw (1989), Christofides (1990), and Verrall (1991; 2004), among others.

[7] Homogeneous data can have a different meaning in mathematics, but here we are defining it to mean having consistent or the same underlying exposures.

[8] Meaning the underlying exposures are changing over time and thus the current cumulative results (observation) for each year are more appropriate for projecting an estimate.

[9] For a more complete discussion of these assumptions of the basic chain ladder model see Zehnwirth (1989).

3.2. The Over-Dispersed Poisson Model

The genesis of this model into an ODP bootstrap framework originated with Renshaw and Verrall (1994) when they proposed modeling the incremental claims $q(w, d)$ directly as the response, with the same linear predictor as Kremer (1982), but using a generalized linear model (GLM) with a log-link function and an over-dispersed Poisson (ODP) error distribution.[10] Then, England and Verrall (1999) discuss how a specific form of this model is identical to the volume weighted chain ladder model, and use bootstrapping (sampling the residuals with replacement) to estimate a distribution of point estimates[11] instead of simulating from a multivariate normal distribution for a GLM. More formally, the following formulas are used to parameterize the GLM.

$$E\left[q(w,d)\right] = m_{w,d} \text{ and } Var\left[q(w,d)\right] = \phi E\left[q(w,d)\right] = \phi m_{w,d}^{z} \tag{3.5}$$

$$\ln\left[m_{w,d}\right] = \eta_{w,d} \tag{3.6}$$

$$\eta_{w,d} = c + \alpha_w + \beta_d, \text{ where: } w = 1, 2, \ldots, n;\ d = 1, 2, \ldots, n; \text{ and } \alpha_1 = \beta_1 = 0. \tag{3.7}$$

In this case the α parameters function as adjustments to the constant, c, level parameter and the β parameters adjust for the development trends after the first development period. The power, z, is used to specify the error distribution with:

$z = 0$ for Normal,
$z = 1$ for Poisson,
$z = 2$ for Gamma, and
$z = 3$ for inverse Gaussian.

Thus, the z parameter specifies not only the mean-variance relationship, but the whole shape of the distribution, including higher moments. Alternatively, we can remove the constant, c, which will cause the α parameters to function as individual level parameters while the β parameters continue to adjust for the development trends after the first development period:

$$\eta_{w,d} = \alpha_w + \beta_d, \text{ where: } w = 1, 2, \ldots, n; \text{ and } d = 2, 3, \ldots, n. \tag{3.8}$$

Standard statistical software can be used to estimate parameters and goodness of fit measures. The parameter ϕ is a scale parameter that is estimated as part of the fitting procedure while setting the variance proportional to the mean (thus "over-dispersed" Poisson for $z = 1$)[12]. For educational purposes, the calculations to solve these equations

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[10] Generalized Linear Modeling can be done with and without link functions and with a variety of error distributions. We are only describing here the particular GLM model that leads to the replication of the chain ladder results. For a more complete treatise on Generalized Linear Modeling, see McCullagh and Nelder (1989).

[11] Some authors refer to this as the standard deviation of the posterior distribution.

[12] While over-dispersed Poisson, or ODP, are commonly used terms for this model, it is certainly possible for the scale parameter to be less than one and thus "under-dispersed" Poisson would be more technically correct in that case. Alternatively, the more general term quasi-Poisson could be used. In addition, we note that the z parameter in equation 3.5, and some later formulas, could be removed for simplicity since the primary focus of this monograph is the ODP Bootstrap model, but it is included so we do not lose sight of the fact that the ODP Bootstrap model is a specialized case of a larger family of models.

for a 10 × 10 triangle are included in the "Bootstrap Models.xlsm" file, but here, and in the "GLM Framework.xlsm" file, the calculations are illustrated for a 3 × 3 triangle for ease of exposition. Consider the following incremental data triangle:

	1	2	3
1	$q(1, 1)$	$q(1, 2)$	$q(1, 3)$
2	$q(2, 1)$	$q(2, 2)$	
3	$q(3, 1)$		

In order to set up the GLM model to fit parameters to the data we need to do a log-link or transform which results in:

	1	2	3
1	$\ln[q(1, 1)]$	$\ln[q(1, 2)]$	$\ln[q(1, 3)]$
2	$\ln[q(2, 1)]$	$\ln[q(2, 2)]$	
3	$\ln[q(3, 1)]$		

The model, as described in (3.8), is then specified using a system of equations with vectors of α_w and β_d parameters as follows:

$$\begin{aligned}
\ln[q(1,1)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\beta_2 + 0\beta_3 \\
\ln[q(2,1)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 0\beta_2 + 0\beta_3 \\
\ln[q(3,1)] &= 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + 0\beta_2 + 0\beta_3 \\
\ln[q(1,2)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 + 0\beta_3 \\
\ln[q(2,2)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 1\beta_2 + 0\beta_3 \\
\ln[q(1,3)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 + 1\beta_3.
\end{aligned} \tag{3.9}$$

Converting this to matrix notation we have:

$$Y = X \times A \tag{3.10}$$

Where:

$$Y = \begin{bmatrix} \ln[q(1,1)] \\ \ln[q(2,1)] \\ \ln[q(3,1)] \\ \ln[q(1,2)] \\ \ln[q(2,2)] \\ \ln[q(1,3)] \end{bmatrix}, \tag{3.11}$$

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \text{ and} \tag{3.12}$$

$$A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_2 \\ \beta_3 \end{bmatrix}. \tag{3.13}$$

In this form we can use iteratively weighted least squares or maximum likelihood[13] to solve for the parameters in the A vector (3.13) that minimize the squared difference between the Y matrix (3.11) and the solution matrix (3.14):

$$\begin{bmatrix} \ln[m_{1,1}] \\ \ln[m_{2,1}] \\ \ln[m_{3,1}] \\ \ln[m_{1,2}] \\ \ln[m_{2,2}] \\ \ln[m_{1,3}] \end{bmatrix}. \tag{3.14}$$

After solving the system of equations we will have:

$$\begin{aligned} \ln[m_{1,1}] &= \eta_{1,1} = \alpha_1 \\ \ln[m_{2,1}] &= \eta_{2,1} = \alpha_2 \\ \ln[m_{3,1}] &= \eta_{3,1} = \alpha_3 \\ \ln[m_{1,2}] &= \eta_{1,2} = \alpha_1 + \beta_2 \\ \ln[m_{2,2}] &= \eta_{2,2} = \alpha_2 + \beta_2 \\ \ln[m_{1,3}] &= \eta_{1,3} = \alpha_1 + \beta_2 + \beta_3. \end{aligned} \tag{3.15}$$

This solution can then be shown as a triangle:

	1	2	3
1	$\ln[m_{1,1}]$	$\ln[m_{1,2}]$	$\ln[m_{1,3}]$
2	$\ln[m_{2,1}]$	$\ln[m_{2,2}]$	
3	$\ln[m_{3,1}]$		

[13] Other methods, such as orthogonal decomposition or Newton-Raphson, can also be used to solve for the parameters. Iteratively weighted least squares and maximum likelihood are both illustrated in the companion Excel files.

These results can then be exponentiated to produce the fitted, or expected, incremental results of the GLM model:

	1	2	3
1	$m_{1,1}$	$m_{1,2}$	$m_{1,3}$
2	$m_{2,1}$	$m_{2,2}$	
3	$m_{3,1}$		

This monograph will refer to this as the "GLM framework" and it is illustrated for a simple 3×3 triangle in the "GLM Framework.xlsm" file. While the GLM framework is used to solve these equations for the fitted results, the usefulness of this framework is that the fitted incremental values (with the Poisson error distribution assumption) will equal the incremental values that can be derived from volume-weighted average development factors, as shown in the "GLM Framework.xlsm" file.[14] That is, it can be reproduced by using the last cumulative diagonal, dividing backwards successively by each volume-weighted average development factor and subtracting to get the fitted incremental results. This monograph will refer to this method as the "simplified GLM" or "ODP Bootstrap." This has three very useful consequences.

First, the GLM portion of the algorithm can be replaced with a simpler development factor algorithm while still being based on the underlying GLM framework. Second, the use of the development factors serves as a "bridge" to the deterministic framework and allows the model to be more easily explainable to others. And, third, for the GLM algorithm the log-link process means that negative incremental values can often cause the algorithm to not have a solution, whereas using development factors will generally allow for a solution.[15]

With a model fitted to the data, the ODP bootstrap process involves sampling with replacement from the residuals. England and Verrall (1999) note that the deviance, Pearson, and Anscombe residuals could all be considered for this process, but the Pearson residuals are the most desirable since they are calculated consistently with the scale parameter. The unscaled Pearson residuals, $r_{w,d}$, and scale parameter, ϕ, are calculated as follows:

$$r_{w,d} = \frac{q(w,d) - m_{w,d}}{\sqrt{m_{w,d}^{z}}}. \tag{3.16}$$

$$\phi = \frac{\sum r_{w,d}^{2}}{N - p}. \tag{3.17}$$

[14] Using other than the Poisson assumption (i.e., $z \neq 1$), the incremental values may be close to the values from the development factors, but they will not be equal.

[15] More specifically, individual negative cell values may not be a problem (by using the negative of the log of the absolute value in 3.14). If the total of all incremental cell values in a development column is negative, then the GLM algorithm will fail. This situation will not cause a problem fitting the model as a link ratio less than one will be perfectly useful. However, this may still cause other problems, e.g., the "GLM framework" and "simplified GLM" may not be equivalent, which we will address in Section 4.

Where N = the number of observations, or incremental data cells in the triangle, which is typically equal to $n \times (n + 1) \div 2$, and p = the number of parameters, which is typically equal to $2 \times (n - 1)$.[16] Sampling with replacement from the residuals can then be used to create new sample triangles of incremental values using formula 3.18. Sampling with replacement assumes that the residuals are independent and identically distributed, but it does not require the residuals to be normally distributed. Indeed, this is often cited as an advantage of the ODP bootstrap model since whatever distributional form the residuals have will flow through to the simulation process. Some authors have referred to this as a "semi-parametric" bootstrap model since we are not parameterizing the residuals.

$$q^*(w,d) = r^* \times \sqrt{m_{w,d}^z} + m_{w,d}. \tag{3.18}$$

The sample triangle of incremental values can then be cumulated, new average development factors can be calculated for the sample and applied to calculate a point estimate for this data, resulting in a distribution of point estimates for some large number of samples. In England and Verrall (1999) this is the end of the process, but at the end of the appendix they note that you should also adjust the resulting distribution by the degrees of freedom adjustment factor (3.19) and the Scale Parameter (3.17), to effectively allow for over-dispersion of the residuals in the sampling process and add process variance to approximate a distribution of possible outcomes.

$$f^{DoF} = \sqrt{\frac{N}{N-p}}. \tag{3.19}$$

Later, in England and Verrall (2002), the authors note that the Pearson residuals (3.16) could be multiplied by the degrees of freedom adjustment factor (3.19) to include the over-dispersion in the residuals. As calculated in (3.20), these adjusted residuals are referred to as scaled Pearson residuals. They also expand the simulation process by adding process variance to the future incremental values from the point estimates. To add this process variance, they assume that each future incremental value $m_{w,d}$ is the mean and the mean times the scale parameter, $\phi m_{w,d}$, is the variance of a gamma distribution.[17] This revised model could now be described as estimating a distribution of possible outcomes, which incorporates process variance and parameter variance in the simulation of the historical and future data.[18]

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[16] The number of data cells could be less than $n \times (n + 1) \div 2$ and the number of parameters could be less than $2 \times (n - 1)$. For example, if the incremental values are zeros for the last three columns in a triangle then these cells would not be included in the total for N and there will be three fewer β parameters since none are needed to fit to these zero values as the development process is completed already.

[17] The Poisson distribution could be used to remain more consistent with the underlying theory of the GLM framework, but it is considerably slower to simulate, so gamma is a close substitute that performs much faster in simulation although it can be more skewed than the Poisson. Indeed, other distributions could be used as well to better approximate the observed "skewness" of the residuals from the diagnostics.

[18] Some authors refer to this as the full predictive distribution of the cash flows.

$$r^{S}_{w,d} = \frac{q(w,d) - m_{w,d}}{\sqrt{m^{z}_{w,d}}} \times f^{DoF}. \tag{3.20}$$

However, Pinheiro et al. (2001; 2003) noted that the bias correction for the residuals using the degrees of freedom adjustment factor (3.20) does not create standardized residuals, which is an important step for making sure that the residuals all have the same variance. In order to have standardized Pearson residuals, the GLM framework requires the use of a hat matrix adjustment factor (3.23).

$$H = X\left(X^{T}WX\right)^{-1}X^{T}W. \tag{3.21}$$

First, the hat matrix (3.21) is calculated using matrix multiplication of the design matrix (3.12) and the weight matrix (3.22).

$$W = \begin{bmatrix} m_{1,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{2,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{3,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{1,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{2,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{1,3} \end{bmatrix} \tag{3.22}$$

$$f^{H}_{w,d} = \sqrt{\frac{1}{1 - H_{i,i}}}. \tag{3.23}$$

The hat matrix adjustment factor (3.23) uses the diagonal of the hat matrix (3.21). In Pinheiro et al. (2003) the authors note two important points about the ODP bootstrap process as described by England and Verrall (1999; 2002). First, the sampling of the residuals should not include any zero-value residuals, which are typically in the corners of the triangle.[19] The exclusion of the zero-value residuals is accounted for in the hat matrix adjustment factor (3.23), but another common explanation is that the zero-value cells will have some variance but we just don't know what it is yet so we should sample from the remaining residuals but not the zeros. Second, the hat matrix adjustment factor (3.23) is a replacement for, and an improvement on, the degrees of freedom factor (3.19).[20]

Thus, the scaled Pearson residuals (3.20) should be replaced by the standardized Pearson residuals:

$$r^{H}_{w,d} = \frac{q(w,d) - m_{w,d}}{\sqrt{m^{z}_{w,d}}} \times f^{H}_{w,d}. \tag{3.24}$$

---

[19] Technically, the two "corner" residuals are zero because they each have a parameter that is unique to that incremental value which causes the fitted incremental value to exactly equal the actual incremental value.

[20] This second point was not addressed clearly in Pinheiro et al. (2001), but as the authors updated and clarified the monograph in Pinheiro et al. (2003) this issue was more clearly addressed.

However, the scale parameter (3.17) is still calculated as before, although the standardized Pearson residuals could be used to approximate the scale parameter as follows:

$$\phi^H = \frac{\sum \left(r_{w,d}^H\right)^2}{N}. \tag{3.25}$$

At this point we have a complete basic "ODP bootstrap" model, as it is often referred to. It is also important to note that the two key assumptions mentioned earlier, each accident year has the same development factor and each accident year has a parameter representing its relative level, are equally applicable to this model.

In order for the reader to test out the different "combinations" of this modeling process the "Bootstrap Models.xlsm" file includes options to allow these historical algorithms to be simulated. The purpose for describing this evolution of the ODP bootstrap model framework is threefold: first, to allow the interested reader to better understand the details of the algorithm and how these papers and their authors have contributed to the evolution of this model framework; second, to illustrate the value of collaborative research via different published papers and the contributions of different authors; and, third, to provide a solid foundation for continuing the evolutionary process and to discuss practical adjustments.

3.3. Variations on the ODP Model

When estimating insurance risk it is generally considered desirable to focus on the claim payment stream in order to measure the variability of the actual cash flows that directly affect the bottom line. Clearly, changes in case reserves and IBNR reserves will also impact the bottom line, but to a considerable extent the changes in IBNR are intended to counter the impact of the changes in case reserves. To some degree, then, the total reserve movements can act to mask the underlying changes due to cash flows. On the other hand, the case reserves contain valuable information about potential future payments so we should not ignore them and use only paid data.

3.3.1. Bootstrapping the Incurred Loss Triangle

The ODP bootstrap model can be used to model both paid and incurred loss data. Using incurred data incorporates case reserves, thus perhaps improving the ultimate estimates. However, the resulting distribution from using incurred data will be possible outcomes of the IBNR, not a distribution of the unpaid.[21] There are two possible approaches for modeling an unpaid loss distribution using incurred loss data: modeling incurred data and convert the ultimate values to a payment pattern, or, modeling paid and case reserves separately.

Using the first approach, a convenient way of converting the results of an incurred data model to a payment stream is to run the paid data model in parallel with the

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21 Using incurred data will also create issues in weighting the results of different models which will be discussed in Section 6.

incurred data model, and use the random payment pattern from each iteration from the paid data model to convert the ultimate values from each corresponding iteration from the incurred data to a payment pattern for each iteration (for each accident year individually). The "Bootstrap Models.xlsm" file illustrates this concept. It is worth noting, however, that this process allows the "added value" of using the case reserves to help predict the ultimate results to work its way into the calculations, thus perhaps improving the ultimate estimates, while still focusing on the payment stream for measuring risk. In effect, it allows a distribution of IBNR to become a distribution of IBNR and case reserves. This process could be made more sophisticated by correlating some part of the paid and incurred models (e.g., the residual sampling and/or process variance portions), but that is beyond the scope of this monograph.

The second approach could be accomplished by applying the ODP bootstrap to the Munich chain ladder model. This has the advantage over the first approach of not modeling the paid losses twice, and of explicitly measuring and imposing a framework around the correlation of the paid and outstanding losses. Since it is so well detailed in Liu and Verrall (2010), it will not be discussed in detail here.

3.3.2. Bootstrapping the Bornhuetter-Ferguson and Cape Cod models

Another common issue with using the ODP bootstrap model is that the distribution for the most recent accident years can produce results with more variance than you would expect when compared to earlier accident years. This is usually because more development factors are used to extrapolate the sampled values for the most recent accident years which, when coupled with random samples of incremental values, can result in more extreme fluctuations in point estimates. This is analogous to one of the weaknesses of the deterministic paid chain ladder method—a low, or high, initial observation can lead to an abnormally low, or high, projected ultimate, respectively.

To help alleviate this problem, the Bornhuetter-Ferguson (1972) or generalized Cape Cod (Struzzieri and Hussian 1998) deterministic methods can be worked into the underlying ODP bootstrap model, and the deterministic assumptions of these methods can also be converted to stochastic assumptions. For example, instead of using deterministic *a priori* loss ratios for the Bornhuetter-Ferguson model, the *a priori* loss ratios can be simulated from a distribution. Similarly, the Cape Cod algorithm can be applied to every ODP bootstrap model iteration to produce a stochastic Cape Cod projection that reflects the unique characteristics of each sample triangle.[22]

The "Bootstrap Models.xlsm" file also illustrates these Bornhuetter-Ferguson and Cape Cod ODP bootstrap models.[23]

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[22] In addition to being consistent between paid and incurred data, to the extent there is commonality with deterministic methods the assumptions should also be consistent. For example, it would not make sense to use one set of *a priori* loss ratio assumptions for a deterministic Bornhuetter-Ferguson method and a different set of mean assumptions for a modified ODP bootstrap model.

[23] More complex implementations of these models could include modifying the underlying assumptions of the GLM framework which would result in a completely different set of residuals, but that is beyond the scope of this monograph.

3.4. The GLM Bootstrap Model

Two limitations of the chain-ladder model, and hence the ODP bootstrap of the chain-ladder model, is that it does not measure or adjust for calendar-year effects, and it includes a significant number of parameters and many would argue that it over-fits the model to the data.

Another approach is to go back to the original GLM framework. Returning to formulas (3.5) to (3.8), the GLM framework does not require a certain number of parameters so we are free to specify only as many parameters as we need to get a robust model, which can address the over-fitting issue. Indeed, it is ONLY when we specify a parameter for EVERY accident year and EVERY development year and specify a Poisson error distribution that we end up exactly replicating the volume weighted average development factors that allow us to substitute the deterministic algorithm instead of solving the GLM fit.

Thus, using the original GLM framework, which this monograph will refer to as the "GLM Bootstrap" model, we can specify a model with only a few parameters, but there are two drawbacks to doing so.[24] First, the GLM must be solved for each iteration of the bootstrap model (which may slow down the simulation process) and, second, the model is no longer directly explainable to others using development factors.[25] While the impact of these drawbacks should be considered, the potential benefits of using the GLM bootstrap can be much greater.

First, having fewer parameters will help avoid over-parameterizing the model.[26] For example, if we use only one accident year parameter then the model specified using a system of equations is as follows (which is analogous to formula 3.9):

$$\begin{aligned}
\ln[q(1,1)] &= 1\alpha_1 + 0\beta_2 + 0\beta_3 \\
\ln[q(2,1)] &= 1\alpha_1 + 0\beta_2 + 0\beta_3 \\
\ln[q(3,1)] &= 1\alpha_1 + 0\beta_2 + 0\beta_3 \\
\ln[q(1,2)] &= 1\alpha_1 + 1\beta_2 + 0\beta_3 \\
\ln[q(2,2)] &= 1\alpha_1 + 1\beta_2 + 0\beta_3 \\
\ln[q(1,3)] &= 1\alpha_1 + 1\beta_2 + 1\beta_3
\end{aligned} \tag{3.26}$$

In this case we will only have one accident year parameter and $n - 1$ development trend parameters, but it will only be coincidence that we would end up with the equivalent of average development factors. Interestingly, this model parameterization moves us away from one of the common basic assumptions (i.e., each accident year has its own level) and substitutes the assumption that all accident years are homogeneous.

[24] Using the GLM framework allows for many other variations in the specification of models and then bootstrapping as described in more detail in England and Verrall (1999; 2002) and others, but this monograph will focus on variations consistent with the framework underpinning the ODP bootstrap model.

[25] However, age-to-age factors could be calculated for the fitted data to compare to the actual age-to-age factors and used as an aid in explaining the model to others.

[26] Over-parameterization will be addressed more completely in Section 5.

Another example of using fewer parameters would be to only use one development year parameter (while continuing to use an accident-year parameter for each year), which would equate to the system of equations in (3.27).

$$
\begin{aligned}
\ln[q(1,1)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\beta_2 \\
\ln[q(2,1)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 0\beta_2 \\
\ln[q(3,1)] &= 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + 0\beta_2 \\
\ln[q(1,2)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 \\
\ln[q(2,2)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 1\beta_2 \\
\ln[q(1,3)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 2\beta_2
\end{aligned} \tag{3.27}
$$

In this example the model parameterization moves away from the other common basic assumption (i.e., each accident year has its own level, but the same development parameter is used for all periods), and again it would be pure coincidence to end up with the equivalent of average development factors.[27] It is also interesting to note that for both of these two examples there will be one additional non-zero residual that can be used in the simulations because in each case one of the incremental values no longer has a unique parameter—i.e., for (3.26) $q(3, 1)$ is no longer uniquely defined by α_3, and for (3.27) $q(1, 3)$ is no longer uniquely defined by β_3.

Of course we can take this simplification to its logical extreme and use a model with only one accident year parameter and one development year parameter, which would result in the system of equations in as shown in (3.28).

$$
\begin{aligned}
\ln[q(1,1)] &= 1\alpha_1 + 0\beta_2 \\
\ln[q(2,1)] &= 1\alpha_1 + 0\beta_2 \\
\ln[q(3,1)] &= 1\alpha_1 + 0\beta_2 \\
\ln[q(1,2)] &= 1\alpha_1 + 1\beta_2 \\
\ln[q(2,2)] &= 1\alpha_1 + 1\beta_2 \\
\ln[q(1,3)] &= 1\alpha_1 + 2\beta_2
\end{aligned} \tag{3.28}
$$

In this example the model parameterization moves away from both of the common basic assumptions (i.e., each accident year has its own level, and the different development parameter is used for all periods), and again it would be pure coincidence to end up with the equivalent of average development factors. In this most "basic" model it is interesting to note that both of the "zero residuals" will now be non-zero and can be used in the simulations because both corners no longer have a unique parameter.

This flexibility allows the modeler to use enough parameters to capture the statistically relevant level and trend changes in the data without forcing a specific number of parameters.[28]

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[27] While we have only one parameter to describe the development period trends, if we convert these to development factors there will be a different factor for each period.

[28] How to determine which parameters are statistically relevant will be discussed in Section 5.

The second benefit, and depending on the data perhaps the most significant, is that this framework affords us the ability to add parameters for calendar-year trends. Adding diagonal, or calendar year trend, parameters to (3.8) we now have:

$$\eta_{w,d} = \alpha_w + \beta_d + \gamma_k, \text{ where: } w = 1, 2, \ldots, n;\ d = 2, 3, \ldots, n; \text{ and } k = 2, 3, \ldots, n. \quad (3.29)$$

A complete system of equations for the (3.29) framework would look like the following:

$$\begin{aligned}
\ln[q(1,1)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 0\beta_2 + 0\beta_3 + 0\gamma_2 + 0\gamma_3 \\
\ln[q(2,1)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 0\beta_2 + 0\beta_3 + 1\gamma_2 + 0\gamma_3 \\
\ln[q(3,1)] &= 0\alpha_1 + 0\alpha_2 + 1\alpha_3 + 0\beta_2 + 0\beta_3 + 1\gamma_2 + 1\gamma_3 \\
\ln[q(1,2)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 + 0\beta_3 + 1\gamma_2 + 0\gamma_3 \\
\ln[q(2,2)] &= 0\alpha_1 + 1\alpha_2 + 0\alpha_3 + 1\beta_2 + 0\beta_3 + 1\gamma_2 + 1\gamma_3 \\
\ln[q(1,3)] &= 1\alpha_1 + 0\alpha_2 + 0\alpha_3 + 1\beta_2 + 1\beta_3 + 1\gamma_2 + 1\gamma_3
\end{aligned} \quad (3.30)$$

However, there is no unique solution for a system with seven parameters and six equations, so some of these parameters will need to be removed. A logical starting point would be to start with a "basic" model with one accident year (level) parameter, one development trend parameter and one calendar trend parameter and then add or remove parameters as needed.[29] The system of equations for this basic model is as follows:

$$\begin{aligned}
\ln[q(1,1)] &= 1\alpha_1 + 0\beta_2 + 0\gamma_2 \\
\ln[q(2,1)] &= 1\alpha_1 + 0\beta_2 + 1\gamma_2 \\
\ln[q(3,1)] &= 1\alpha_1 + 0\beta_2 + 2\gamma_2 \\
\ln[q(1,2)] &= 1\alpha_1 + 1\beta_2 + 1\gamma_2 \\
\ln[q(2,2)] &= 1\alpha_1 + 1\beta_2 + 2\gamma_2 \\
\ln[q(1,3)] &= 1\alpha_1 + 2\beta_2 + 2\gamma_2
\end{aligned} \quad (3.31)$$

A third benefit of the GLM bootstrap model is that it can be used to model data shapes other than triangles. For example, missing incremental data for the first few diagonals would mean that the cumulative values could not be calculated and the remaining values in those first few rows would not be useful for the ODP bootstrap. However, since the GLM bootstrap uses the incremental values the entire trapezoid can be used to fit the model parameters.[30]

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[29] A simple algorithm to add and/or remove parameters in a search for the "optimal" set of parameters is included in the "Bootstrap Models.xlsm" file, but more complex algorithms are outside the scope of this monograph. We focus on the "mechanical" aspects of searching for the "optimal" set of parameters in Section 5 in order to enhance the educational benefits.

[30] This issue will be examined in more detail in Section 4.

It should also be noted that the GLM bootstrap model allows the future expected values to be directly estimated from the parameters of the model for each sample triangle in the bootstrap simulation process. However, we must solve the GLM within each iteration for the same parameters as we originally set up for the model rather than using development factors to project future expected values (which is a way of fitting the model to each sample triangle).

The additional modeling power that this flexible GLM bootstrap model adds to the actuary's toolkit cannot be overemphasized. Not only does it allow one to move away from the two basic assumptions of a deterministic chain ladder method, it allows for the ability to match the model parameters to the statistical features you find in the data, rather than "force" the data to fit the model, often with far fewer parameters and to extrapolate those features. For example, modeling with fewer development trend parameters means that the last parameter can be assumed to continue past the end of the triangle which will give the modeler a "tail" of the incremental values beyond the end of the triangle without the need for a specific tail factor.

While the monograph continues to illustrate the GLM bootstrap with a 3 × 3 triangle, also included in the companion Excel files are a set of "GLM Bootstrap 6___.xlsm" files that illustrate the calculations for these different models using a 6 × 6 triangle. Also, the "Bootstrap Models.xlsm" file contains a "GLM bootstrap" model for a 10 × 10 triangle that can be used to specify any combination of accident year, development year, and calendar year parameters, including setting parameters to zero. The GLM bootstrap model is akin to the incremental log model described in Barnett and Zehnwirth (2000), so we will leave it to the reader to explore this flexibility by using the Excel file.

4. Practical Issues

Now that the basic ODP bootstrap model has been expanded in a variety of ways, it is important to address some of the key assumptions of the ODP model and some common data issues.

4.1. Negative Incremental Values

As noted in Section 3.2, because of the log-link used in the GLM framework the incremental values must be greater than zero in order to parameterize a model. However, a slight modification to the log-link function will help this common problem become a little less restrictive. If we use (4.1) as the log-link function, then individual negative values are only an issue if the total of all incremental values in a development column is negative, as the GLM algorithm will not be able to find a solution in that case.

$$\begin{gathered} \ln\left[q(w,d)\right] \text{ for } q(w,d) > 0, \\ 0 \text{ for } q(w,d) = 0, \\ -\ln\left[abs\left\{q(w,d)\right\}\right] \text{ for } q(w,d) < 0. \end{gathered} \tag{4.1}$$

Using (4.1) in the GLM bootstrap will help in many situations, but it is quite common for entire development columns of incremental values to be negative, especially for incurred data. To give the GLM framework the ability to solve for a solution in this case we need to make another modification to the basic model to include a constant. Whenever a column or columns of incremental values sum to a negative value, we can find the largest negative[31] in the triangle, set ψ equal to the largest negative and adjust the log-link function by making all the incremental values positive.

$$\begin{gathered} q^{+}(w,d) = q(w,d) - \psi \\ \ln\left[q^{+}(w,d)\right] \text{ for all } q(w,d) \end{gathered} \tag{4.2}$$

Using the adjusted log-link function (4.2) we can solve the GLM using formulas (3.7), (3.8), or (3.27). Then we use (4.3) to adjust the fitted incremental values

[31] The largest negative value can either be the largest negative among the sums of development columns (in which case there may still be individual negative values in the adjusted triangle) or the largest negative incremental value in the triangle.

and the constant ψ is used to reduce each fitted incremental value by the largest negative.

$$m_{w,d} = \overset{+}{m}_{w,d} + \psi \tag{4.3}$$

The combination of formulas (4.2) and (4.3) allow the GLM bootstrap to handle all negative incremental values, which overcomes a common criticism of the ODP bootstrap. Incidentally, these formulas can also be used to allow the incremental log model described by Barnett and Zehnwirth (2000) to handle negative incremental values. As long as these formulas are used sparingly, the author believes that the resulting distribution will not be adversely affected.

When using the ODP bootstrap simulation process, the solution to negative incremental values needs to focus on the residuals and sampled incremental values since a development factor less than 1.00 will create negative incremental values in the fitted values. More specifically, we need to modify formulas (3.16) and (3.18) as follows: [32]

$$r_{w,d} = \frac{q(w,d) - m_{w,d}}{\sqrt{abs\left\{m^{z}_{w,d}\right\}}}. \tag{4.4}$$

$$q^{*}(w,d) = r^{*} \times \sqrt{abs\left\{m^{z}_{w,d}\right\}} + m_{w,d}. \tag{4.5}$$

While the fitted incremental values and residuals using the development factor simplification (ODP bootstrap) will generally not match the GLM framework solution using (4.1) or (4.2) and (4.3) they should be reasonably close. While the purists may object to these practical solutions, we must keep in mind that every model is an approximation of reality so our goal is to find reasonably close models that replicate the statistical features in the data rather than only restrict ourselves to "pure" models. After all, the assumptions of the "pure" models are themselves approximations.

4.1.1. Negative Values During Simulation

Even though we have solved problems with negative values when parameterizing a model, negative values can still affect the process variance in the simulation process. When each future incremental value (using $m_{w,d}$ as the mean and the mean times the scale parameter, $\phi m_{w,d}$, as the variance) is sampled from a gamma distribution to add process variance, the parameters of a gamma distribution must be positive. In this case we have two options for using the gamma distribution to simulate from a negative incremental value, $m_{w,d}$.

$$-Gamma\left[abs\left\{m_{w,d}\right\}, \phi abs\left\{m_{w,d}\right\}\right] \tag{4.6}$$

$$Gamma\left[abs\left\{m_{w,d}\right\}, \phi abs\left\{m_{w,d}\right\}\right] + 2m_{w,d} \tag{4.7}$$

[32] The use of other types of residuals, as noted in Section 3.2, may also help address the issue of negative incremental values, but their exposition is left to the interested reader.

Using formula (4.6) is more intuitive as we are using absolute values to simulate from a gamma distribution and then changing the sign of the result. However, since the gamma distribution is skewed to the right, the resulting distribution using (4.6) will be skewed to the left. Using formula (4.7) is a little less intuitive, but seems more logical since adding twice the mean, $m_{w,d}$, will result in a distribution with a mean of $m_{w,d}$ while keeping it skewed to the right (since $m_{w,d}$ is negative).

Negative incremental values can also cause extreme outcomes. This is most prevalent when resampled triangles are created with negative incremental losses in the first few development periods, causing one column of cumulative values to sum close to zero and the next column to sum to a much larger number and, consequentially, produce development factors that are extremely large. This can result in one or more extreme iterations in a simulation (for example, outcomes that are multiples of 1,000s of the central estimate). These extreme outcomes cannot be ignored, even if the high percentiles are not of interest, because they may significantly affect the mean of the distribution.

In these instances, you have several options. You can 1) remove these iterations from your simulation and replace them with new iterations, 2) recalibrate your model, or 3) limit incremental values to a minimum of zero (or some other minimum value).

The first option is to identify the extreme iterations and remove them from your results. Care must be taken that only truly unreasonable extreme iterations are removed, so that the resulting distribution does not understate the probability of extreme outcomes.

The second option is to recalibrate the model to fix this issue. First you must identify the source of the negative incremental losses. The most theoretically sound method to deal with negative incremental values is to consider the source of these losses. For example, it may be from the first row in your triangle, which was the first year the product was written, and therefore exhibit sparse data with negative incremental amounts. One option is to remove this row from the triangle if it is causing extreme results and does not improve the parameterization of the model. Or, if they are caused by reinsurance or salvage and subrogation, then you can model the losses gross of salvage and subrogation, model the salvage and subrogation separately, and combine the iterations assuming the values are correlated.

The third option is to constrain the model output by limiting incremental losses to a minimum of zero, where any negative incremental is replaced with a zero incremental.[33] For each of these options, keep in mind that this is a form of diagnosing a model by reviewing the simulated results and then searching for a practical solution before abandoning a model altogether. This does not mean that you should never abandon a model in favor of a practical adjustment. Indeed, the higher the frequency of the underlying issue (negative incremental values in this case) the more likely that the model does not really fit the data.

---

[33] While zero is a convenient minimum or lower bound, a small positive number could also be used, in which case any values less than the minimum are changed to the minimum.

4.2. Non-Zero Sum of Residuals

The standardized residuals that are calculated in the ODP bootstrap model are essentially error terms, and should in theory be independent and identically distributed with a mean of zero. However, the residuals are random observations of the true residual distribution, so the average of all the residuals is usually non-zero. If significantly different than zero, then the fit of the model should be questioned. If the average of the residuals is close to zero, then the question is whether they should be adjusted so that their average is zero. For example, if the average of the residuals is positive, then re-sampling from the residual pool will not only add variability to the resampled incremental losses, but may increase the resampled incremental losses such that the average of the resampled loss will be greater than the fitted loss.

It could be argued that the non-zero average of residuals is a characteristic of the data set, and therefore should not be removed. For example, standardized residuals implies a normal distribution with zero mean, but skewness in the residuals does not necessarily imply an average of zero. However, if a zero residual average is desired, then one option is the addition of a single constant to all non-zero residuals, such that the sum of the shifted residuals is zero.

4.3. Using an *N*-Year Weighted Average

It is quite common for actuaries to use weighted averages that are less than all years in their chain-ladder and related methods. Similarly, both the ODP bootstrap and the GLM bootstrap can be adjusted to only consider the data in the most recent diagonals. For the GLM framework (and the GLM bootstrap model), we can use only the most recent $L + 1$ diagonals (since an L-year average uses $L + 1$ diagonals) to parameterize the model. The shape of the data to be modeled essentially becomes a trapezoid instead of a triangle, the excluded diagonals are given zero weight in the model and we have fewer calendar year trend parameters if we are using formula (3.29). When running the GLM bootstrap simulations we will only need to sample residuals for the trapezoid that was used to parameterize the model as that is all that will be needed to estimate parameters for each iteration.

For the ODP bootstrap model, we can calculate L-year average factors instead of all-year factors and only have residuals for the most recent $L + 1$ diagonals. However, when running the ODP bootstrap simulations we would still need to create a whole resampled triangle so that we can calculate cumulative values.[34] But, for consistency, we would want to use L-year average factors for projecting the future expected values from these resampled triangles.

The calculations for the GLM bootstrap are illustrated in the companion "GLM Bootstrap 6 with 3yr avg.xlsm" file. Note that because the GLM bootstrap estimates parameters for the incremental data, the fitted values will no longer match the fitted values from the ODP bootstrap using volume-weighted average development factors.

[34] The fitted values for the "unused" diagonals would be calculated using the L-year average ratios, but the corresponding residuals for those diagonals are all excluded from the sampling process.

Depending on the data, the fitted values from the simplified GLM (ODP bootstrap) may or may not be a reasonable approximation to the GLM framework (GLM bootstrap).

Note that this discussion of using L-year average factors assumes volume weighted averages to be consistent with the GLM framework. This also assumes that all of the diagnostic tests will be adjusted to reflect the use of the last $L + 1$ diagonals, although this is beyond the scope of the monograph. Finally, other types of averages could be used (i.e., straight average, average excluding high & low, etc.) to be more consistent with what actuaries might use in a deterministic analysis, but these typically move further away from the GLM framework and are beyond the scope of this monograph.

4.4. Missing Values

Sometimes the loss triangle will have missing values. For example, values may be missing from the middle of the triangle, or a triangle may be missing the oldest diagonals, if loss data was not kept in the early years of the book of business.

If values are missing, then the following calculations will be affected:

- Loss development factors
- Fitted triangle—if the missing value lies on the most recent diagonal
- Residuals
- Degrees of freedom

There are several solutions. The missing value may be estimated using the surrounding values. Or, the loss development factors can be modified to exclude the missing values, and there will not be a corresponding residual for those missing values. Subsequently, when triangles are resampled, the simulated incremental corresponding to the missing value should still be resampled so that the cumulative values in those rows can be calculated, but they would still be excluded from the projection process (i.e., not included with the sample age-to-age factors) to reproduce the uncertainty in the original dataset.

If the missing value lies on the most recent diagonal, the fitted triangle cannot be calculated in the usual way. A solution is to estimate the value, or use the value in the second most recent diagonal to construct the fitted triangle. These are not strictly mathematically correct solutions, and judgment will be needed as to their effect on the resulting distribution. Of course for the GLM bootstrap model, the missing data only reduces the number of observations used in the model.

4.5. Outliers

There may be a few extreme or incorrect values in the original triangle dataset that could be considered outliers. These may not be representative of the variability of the dataset in the future and, if so, the modeler may want to remove their impact from the model.

There are several solutions. These values could be removed, and dealt with in the same manner as missing values. Another alternative is to identify outliers and exclude them from the average development factors (either the numerator, denominator, or both) and residual calculations, as when dealing with missing values, but re-sample the corresponding incremental when re-sampling triangles. In this case we have removed the extreme impact of the incremental cell, but we still want to include a non-extreme variability, which is different from a missing data cell since in that case the additional uncertainty of that missing data can be included by continuing to exclude that cell in the projection process.

The calculations for the GLM bootstrap are illustrated in the companion "GLM Bootstrap 6 with Outlier.xlsm" file. Again the GLM bootstrap fitted values will no longer exactly match the fitted values from the ODP Bootstrap using volume weighted average development factors, but they should normally be close.

If there are a significant number of outliers, then this could be an indication that the model is not a good fit to the data. With the GLM bootstrap, new parameters could be chosen, or the distribution of the error term can be changed (i.e., change the z parameter). Under the ODP bootstrap model, an L-year weighted average could be used, instead of an all year weighted average, which may provide a better fit to the data, or, heteroscedasticity may exist. Remember, though, that for the ODP bootstrap model there is no distribution assumption for the residuals so a significant number of residual outliers could just mean that the residuals are quite skewed. One of the nice features of the ODP bootstrap is that the skewness in the residuals will be reflected in the simulation process which will result in a skewed distribution of possible outcomes.[35] Thus, removing any outliers (i.e., giving them zero weight) should be done with caution and would most commonly be done only after understanding the underlying data.

4.6. Heteroscedasticity

As noted earlier, the ODP bootstrap model is based on the assumption that the standardized Pearson residuals are independent and identically distributed. It is this assumption that allows the model to take a residual from one development period/accident period and apply it to the fitted loss in any other development period/accident period, to produce the sampled values. In statistical terms this is referred to as homoscedasticity (the residuals have the same variance) and it is important that this assumption is validated.

A common problem is when some development periods have residuals that appear to be more variable than others—i.e., they appear to have different variances. This is referred to as heteroscedasticity. With heteroscedasticity, it is no longer possible to take a residual from one development/accident period and deem it suitable to be applied to any other development/accident period. In making this assessment, you must account for the credibility of the observed differences in variance, and also to note that there are fewer residuals as the development years become older, so comparing development years is difficult, particularly near the tail-end of the triangle.[36]

[35] Other methods of handling outliers could also be introduced, e.g., tempering residuals that are further away from the interquartile range, but the key to any approach is to understand what the residuals represent so an explicit assumption can be made and the "best" solution can be used.

[36] Section 5 will illustrate how to use residual graphs and other statistical tests to evaluate heteroscedasticity.

The existence of heteroscedasticity may suggest that the model is not a good fit for the data. Under an ODP bootstrap, there are a number of ad-hoc adjustments that can be made to address heteroscedasticity, but they may or may not improve the fit of the model to the data. They also often result in even more parameters in a model which could already be over-parameterized. In contrast, under a GLM bootstrap the flexibility of choosing the number of parameters to use, the ability to account for any calendar year trends, and the flexibility to choose the distribution of the error term mean that there are many ways within the model framework itself to improve the fit to the data. Therefore, this flexibility could remove the heteroscedasticity problem or at least reduce it.

Nevertheless, if the ODP bootstrap model is still to be used, then to adjust for heteroscedasticity in your data there are at least three options, 1) stratified sampling, 2) calculating hetero-adjustment (or variance) parameters, or 3) calculate non-constant scale parameters. Stratified sampling is accomplished by grouping those development periods with homogeneous variances and then sampling only from the residuals in each group. While this process is straightforward, some groups may only have a few residuals in them, which limits the amount of variability in the possible outcomes compared to the other two options and at least partially defeats the benefits of random sampling with replacement.

The second option is to group those development periods with homogeneous variances and calculate the standard deviation of the residuals in each of the groups. Then calculate h_i, which is the "hetero-adjustment" factor, for each group, i:

$$h_i = \frac{stdev\left(\bigcup_1^j r_{w,d}^H\right)}{stdev\left(\bigcup_i r_{w,d}^H\right)}, \text{ for each } 1 \le i \le j \tag{4.8}$$

All residuals in group i are multiplied by h_i.

$$r_{w,d}^{iH} = \frac{q(w,d) - m_{w,d}}{\sqrt{m_{w,d}}} \times f_{w,d}^H \times h^i \tag{4.9}$$

Now all groups have the same standard deviation and we can sample with replacement from among all $r_{w,d}^{iH}$. The original distribution of residuals has been altered, but this can be remedied. When the adjusted residuals are resampled, the residual is divided by the hetero-adjustment factor, h_i, that applies to the development year of the incremental loss, as shown in (4.10).

$$q^{i^*}(w,d) = \frac{r^*}{h^i} \times \sqrt{m_{w,d}} + m_{w,d}. \tag{4.10}$$

By doing this, the heteroscedastic variances we observed in the data are replicated when the sample triangles are created, but we are able to freely resample with replacement from the entire pool of heteroscedasticity adjusted residuals. Also note that these factors are new parameters so it will affect the degrees of freedom, which impacts the scale parameter (3.17) and the degrees of freedom adjustment factor (3.19).[37] Finally, the hetero-adjustment factors should also be used to adjust the variance by development period when simulating the future process variance.

The third option is to modify the formula for the scale parameter (3.17) so that we have a different scale parameter for each hetero group, as illustrated in (4.11) and (4.12).[38] In (4.12) n_i is the number of residuals in each hetero group.

$$\phi = \frac{\sum r_{w,d}^2}{N-p} = \frac{N}{N-p} \times \frac{\sum r_{w,d}^2}{N} = \frac{\sum\left(\sqrt{\frac{N}{N-p}} \times r_{w,d}\right)^2}{N} \tag{4.11}$$

$$\phi_i = \frac{\sum_{i=1}^{n_i}\left(\sqrt{\frac{N}{N-p}} \times r_{w,d}\right)^2}{n_i} \tag{4.12}$$

For this option, the different scale parameters also amount to new parameters so the degrees of freedom adjustment factor would likewise be impacted. In this case, the scale parameters adjust the future process variance, but we also need to calculate parameters to adjust the residuals as shown in (4.13). These hetero-adjustment factors, h_i, can also be used to adjust the residuals in (4.9) and used in calculating the resampled loss in (4.10), similar to the second option.

$$h_i = \frac{\sqrt{\phi}}{\sqrt{\phi_i}} \tag{4.13}$$

While the hetero-adjustment factors in (4.13) are a bit more theoretically sound, in practice the factors in (4.8) are likely to be very close so the differences are not likely to have much impact. Both of these options are illustrated in the "Bootstrap Models.xlsm" file.

Of course no matter which formula is used, care needs to be exercised as hetero groups are used toward the tail of the triangle where fewer and fewer observations stretch the credibility of the resulting factors.[39] Finally, while use of the GLM bootstrap should reduce the need for hetero factors, the same three options could also be used for that model too.

4.7. Heteroecthesious Data

The basic ODP bootstrap model requires both a symmetrical shape (e.g., annual by annual, quarterly by quarterly, etc. triangles) and homoecthesious data (i.e., similar

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[37] Some authors have suggested adding a factor for each development period to insure homoscedasticity. However, this adds many more parameters to a model that can already suffer from the criticism of over-parameterization. Thus, a balance between the need for hetero parameters and parsimony is appropriate. This will be discussed in more detail in Section 5.

[38] For a more detailed development of this third option see England and Verrall (2006). In particular, see Appendix A.1 on pages 266–268.

[39] In the discussion of diagnostics in Section 5 it will be noted that the use of the AIC and BIC statistics will effectively reflect the credibility of the development periods.

exposures).[40] As discussed above, using an *L*-year weighted average in the ODP bootstrap model or adjusting to a trapezoid shape allow us to "relax" the requirement of a symmetrical shape. Other non-symmetrical shapes (e.g., annual × quarterly data) can also be modeled with either the ODP bootstrap or GLM bootstrap, but they will not be discussed in detail in this monograph.

Most often, the actuary will encounter heteroecthesious data (i.e., incomplete or uneven exposures) at interim evaluation dates, with the two most common data triangles being either a partial first development period or a partial last calendar period. For example, with annual data evaluated as of June 30, partial first development period data would have all development periods ending at 6, 18, 30, etc. months, while partial last calendar period data would have development periods as of 12, 24, 36, etc. months for all of the data in the triangle except the last diagonal, which would have development periods as of 6, 18, 30, etc. months. In either case, not all of the data in the triangle has full annual exposures—i.e., it is heteroecthesious data.

4.7.1. Partial First Development Period Data

For partial first development period data, the first development column has a different exposure period than the rest of the columns (e.g., in the earlier example the first column has six months of development exposure while the rest have 12). In a deterministic analysis this is not a problem as the development factors will reflect the change in exposure. For parameterizing an ODP bootstrap model, it also turns out to be a moot issue, since the Pearson residuals use the square root of the fitted value to make them all "exposure independent."

The only adjustment for this type of heteroecthesious data is the projection of future incremental values. In a deterministic analysis, the most recent accident year needs to be adjusted to remove exposures beyond the evaluation date. For example, continuing the previous example the development periods at 18 months and later are all for an entire year of exposure whereas the six month column is only for six months of exposure. Thus, the 6–18 month development factor will effectively extrapolate the first six months of exposure in the latest accident year to a full accident year's exposure. Accordingly, it is common practice to reduce the projected future payments by half to remove the exposure from June 30 to December 31.[41]

The simulation process for the ODP bootstrap model can be adjusted similarly to the way a deterministic analysis would be adjusted. After the development factors from each sample triangle are used to project the future incremental values the last accident year's values can be reduced (in the previous example by 50%) to remove the future exposure and then process variance can be simulated as before. Alternatively, the future incremental values can be reduced after the process variance step.

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[40] To the author's knowledge, the terms *homoecthesious* and *heteroecthesious* are new. They are a combination of the Greek *homos* (or όμός) meaning the same or *hetero* (or έτερο) meaning different and the Greek *ekthesë* (or έκθεση) meaning exposure.

[41] Reduction by half is actually an approximation since we would also want to account for the differences in development between the first and second half years.

4.7.2. Partial Last Calendar Period Data

For partial last calendar period data, most of the data in the triangle has annual exposures and annual development periods, except for the last diagonal which, continuing our example, only has a 6-month development period. For a deterministic analysis, it is common to exclude the last diagonal when calculating average development factors, then interpolate those factors to project the future values. Similarly to the adjustments for partial first development period data, we can adjust the calculations and steps in the ODP bootstrap model. Instead of ignoring the last diagonal during the parameterization of the model, an alternative is to adjust or annualize the exposures in the last diagonal to make them consistent with the rest of the triangle. The fitted triangle can be calculated from this annualized triangle to obtain residuals.

During the ODP bootstrap simulation process, development factors can be calculated from the fully annualized sample triangles and interpolated. Then, the last diagonal from the sample triangle can be adjusted to de-annualize the incremental values in the last diagonal—i.e., reversing the annualization of the original last diagonal. The new cumulative values can be multiplied by the interpolated development factors to project future values. Again, the future incremental values for the last accident year must be reduced (in the previous example by 50%) to remove the future exposure.[42]

4.8. Exposure Adjustment

Another common issue in real data is exposures that have changed dramatically over the years. For example, in a line of business that has experienced rapid growth or is being run off. If the earned exposures exist for this data, then a useful option for the ODP bootstrap model is to divide all of the claim data by the exposures for each accident year—i.e., effectively using pure premium development instead of total loss development. This may improve the fit of the model to the data.

During the ODP bootstrap simulation process, all of the calculations would be done using the exposure-adjusted data and only after the process variance step has been completed would you multiply the results by the exposures by year to restate them in terms of total values again.

When adjusting the GLM bootstrap for exposure, the model is fitted to exposure adjusted losses, similar to the ODP bootstrap model using exposure. However, under the GLM, the fit to the exposure adjusted losses are also exposure-weighted. That is, exposure adjusted losses with higher exposure are assumed to have lower variance. For more details, see Anderson et al. (2007).

For the GLM bootstrap, exposure adjustment could allow fewer accident year parameter(s) to be used.

4.9. Tail Factors

One of the most common data issues is that claim development is not complete within the loss triangle and tail factors are commonly used to extrapolate beyond the end of

[42] These heteroecthesious data issues are not illustrated in the "Bootstrap Models.xlsm" file.

the data triangle. There are many common methods for calculating tail factors and a useful reference in this regard is the CAS Tail Factor Working Party Report (2013). Tail factors can be added to the ODP bootstrap algorithm and converted from deterministic to stochastic by assuming that the tail factor parameter follows a distribution. Once this is added, other considerations such as process variance, hetero-adjustment factors, etc. can all be extended to include the tail factors.

A key ingredient for all of these considerations is to verify that the simulations in the tail are reasonable. For example, the tail factor itself represents the accumulation of incremental factors (i.e., an age-to-ultimate factor) and using just a single factor may not produce appropriate incremental results so the "extrapolation" of "incremental tail factors" may be more appropriate. In the "Bootstrap Models.xlsm" file, the tail factors can be extrapolated for up to 5 years so that one possibility for how these concepts can be implemented is included in the companion files.

A rough rule of thumb for the tail factor standard deviation is 50% or less of the tail factor minus one (assuming the tail factor is greater than one). However, this should be compared to the standard deviations of the age-to-age factors leading up to the tail in both the actual data triangle and in the simulated results.

As noted at the end of Section 3.4, for the GLM bootstrap model the last development parameter can continue to apply past the end of the data triangle until the trend results in no further claim activity, thus indirectly creating a tail factor. In addition to the last development parameter, the last calendar period parameter would also extend past the end of the tail until the combination of the two trends resulted in no further claim activity.

4.10. Fitting a Distribution to ODP Bootstrap Residuals

Because the number of data points used to parameterize the ODP bootstrap model are limited (in the case of a 10×10 triangle to 55 data points or 53 residuals), it is hard to determine whether the most extreme observation is a one-in-100 or a one-in-1,000 event (or simply, in this example, a one-in-53 event). Of course, the nature of the extreme observations in the data will also affect the level of extreme simulations in the results. Judgment is involved here, but the modeler will either need to be satisfied with the level of extreme simulations in the results or modify the ODP bootstrap algorithm.

One way to overcome a lack of extreme residuals for the ODP bootstrap model would be to fit a distribution to the residuals and sample from the distribution instead of from the residuals themselves (e.g., use a normal distribution if the residuals are found to be normally distributed). This option is beyond the scope of the companion Excel files, but this could be referred to as parametric bootstrapping of the ODP bootstrap model. Note however, that as there are a wide variety of other types of models that can be bootstrapped, either with or without residuals, parametric bootstrapping can be done in other ways.

5. Diagnostics

The quality of any model depends on the quality of the underlying assumptions. When a model fails to "fit" the data, it cannot produce a good estimate of the distribution of possible outcomes.[43] However, a balance must be considered for parsimony of parameters and the goodness-of-fit. Over-parameterization may cause the model to be less predictive of future losses. On the other hand, no model will perfectly "fit" the data, so the best you can hope for with any model is that it reasonably represents the data and your understanding of the processes that impact the data. Therefore, diagnostically evaluating the assumptions underlying a model is important for evaluating whether it will produce reasonable results or not and whether it should stay in your selected group of reasonable models which could receive some weight.

The CAS Working Party, in the third section of their report on quantifying variability in reserve estimates (2005), identified 20 criteria or diagnostic tools for gauging the quality of a stochastic model. The Working Party also noted that, in trying to determine the optimal fit of a model, or indeed an optimal model, no single diagnostic tool or group of tools can be considered definitive. Depending on the statistical features found in the data, a variety of diagnostic tools are necessary to best judge the quality of the model assumptions and to adjust the parameters of the model. This monograph will discuss some of these tools in detail as they relate to the ODP bootstrap and the GLM bootstrap models.

The key diagnostic tests are designed for three purposes: to test various assumptions in the model, to gauge the quality of the model fit to the data, and/or to help guide the adjustment of model parameters. Some tests are relative in nature, enabling results from one set of model parameters to be compared to those of another, for a specific model, allowing a modeler to improve the fit of the model. For the most part, however, the tests can't be used to compare different models. The objective, consistent with the goals of a deterministic analysis, is ***not*** to find the one best model, but rather a set of reasonable models.

Some diagnostic measures include statistical tests, providing a pass/fail determination for some aspects of the model assumptions. This can be useful even though a "fail" does not necessarily invalidate an entire model; it only points to areas where improvements can be made to the model or its parameterization. The goal is to find the sets of models

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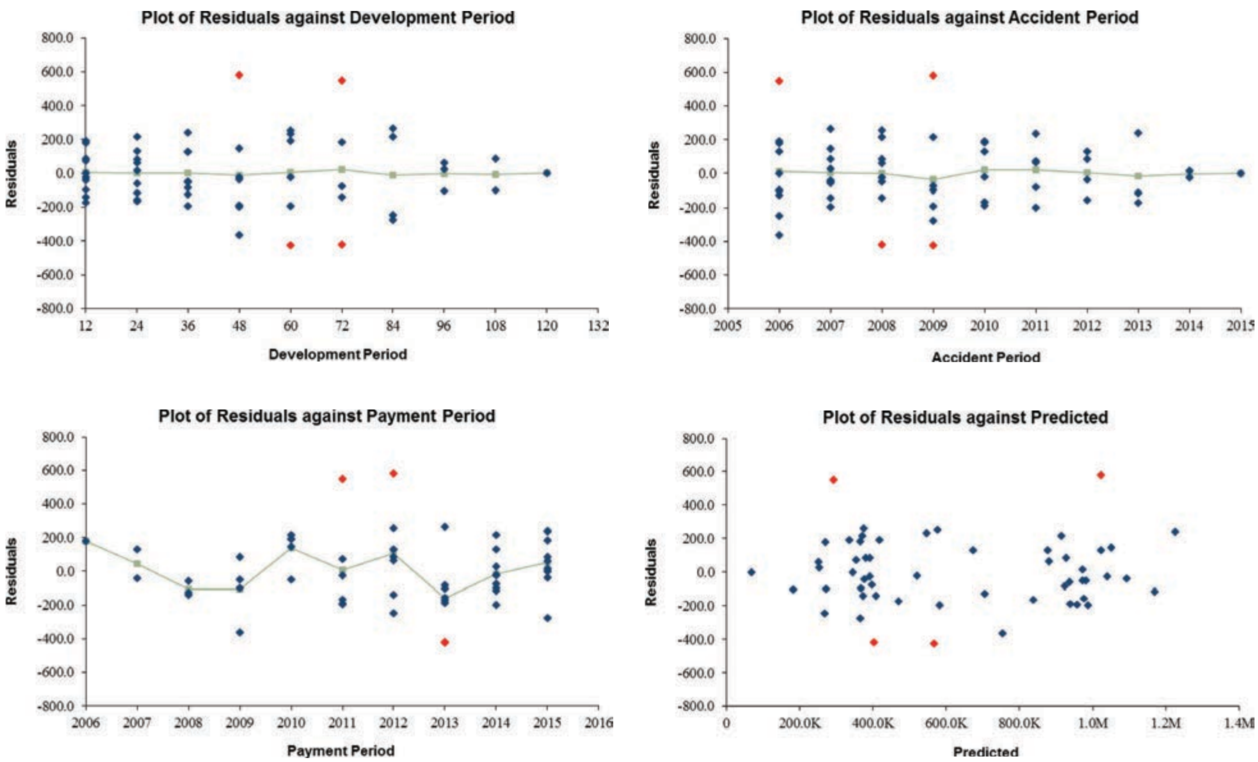
[43] While the examples are different, significant portions of Sections 5 and 6 are based on Milliman (2014) and IAA (2010).

Figure 5.1. Residual Graphs Prior to Heteroscedasticity Adjustment



and parameters that will yield the most realistic, most consistent simulations, based on statistical features found in the data.

To illustrate some of the diagnostic tests for the ODP bootstrap model we will consider data from England and Verrall (1999).[44]

5.1. Residual Graphs

The ODP bootstrap model does not require a specific type of distribution for the residuals, but they are assumed to be independent and identically distributed. Because residuals will be sampled with replacement during the simulations, this requirement is important and thus it is necessary to test this assumption. Graphing residuals is a good way to do this.

Going clock-wise, and starting from the lower-left-hand corner, the graphs in Figure 5.1 show the residuals (blue and red dots[45]) by calendar period, development period, and accident period and against the fitted incremental loss (in the lower-right-hand corner). In addition, the graphs include a trend line (in green) that highlights the averages for each period.

At first glance, the residuals in the graphs appear reasonably random, indicating the model is likely a good fit of the data. But a closer look may also reveal potential features in the data that may indicate ways to improve the model fit.

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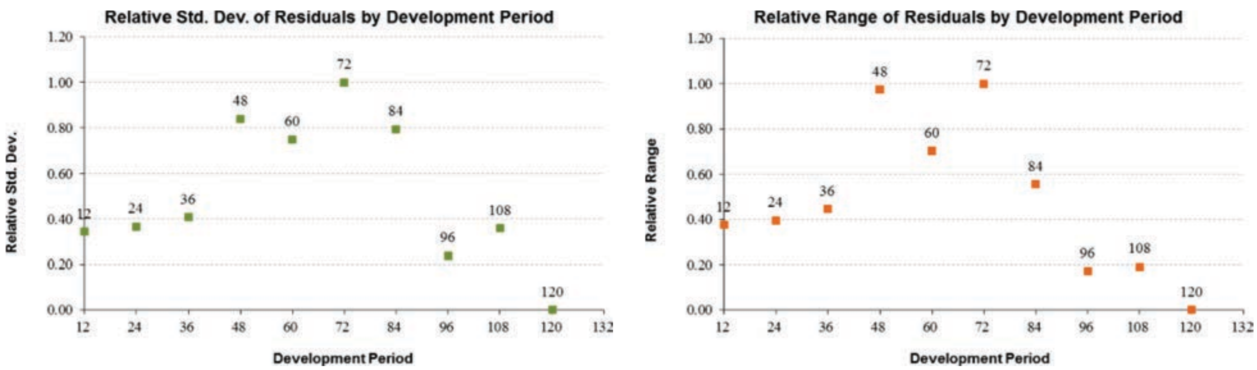
[44] The data triangle was originally used by Taylor and Ashe (1983) and has been used by other authors. This data is included in the "Bootstrap Models.xlsm" file.

[45] In the graphs that follow, the red dots are outliers as identified in Figure 5.7.

Figure 5.2. Residual Relativities Prior to Heteroscedasticity Adjustment



The graphs in Figure 5.1 do not appear to indicate issues with un-modeled trends by accident period or development period (that is, the green "average" lines appear flat at zero). That's because the ODP bootstrap specifies a parameter for every accident and development period. The development-period graph does, however, reveal a potential heteroscedasticity issue associated with the data—i.e., different variances. Note how the upper left graph appears to show a variance of the residuals in the first three periods that differs from those of the middle four or last two periods.

Adjustments for heteroscedasticity can be made with the "Bootstrap Models.xlsm" file, which enables us to recognize groups of development periods and then adjust the residuals to a common standard deviation value, as described in Section 4.6. As an aid to visualizing how to group the development periods into "hetero" groups, graphs of the standard deviation and range relativities can be developed. Figure 5.2 represents pre-adjusted relativities for the residuals shown in Figure 5.1 (i.e., prior to adjustment for factors calculated using either formulas 4.8 or 4.13 and 4.9).

The relativities illustrated in Figure 5.2 help to clarify the changing variability. However, further testing will be required to assess the optimal groups, which can be performed using the other diagnostic tests noted below.

The residual plots in Figure 5.3 originate from the same data model after adjusting for heteroscedasticity using the third option described in Section 4.6 (i.e., using formulas 4.13 and 4.9). The "hetero" groups chosen are for the first three, middle four, and last two development periods, respectively. Determining whether this adjustment has improved the model will require review of other diagnostic tests.

Comparing the residual plots in Figures 5.1 and 5.3 shows that the residuals now appear to exhibit the same standard deviation, or homoscedasticity. More consistent relativities may also be seen in a comparison of the residual relativities in Figures 5.2 and 5.4.

5.2. Normality Test

The ODP bootstrap model does not depend on the residuals being normally distributed, but even so, comparing residuals against a normal distribution remains a useful test, enabling comparison of parameter sets and gauging skewness of the residuals. This test uses both graphs and calculated test values. Figure 5.5 is based on the data used earlier, before and after the adjustment for heteroscedasticity.

Figure 5.3. Residual Graphs After Heteroscedasticity Adjustment

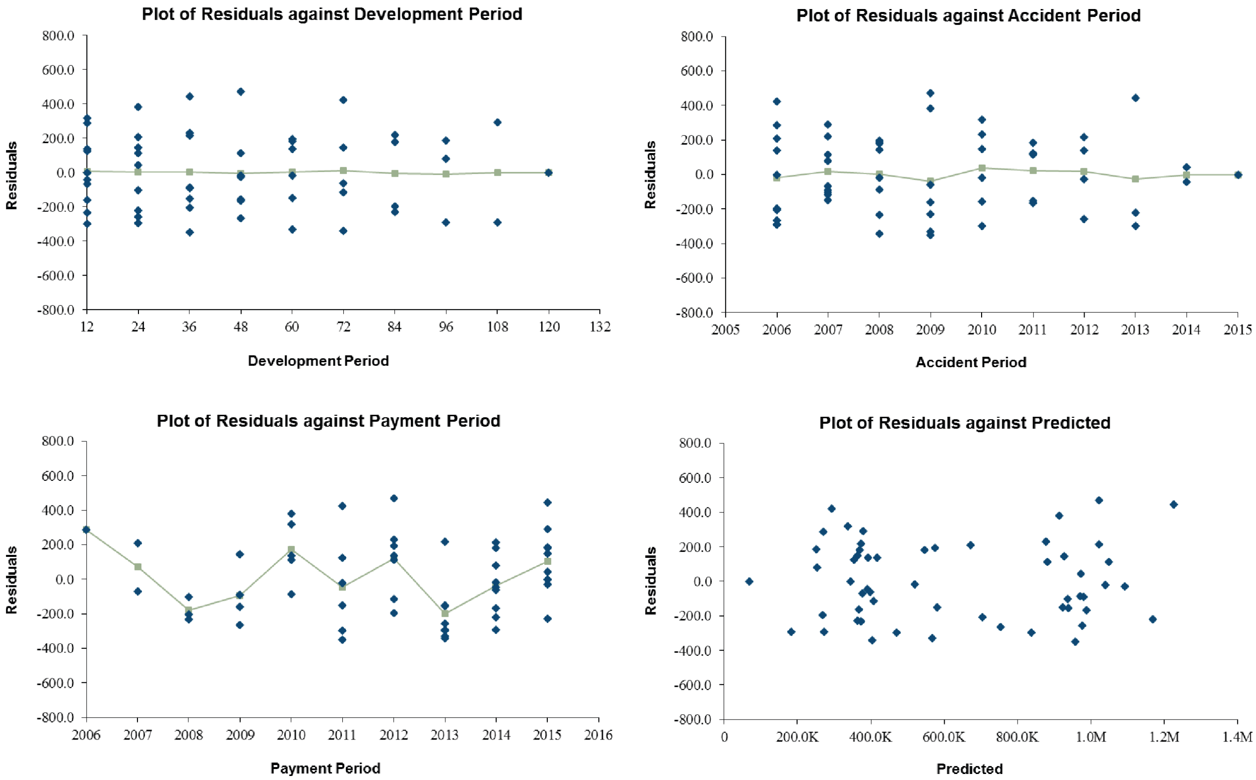


Figure 5.4. Residual Relativities After Heteroscedasticity Adjustment

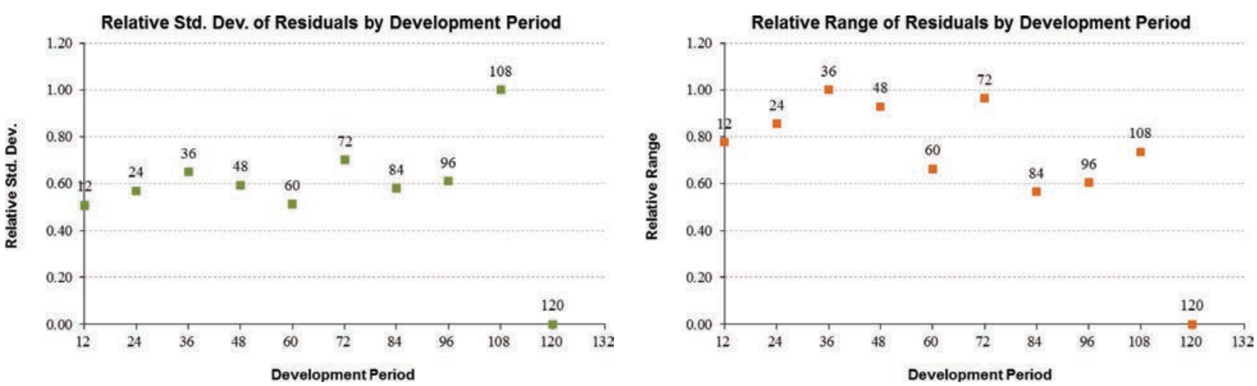
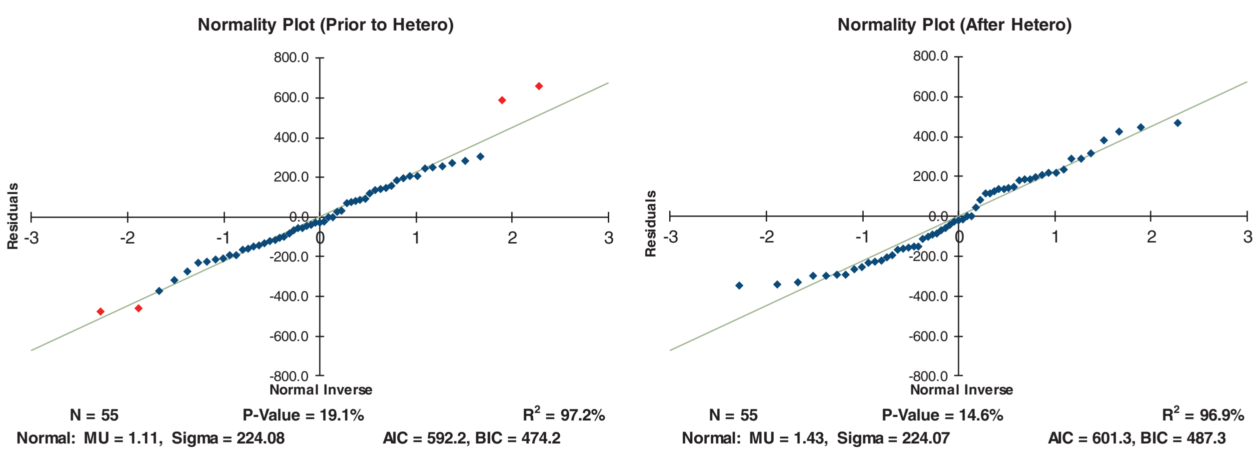


Figure 5.5. Normality Plots Prior to and After Heteroscedasticity Adjustment



Even before the heteroscedasticity adjustment, the residual plots appear close to normally distributed, with the data points tightly distributed around the diagonal line. The *P*-value, a statistical pass-fail test for normality, came in at 19.1%, which exceeds the value generally considered a "passing" score of the normality test, which is greater than 5.0%.[46] The graphs in Figure 5.5 also show *N* (the number of data points) and the R^2 test. After the hetero adjustment, the *P*-value and R^2 get slightly worse, which indicates that the heteroscedasticity adjustment has not improved the results of the diagnostic tests.

While the *P*-value and R^2 tests assess the goodness of fit of the model to the data, they do not penalize for added parameters. Adding more parameters will almost always improve the fit of the model to the data, but the goal is to have a good fit with as few parameters as possible. Two other tests, the Akaike Information Criteria (AIC) and the Bayesian Information Criteria (BIC), address this limitation, using the difference between each residual and its normal counterpart from the normality plot to calculate the Residual Sum Squared (RSS) and include a penalty for additional parameters, as shown in (5.1) and (5.2), respectively.[47]

$$\text{AIC} = 2 \times p + n \times \left[\ln\left(\frac{2 \times \pi \times \text{RSS}}{n} \right) + 1 \right] \tag{5.1}$$

$$\text{BIC} = n \times \ln\left(\frac{\text{RSS}}{n} \right) + p \times \ln(n) \tag{5.2}$$

A smaller value for the AIC and BIC tests indicate residuals that fit a normal distribution more closely, and this improvement in fit overcomes the penalty of adding a parameter.

In our example, with some trial and error, a better "hetero" grouping was found with the diagnostic results shown in Figure 5.6.[48] For the new "hetero" groups, all of the statistical tests improved significantly.

While it might be tempting to add a hetero group for each development column to improve normality, in general normality can be improved with far fewer groups which also helps keep the model from being over-parameterized. As an example, if we use 9 hetero groups for the Taylor and Ashe (1983) data the *P*-value is 14.3%, which is worse than no groups and only slightly better than the original 3 groups, but the AIC and BIC increase significantly.

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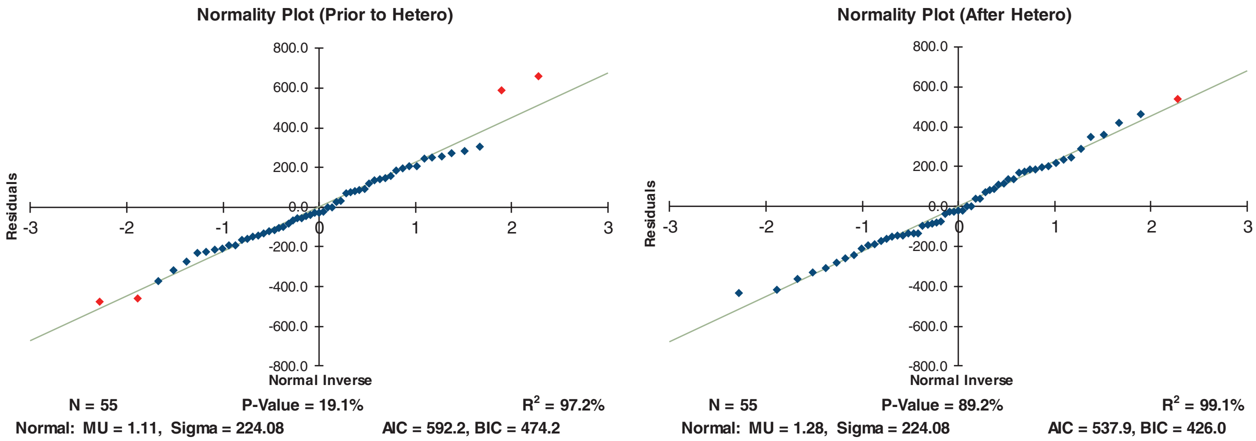
46 Remember that this doesn't indicate whether the ODP bootstrap model itself passes or fails—the ODP bootstrap model doesn't require the residuals to be normally distributed. While not included in the "Bootstrap Models.xlsm" file, as discussed in Section 4.10 it could be used to determine whether to switch to a parametric bootstrap process using a normal distribution.

47 There are different versions of the AIC and BIC formula from various authors and sources, but the general idea of each version is consistent. Other similar formulas could also be used.

48 In the "Bootstrap Models.xlsm" file the Taylor & Ashe data was entered as both paid and incurred. The first set of "hetero" groups are illustrated for the "paid" data and the second set of "hetero" groups are illustrated for the "incurred" data. The "best" groups were found using the optimization tool shown in the "Groups" sheet.

Figure 5.6. Normality Plots Prior to and After Heteroscedasticity Adjustment



5.3. Outliers

Identifying outliers in the data provides another useful test in determining model fit. Outliers can be represented graphically in a box-whisker plot, which shows the inter-quartile range (the 25th to 75th percentiles) and the median (50th percentile) of the residuals—the so-called box. The whiskers then extend to the largest values within three times this inter-quartile range.[49] Values beyond the whiskers may generally be considered outliers and are identified individually with a point.

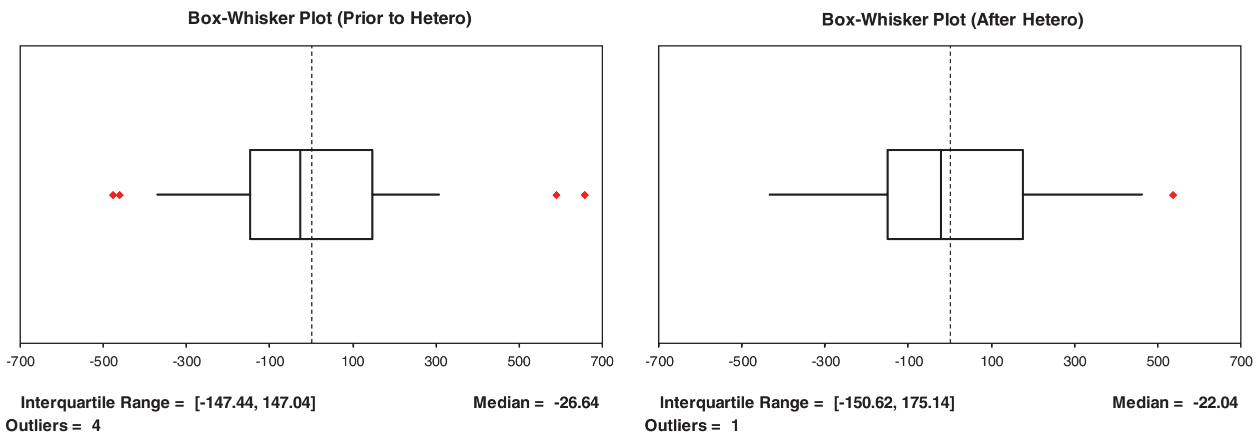
Figure 5.7 shows an example of the residuals for the second set of "hetero" groups (Figure 5.6). A pre-hetero adjustment plot returns four outliers (red dots) in the data model, corresponding to the two highest and two lowest values in the previous graphs in Figures 5.1, 5.3, 5.5, and 5.6.

Even after the hetero adjustment, the residuals still appear to contain one outlier. Now comes a very delicate and often tricky matter of actuarial judgment. If the data in those cells genuinely represent events that cannot be expected to happen again, the outlier(s) may be removed from the model (by giving it/them zero weight). But extreme caution should be taken even when the removal of outliers seems warranted. The possibility always remains that apparent outliers may actually represent realistic extreme values, which, of course, are critically important to include as part of any sound analysis.

Additionally, when residuals are not normally distributed a significant number of outliers tend to result, which may only be an artifact of the distributional shape of the residuals. In this case it is preferable to let these stand in order to enable the simulation process to replicate this shape. Finally, a significant number of residuals can also mean the underlying model is not a good fit to the data so other models should be used (see Section 4.5 for a discussion) or this model given less weight (see Section 6).

[49] Various authors and textbooks use widths for the whiskers which tend to span from 1.5 to 3 times the inter-quartile range. Changing the multiplier will therefore make the box-whisker plot more or less sensitive to outliers. It is also possible to illustrate "mild" outliers with a multiplier of 1.5 and the more "extreme" outliers with a multiplier of 3 using different colors and/or symbols in the graphs. Of course the actual multipliers can be adjusted based on personal preference.

Figure 5.7. Box-Whisker Plots Prior to and After Heteroscedasticity Adjustment



While the three diagnostic tests shown above demonstrate techniques commonly used with most types of models, they are not the only tests available.[50] Next, we'll take a look at the flexibility of the GLM bootstrap and some of the diagnostic elements of the simulation results. For a more extensive list of other tests available, see the report, CAS Working Party on Quantifying Variability in Reserve Estimates (2005).

5.4. Parameter Adjustment

As noted in Section 5.1 the relatively straight average lines in the development and accident period graphs are a reflection of having a parameter for every accident and development period. In most instances, this is also a strong indication that the model may be over-parameterized. Using the "GLM Bootstrap" model in the "Bootstrap Models.xlsm" file we can illustrate the power of removing some of the parameters.

Starting with a "basic" model which includes only one parameter for accident, development and calendar periods (i.e., only one α, β and γ parameter), and adding vertical brown bars to signify a parameter and vertical red lines to signify no parameter (i.e., parameter of zero), the residual graphs for the "GLM Bootstrap" model are shown in Figure 5.8.

The brown bars in the basic model residual graphs represent the parameters and statistics shown in Table 5.1.

Now for this "basic" model the green average lines show trends in the underlying data that are not yet captured by the model as well as a parameter for calendar year trend that is not significant. For example, the overall development period trend parameter is −11%, but the underlying data shows a positive trend for the first 2 or 3 periods followed by a stronger negative trend for the remaining development periods. Another way to see that this basic model does not yet provide a good fit to the underlying data is to compare the implied development pattern with that of the ODP bootstrap model, as shown in Figure 5.9.

[50] For example, see Venter (1998).

Figure 5.8. Residual Graphs for "Basic" GLM Bootstrap Model

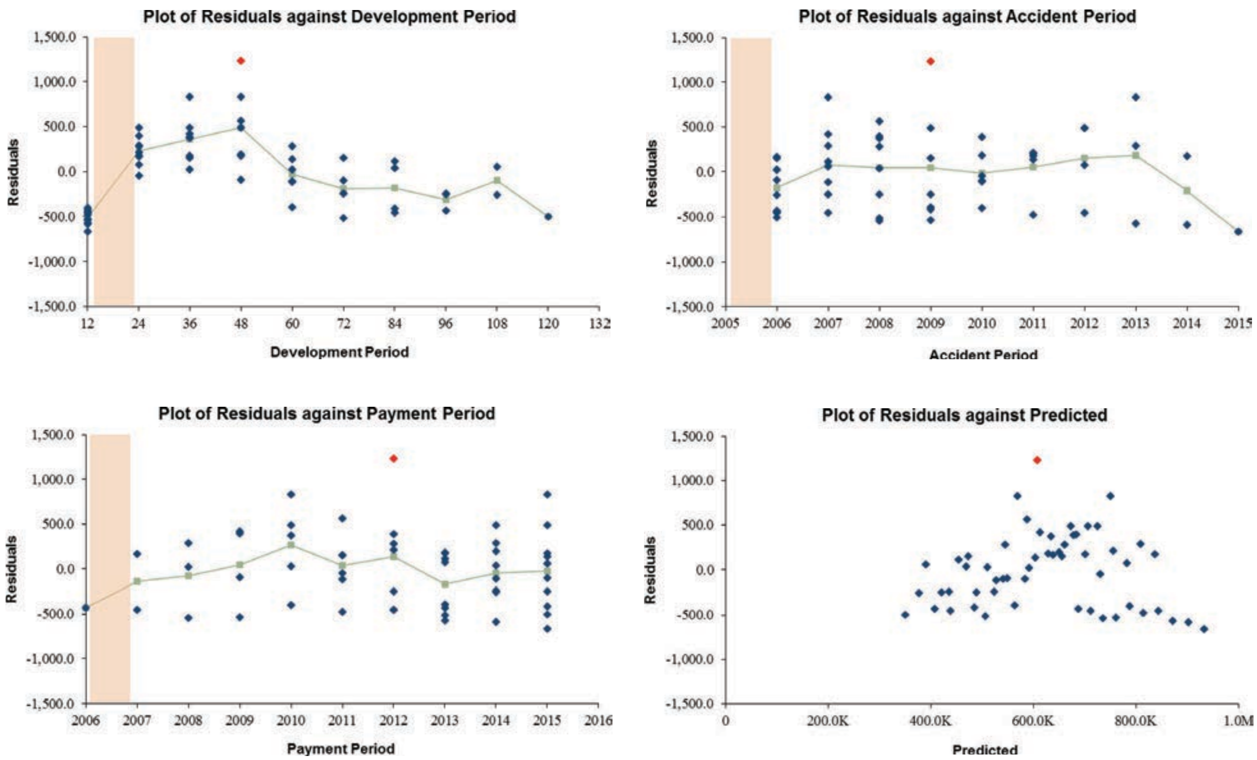
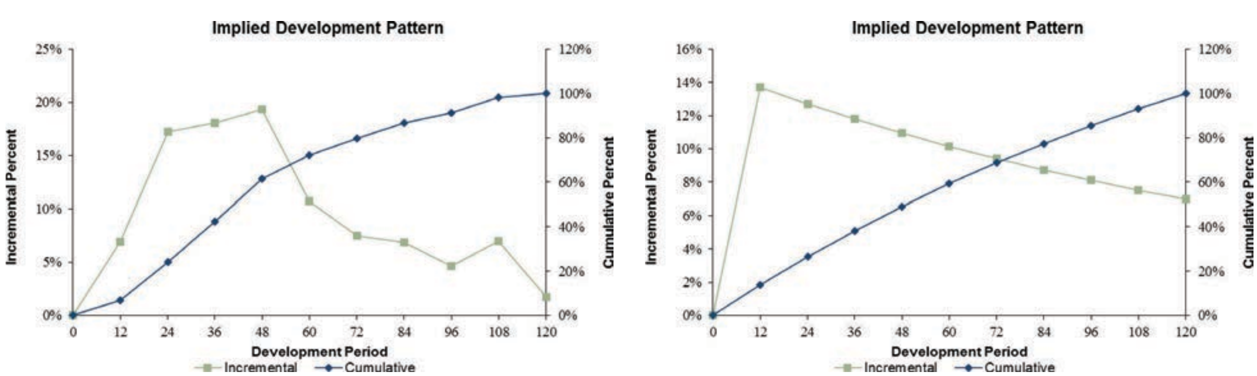


Table 5.1. Parameters and Statistics for "Basic" GLM Bootstrap Model

Parm	Value	Exp(Value)	*t*-Stat	Periods
α_1	13.44	686,938	73.92	Accident Years 2006–2015
β_1	(0.11)		(3.19)	Development Periods 12–132
γ_1	0.03		1.08	Calendar Years 2006–2015

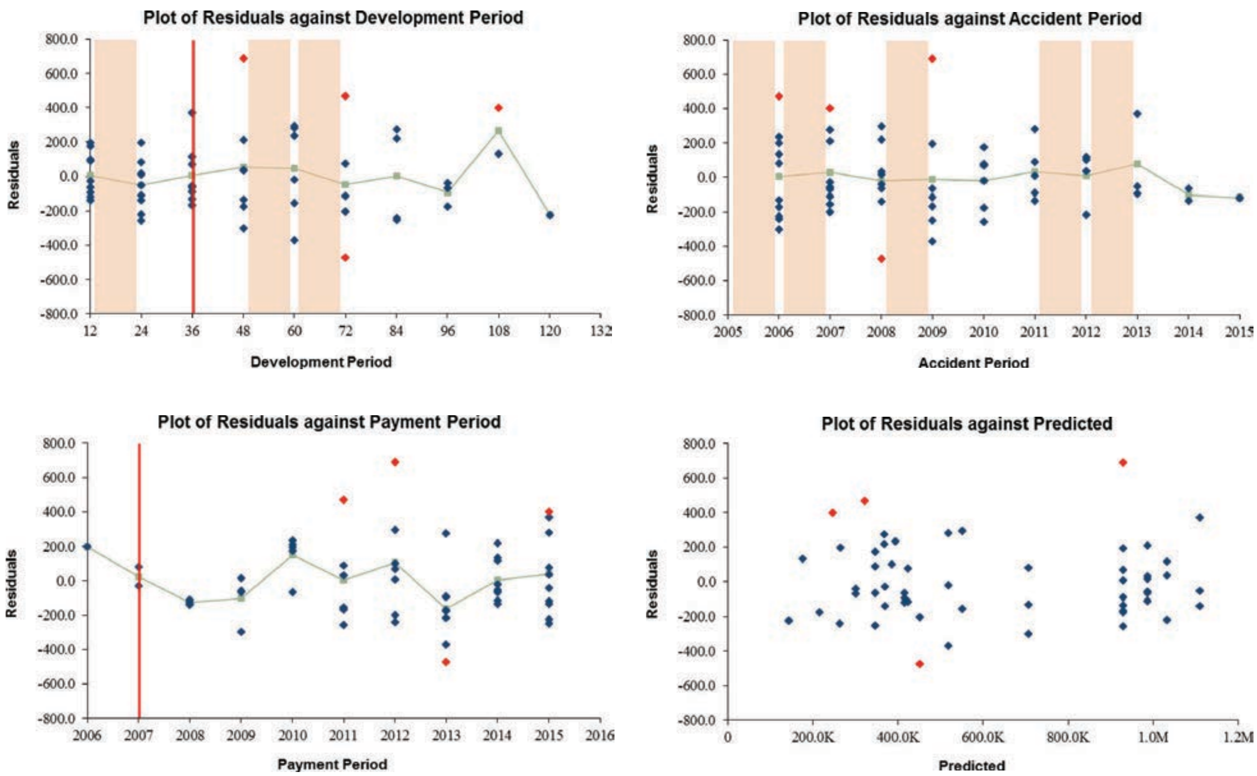
Figure 5.9. Implied Development Patterns



ODP Bootstrap Model

"Basic" GLM Bootstrap Model

Figure 5.10. Residual Graphs for GLM Bootstrap Model



With a little trial and error we can find a reasonably good fit to the data using only five accident, three development and no calendar parameters as shown in Figure 5.10.[51]

In addition to checking the remaining trends in the data with the green average lines, *t*-statistics for each new parameter can be checked to make sure each parameter is statistically significant.[52] The final parameters and statistics for the GLM Bootstrap model are shown in Table 5.2.

Using the "optimal" set of "hetero" groups we can also check the normality graphs and statistics in Figure 5.11 and outliers in Figure 5.12.[53] Comparing the statistics to the ODP bootstrap values shown in Figures 5.6 and 5.7, most values improved while some did not, yet the GLM Bootstrap model is far more parsimonious.

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[51] In the "Bootstrap Models.xlsm" file the optimization tool in the "GLM" sheet can be used to help find a good fit for the parameters of the GLM bootstrap. The algorithm for this tool starts with the ODP bootstrap parameters and then removes the least significant parameters until only significant parameters remain. Then, if there are few enough Alpha and Beta parameters, the Gamma parameters are added and removed if not significant. The tool does not test to see if a parameter should be zero, so some improvements can sometimes occur by forcing parameters to equal zero (e.g., compare the parameters from Figure 5.10 to the parameters in the optimization tool). Finally, it is possible to have a better model fit (i.e., lower AIC and/or BIC) with more parameters even though some of the parameters may not be significant, so judgment is still appropriate for selection of parameters.

[52] The *t*-statistic indicates that a parameter is statistically significant if the absolute value is greater than 2.

[53] When using the GLM bootstrap, any selected outliers and hetero groups used for the ODP bootstrap should be reset and then re-evaluated as they will likely be different for the GLM bootstrap. For the "after hetero" portions of Figures 5.11 and 5.12 the optimization tool in the "Groups" sheet was used.

Table 5.2. Parameters and Statistics for GLM Bootstrap Model

Parm	Value	Exp(Value)	t-Stat	Periods
α_1	12.48	264,036	79.26	Accident Year 2006
α_2	12.82	368,718	2.48	Accident Years 2007–2008
α_3	12.76	347,009	2.11	Accident Years 2009–2011
α_4	12.86	385,644	2.35	Accident Year 2012
α_5	12.93	414,414	3.29	Accident Years 2013–2015
β_1	0.98		7.88	Development Periods 12–24
—	0.00			Development Periods 24–48
β_2	(0.58)		(4.88)	Development Periods 48–60
β_3	(0.20)		(3.29)	Development Periods 60–132
—	0.00			Calendar Year 2006–2015

Figure 5.11. Normality Plots for GLM Bootstrap Model

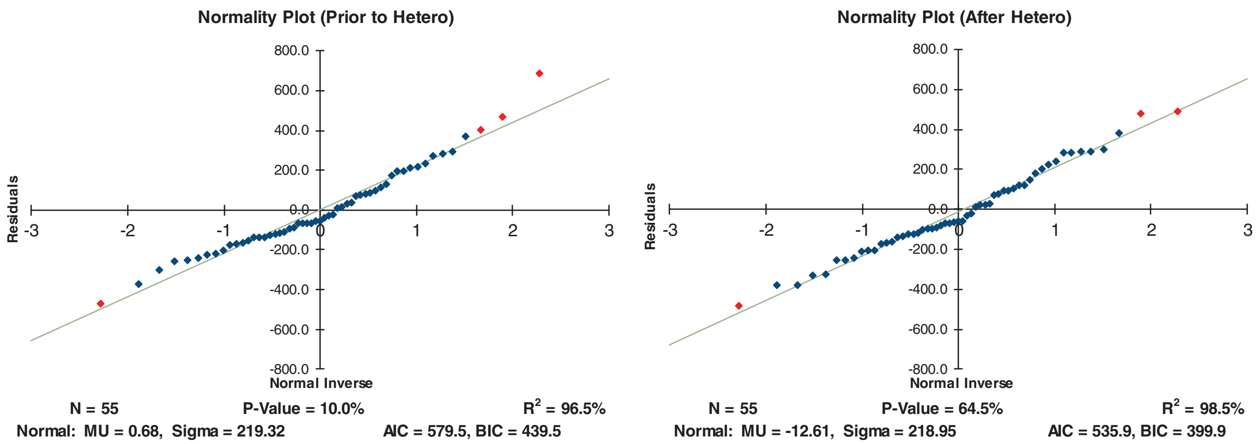


Figure 5.12. Box-Whisker Plots for "GLM Bootstrap" Model

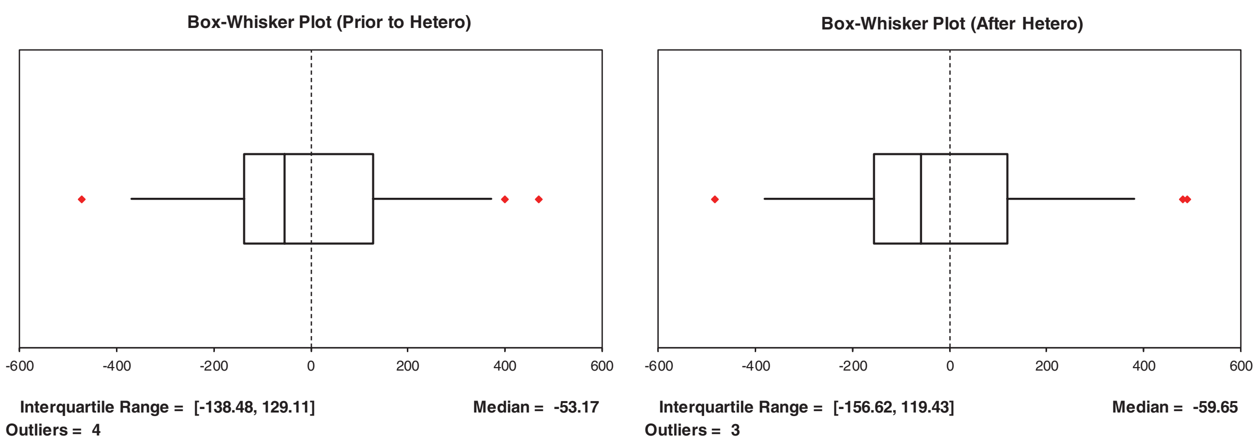
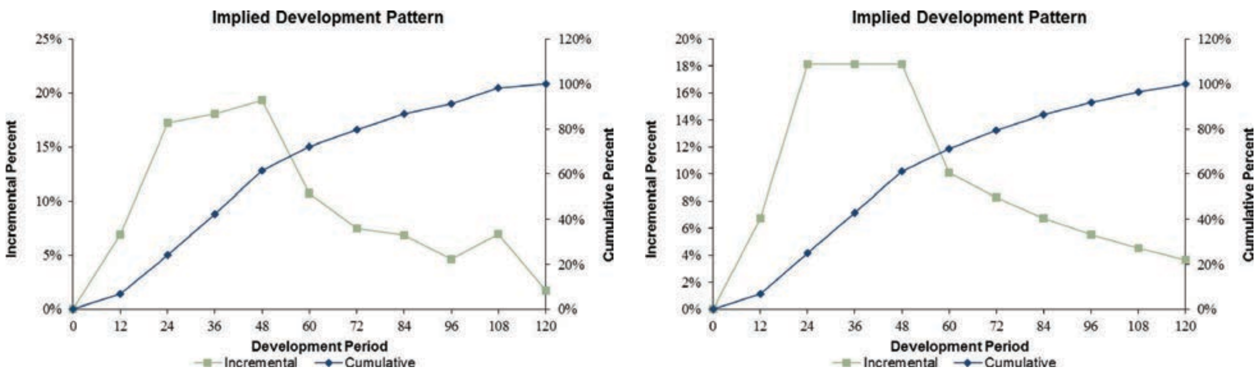


Figure 5.13. Implied Development Patterns



ODP Bootstrap Model

"Basic" GLM Bootstrap Model

As one final check on the trends in this GLM bootstrap model, we can compare a graph of the implied development patterns with the patterns from the chain ladder in the ODP bootstrap model, as shown in Figure 5.13. Because the chain ladder model used a parameter for each development period the implied development pattern can appear a bit jagged, which is why it is often "smoothed" out in practice by selecting development factors. Interestingly, the GLM bootstrap model looks quite similar, yet with much smoother trends in the development patterns. As noted earlier, the last GLM bootstrap development (and calendar trend) parameter can be assumed to extend until the projected model incremental values equal zero which could then be compared to tail factors used in the ODP bootstrap model.[54]

5.5. Model Results

Once the parameter diagnostics have been reviewed, simulations should be run for each model. These simulation results provide an additional diagnostic tool to aid in evaluation of the model, as described in Section 3 of CAS Working Party (2005). As an example, we will review the results for the Taylor and Ashe (1983) data using the ODP bootstrap model. The estimated-unpaid results shown in Figure 5.14 were simulated using 10,000 iterations with the hetero adjustments from Figure 5.6.

5.5.1. Estimated-Unpaid Results

It's recommended to start a diagnostic review of the estimated unpaid results with the standard error (standard deviation) and coefficient of variation (standard error divided by the mean), shown in Figure 5.14. Keep in mind that the standard error should increase when moving from the oldest years to the most recent years, as the standard errors (value scale) should follow the magnitude of the mean of unpaid estimates. In Figure 5.14, the standard errors conform to this pattern. At the same time, the standard error for the total of all years should be larger than any individual year.

[54] Results for the GLM bootstrap model, as illustrated in Figures 5.9 through 5.12, are shown in Appendix E, although no extrapolation was included to be consistent with the ODP bootstrap results.

Figure 5.14. Estimated Unpaid Model Results

Taylor & Ashe Data
Accident Year Unpaid
Paid Chain Ladder Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-	-	-	-	-	-	-	-
2007	94,649	96,571	102.0%	(119,298)	541,054	71,176	147,232	278,360	374,056
2008	473,619	199,302	42.1%	(25,494)	1,217,544	454,644	590,676	830,916	1,018,835
2009	714,763	250,044	35.0%	140,156	1,642,391	684,461	882,486	1,146,017	1,396,652
2010	981,305	271,726	27.7%	324,024	2,062,359	951,467	1,148,872	1,475,857	1,731,112
2011	1,414,007	364,527	25.8%	468,645	2,829,838	1,392,288	1,642,974	2,059,339	2,349,855
2012	2,173,552	489,442	22.5%	806,008	4,293,160	2,142,306	2,489,525	3,033,205	3,345,364
2013	3,969,749	768,637	19.4%	1,655,462	6,369,285	3,913,503	4,501,100	5,307,862	5,989,765
2014	4,317,349	887,688	20.6%	1,874,779	7,677,306	4,260,113	4,898,209	5,844,560	6,516,905
2015	4,703,420	2,176,343	46.3%	445,056	13,859,166	4,493,023	6,127,676	8,500,947	10,529,157
Totals	18,842,414	2,902,735	15.4%	11,312,275	29,464,222	18,594,140	20,734,478	23,885,153	26,388,103

Also, the coefficients of variation should generally decrease when moving from the oldest years to the more recent years and the coefficient of variation for all years combined should be less than for any individual year. With the exception of the 2014 and 2015 accident years, the coefficients of variation in Figure 5.14 seem to also conform, although some random fluctuations may be seen.

The main reason for the decrease in the coefficient of variation has to do with the independence in the incremental claim-payment stream. Because the oldest accident year typically has only a few incremental payments remaining, or even just one, the variability is nearly all reflected in the coefficient. For more current accident years, random variations in the future incremental payment stream may tend to offset one another, thereby reducing the variability of the total unpaid loss.[55]

While the coefficients of variation should go down, they could also start to rise again in the most recent years, as seen in Figure 5.14 for 2014 and 2015. Such reversals are from a couple of issues:

- With an increasing number of parameters used in the model, the parameter uncertainty tends to increase when moving from the oldest years to the more recent years. In the most recent years, parameter uncertainty can grow to overpower process uncertainty, which may cause the coefficient of variation to start rising again. At a minimum, increasing parameter uncertainty will slow the rate of decrease in the coefficient of variation.
- The model may be overestimating the uncertainty in recent accident years if the increase is significant. In that case, another model algorithm (e.g., Bornhuetter-Ferguson or Cape Cod) may need to be used instead of a chain-ladder model.

Keep in mind also that the standard error or coefficient of variation for the total of all accident years will be less than the sum of the standard error or coefficient of variation for the individual years. This is because the model assumes that accident years are independent.

[55] To visualize this reducing Coefficient of Variation, recall that the standard deviation for the total of several independent variables is equal to the square root of the sum of the squares.

Minimum and maximum results are the next diagnostic element in our analysis of the estimated unpaid claims in Figure 5.14, representing the smallest and largest values from all iterations of the simulation. These values will need to be reviewed in order to determine their veracity. If any of them seem implausible, the model assumptions would need to be reviewed. Their effects could materially alter the mean indication. Sometimes implausible extreme iterations are the result of negative incremental values in those "rare" iterations and the limiting incremental value options discussed in Section 4.1 can be used to constrain the model simulation process.

5.5.2. Mean, Standard Deviation and CoV of Incremental Values

The mean, standard deviation and coefficients of variation for every incremental value from the simulation process also provide useful diagnostic results, enabling us to dig deeper into potential coefficient of variation issues that may be found in the estimated unpaid results. Consider, for example, the mean, standard deviation and coefficient of variation results shown in Figures 5.15, 5.16 and 5.17, respectively.

The mean values in Figure 5.15 appear consistent throughout and support the increases in estimated unpaid by accident year that are shown in Figure 5.14. In fact, the future mean values, which lay beyond the stepped diagonal line in Figure 5.15, sum to the results in Figure 5.14. The standard deviation values in Figure 5.16 also

Figure 5.15. Mean of Incremental Values

Taylor & Ashe Data
Accident Year Incremental Values by Development Period
Paid Chain Ladder Model

Accident	Mean Values									
Year	12	24	36	48	60	72	84	96	108	120+
2006	278,309	678,678	706,559	769,219	414,449	296,763	266,301	182,021	270,614	66,922
2007	380,244	940,173	979,875	1,076,297	588,887	408,707	372,144	251,228	381,983	94,649
2008	376,488	936,096	971,651	1,038,686	584,856	405,028	367,109	256,617	379,226	94,393
2009	358,750	918,068	955,061	1,023,741	565,152	405,626	367,359	249,479	372,693	92,592
2010	328,119	837,454	881,193	941,139	514,722	373,168	332,243	226,148	339,996	82,918
2011	353,894	879,226	924,325	986,018	540,281	386,069	348,473	234,329	357,224	87,913
2012	386,915	980,382	1,016,136	1,104,983	595,138	436,918	393,002	267,350	389,062	92,083
2013	477,460	1,175,498	1,227,022	1,334,527	739,306	511,050	461,997	320,655	480,476	121,737
2014	396,237	973,510	1,023,124	1,106,316	597,274	431,428	390,159	264,060	404,885	100,103
2015	342,385	875,509	913,011	977,993	539,429	389,906	344,466	230,160	347,729	85,218

Figure 5.16. Standard Deviation of Incremental Values

Taylor & Ashe Data
Accident Year Incremental Values by Development Period
Paid Chain Ladder Model

Accident	Standard Error Values									
Year	12	24	36	48	60	72	84	96	108	120+
2006	132,756	127,296	126,502	280,755	159,020	136,284	105,608	84,429	104,410	50,555
2007	154,318	150,888	145,947	329,237	187,519	159,277	117,183	101,220	122,902	96,571
2008	151,882	147,943	153,986	332,283	193,190	160,114	121,272	101,681	167,482	98,760
2009	146,220	150,178	149,690	327,782	186,733	158,176	119,597	125,035	171,497	98,042
2010	145,531	138,894	144,262	300,660	178,639	151,920	139,437	118,041	156,163	87,924
2011	146,339	141,271	148,740	317,044	185,534	183,768	145,838	122,161	155,734	95,224
2012	153,454	152,178	153,054	338,980	242,220	199,497	163,440	139,434	168,716	97,719
2013	173,003	165,002	168,993	447,745	261,465	215,809	165,965	141,086	201,662	121,867
2014	156,130	151,172	235,610	410,825	235,604	210,647	163,372	131,985	177,616	103,507
2015	142,319	418,523	436,805	577,315	322,537	254,332	205,111	153,930	222,174	98,010

Figure 5.17. Coefficient of Variation of Incremental Values

Taylor & Ashe Data
Accident Year Incremental Values by Development Period
Paid Chain Ladder Model

Accident	Coefficient of Variation Values									
Year	12	24	36	48	60	72	84	96	108	120+
2006	47.7%	18.8%	17.9%	36.5%	38.4%	45.9%	39.7%	46.4%	38.6%	75.5%
2007	40.6%	16.0%	14.9%	30.6%	31.8%	39.0%	31.5%	40.3%	32.2%	102.0%
2008	40.3%	15.8%	15.8%	32.0%	33.0%	39.5%	33.0%	39.6%	44.2%	104.6%
2009	40.8%	16.4%	15.7%	32.0%	33.0%	39.0%	32.6%	50.1%	46.0%	105.9%
2010	44.4%	16.6%	16.4%	31.9%	34.7%	40.7%	42.0%	52.2%	45.9%	106.0%
2011	41.4%	16.1%	16.1%	32.2%	34.3%	47.6%	41.9%	52.1%	43.6%	108.3%
2012	39.7%	15.5%	15.1%	30.7%	40.7%	45.7%	41.6%	52.2%	43.4%	106.1%
2013	36.2%	14.0%	13.8%	33.6%	35.4%	42.2%	35.9%	44.0%	42.0%	100.1%
2014	39.4%	15.5%	23.0%	37.1%	39.4%	48.8%	41.9%	50.0%	43.9%	103.4%
2015	41.6%	47.8%	47.8%	59.0%	59.8%	65.2%	59.5%	66.9%	63.9%	115.0%

appear consistent, although the future periods seem to have larger standard deviations than historical periods. But the standard deviations can't be added because the standard deviations in Figure 5.14 represent those for aggregated incremental values by accident year, which are less than perfectly correlated.

The differences between the future and historical coefficients of variation in Figure 5.17 help clarify any issues with the model results. For example, notice how the differences by development period are more significant in the bottom two rows in Figure 5.17. This is consistent with the increases in the accident year 2014 and 2015 coefficients of variation noted in Figure 5.14, so they can be used to diagnose the causes noted above when compared to the same results for different models.

6. Using Multiple Models

So far we have focused only on one model. In practice, multiple stochastic models should be used in the same way that multiple methods should be used in a deterministic analysis. First the results for each model must be reviewed and finalized, after an iterative process of diagnostic testing and reviewing model output to make sure the model "fits" the data, has reasonable assumptions and produces reasonable results. Then these results can be combined by assigning a weight to the results of each model.

Two primary methods exist for combining the results for multiple models:

- **Run models with the same random variables.** For this algorithm, every model uses the exact same random variables. In the "Bootstrap Models.xlsm" file, the random values are simulated before they are used to simulate results, which means that this algorithm may be accomplished by reusing the same set of random variables for each model. At the end, the incremental values for each model, for each iteration by accident year (that have a partial weight), can be weighted together.
- **Run models with independent random variables.** For this algorithm, every model is run with its own random variables. In the "Bootstrap Models.xlsm" file the random values are simulated before they are used to simulate results, which means that this algorithm may be accomplished by simulating a new set of random variables for each model.[56] At the end, the weights are used to randomly select a model for each iteration by accident year so that the result is a weighted "mixture" of models.

Both algorithms are similar to the process of weighting the results of different deterministic methods to arrive at an actuarial best estimate. The process of weighting the results of different stochastic models produces an actuarial best estimate of a distribution. In practice it is also common to further "adjust" or "shift" the weighted results by year after considering case reserves and the calculated IBNR. This "shifting" can also be done for weighted distributions, either additively to maintain the exact shape and width of the distribution by year or multiplicatively to maintain the exact shape of the distribution but adjusting the width of the distribution.

[56] In general, in order to simulate new random values a new seed value must be selected, otherwise the same random values will be simulated. In the "Bootstrap Models.xlsm" file the seed value is incremented for each model and data type so that different seed values are being used as long as new random numbers are generated for each model and data type.

Figure 6.1. Model Weights by Accident Year

Accident	Model Weights by Accident Year								
Year	Paid CL	Incd CL	Paid BF	Incd BF	Paid CC	Incd CC	Paid GLM	Incd GLM	TOTAL
2006	50.0%	50.0%							100.0%
2007	50.0%	50.0%							100.0%
2008	50.0%	50.0%							100.0%
2009	50.0%	50.0%							100.0%
2010	50.0%	50.0%							100.0%
2011	50.0%	50.0%							100.0%
2012	50.0%	50.0%							100.0%
2013	16.7%	16.7%	16.7%	16.7%	16.7%	16.7%			100.0%
2014	16.7%	16.7%	16.7%	16.7%	16.7%	16.7%			100.0%
2015	16.7%	16.7%			16.7%	16.7%	16.7%	16.7%	100.0%

The second method of combining multiple models will be illustrated using combined Schedule P data for five top 50 companies.[57] Data for all Schedule P lines with 10 years of history may be found in the "Industry Data.xlsm" file, but this example will be confined to Parts A, B, and C. For each line of business ODP bootstrap models were run for paid and incurred data (labeled Chain Ladder), as well as paid and incurred data for the Bornhuetter-Ferguson and Cape Cod models described in Section 3.3 and the GLM bootstrap model described in Section 3.4.[58] For this section, only the results for Part A (Homeowners/Farmowners) will be reviewed.[59]

By comparing the results for all eight models (or fewer, depending on how many are used)[60] a qualitative assessment of the relative merits of each model may be determined. Bayesian methods can be used to determine weighting based on the quality of each model's forecasts. The weights can be determined separately for each year. The table in Figure 6.1 shows an example of weights for the Part A data.[61] The weighted results are displayed in the "Best Estimate" column of Figure 6.2. As a parallel to a deterministic analysis, the means from the eight models could be used to derive a reasonable range from the modeled results (i.e., from $4,099 to $5,650) as shown in Figure 6.3. Alternatively, if we only consider results by accident year which are given some weight when deriving the best estimate, then the "weighted range" may be a more representative view of the uncertainty of the actuarial central estimate.[62]

When selecting weights for stochastic models, the standard deviations should also be considered in addition to the means by model since the weighted best estimate should reflect the actuary's judgments about the entire distribution not just a central

[57] The five companies represent large, medium and smaller companies that have been combined to maintain anonymity. For each Part, a unique set of five companies were used.

[58] An additional benefit of converting the incurred data models to a random payment stream as discussed in Section 3.3.1 is that they can be combined with other model results.

[59] Only selected weighted results are displayed and discussed in Section 6. A more complete set of results, including results for each model, are included in Appendix A.

[60] Other models in addition to the ODP bootstrap and GLM bootstrap models could also be included in the weighting process as long as the simulated results are in the form of random incremental payment streams.

[61] For simplicity, the weights are judgmental and not derived using Bayesian methods.

[62] The "modeled range" in Figure 6.3 is derived using each model that is given at least some weight for any accident year—i.e., if the model is used. In contrast, the "weighted range" is derived using only the models given weight for each accident year, which are highlighted in grey in Figure 6.2 and 6.4.

Figure 6.2. Summary of Mean Results by Model

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Summary of Results by Model

Accident Year	Mean Estimated Unpaid								
	Chain Ladder		Bornhuetter Ferguson		Cape Cod		GLM Bootstrap		Best Est.
	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred	(Weighted)
2006	-	-	-	-	-	-	-	-	-
2007	3	3	2	2	3	3	9	12	3
2008	41	42	28	27	32	33	27	27	41
2009	45	46	37	39	43	45	40	45	46
2010	63	62	60	59	66	71	62	73	64
2011	103	103	96	98	109	115	106	113	103
2012	222	226	169	168	191	199	213	169	224
2013	294	306	327	334	373	385	280	307	335
2014	679	723	722	753	835	871	646	650	752
2015	3,851	3,912	2,660	2,885	3,225	3,430	3,738	4,255	3,742
Totals	5,300	5,422	4,099	4,366	4,878	5,151	5,120	5,650	5,308

Figure 6.3. Summary of Ranges by Accident Year

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Summary of Results by Model

Accident Year	Best Est. (Weighted)	Ranges			
		Weighted		Modeled	
		Minimum	Maximum	Mininum	Maximum
2006	-				
2007	3	3	3	2	12
2008	41	41	42	27	42
2009	46	45	46	37	46
2010	64	62	63	59	73
2011	103	103	103	96	115
2012	224	222	226	168	226
2013	335	294	385	280	385
2014	752	679	871	646	871
2015	3,742	3,225	4,255	2,660	4,255
Totals	5,308	4,674	5,992	4,099	5,650

estimate. Thus, coefficients of variation by model can be used for this purpose as illustrated in Figure 6.4.

With our focus on the entire distribution, the weights by year were used to randomly sample the specified percentage of iterations from each model. A more complete set of the results for the "weighted" iterations can be created similar to the tables shown in Section 5. The companion "Best Estimate.xlsm" file can be used to weight eight different models together in order to calculate a weighted best estimate. An example for Part A is shown in the table in Figure 6.5.

As one final check of the weighted results it would be common to review the implied IBNR to make sure there are no issues as shown in Figure 6.6. By reviewing this reconciliation, and perhaps also comparing it to deterministic results, additional adjustments could be made to various assumptions. For example, from year 2006 in Figure 6.6 it may be more realistic to revisit the tail factor assumption so that the unpaid estimate is more consistent with the case reserves. Finally, after the interactive process of reviewing results and adjusting assumptions is complete, it may still be

Figure 6.4. Summary of CoV Results by Model

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Summary of Results by Model

Accident Year	Coefficient of Variation							
	Chain Ladder		Bornhuetter Ferguson		Cape Cod		GLM Bootstrap	
	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred
2006								
2007	264.9%	309.9%	310.2%	318.6%	276.2%	326.5%	86.4%	91.5%
2008	74.7%	101.0%	89.2%	109.3%	86.1%	95.6%	177.0%	184.0%
2009	65.5%	93.2%	69.7%	93.5%	69.2%	89.0%	119.3%	118.9%
2010	49.4%	75.6%	52.2%	78.0%	47.2%	72.7%	78.5%	78.1%
2011	34.9%	62.4%	35.7%	64.6%	33.5%	59.5%	51.3%	50.9%
2012	26.1%	49.5%	31.3%	51.4%	28.1%	50.2%	33.6%	41.5%
2013	27.3%	57.5%	26.9%	59.3%	23.3%	56.2%	27.9%	34.9%
2014	18.9%	48.8%	21.8%	51.0%	17.1%	46.7%	20.3%	26.3%
2015	9.2%	39.2%	14.4%	40.5%	8.0%	39.4%	9.0%	16.0%
Totals	8.4%	29.0%	11.1%	28.9%	7.9%	27.5%	8.7%	13.3%

Figure 6.5. Estimated Unpaid Model Results (weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Best Estimate (Weighted)

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	3	9	292.0%	-	173	0	1	17	42
2008	41	37	88.6%	-	391	32	57	111	168
2009	46	37	81.0%	1	522	36	60	114	175
2010	64	41	63.6%	4	537	55	81	139	205
2011	103	50	48.8%	10	636	94	125	193	276
2012	224	89	40.0%	36	917	211	266	382	529
2013	335	148	44.3%	25	1,460	315	401	594	865
2014	752	293	39.0%	106	2,881	725	873	1,265	1,789
2015	3,742	982	26.2%	1,094	10,700	3,654	4,118	5,392	7,059
Totals	5,308	1,044	19.7%	2,116	12,445	5,224	5,758	7,074	8,675
Normal Dist.	5,308	1,044	19.7%			5,308	6,013	7,026	7,738
logNormal Dist.	5,309	1,034	19.5%			5,211	5,935	7,158	8,164
Gamma Dist.	5,308	1,044	19.7%			5,240	5,971	7,135	8,035
TVaR						6,035	6,593	8,140	10,091
Normal TVaR						6,142	6,636	7,463	8,092
logNormal TVaR						6,121	6,691	7,780	8,733
Gamma TVaR						6,137	6,688	7,689	8,516

Figure 6.6. Reconciliation of Total Results (weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Reconciliation of Total Results
Best Estimate (Weighted)

Accident Year	Paid To Date	Incurred To Date	Case Reserves	IBNR	Estimate of Ultimate	Estimate of Unpaid
2006	5,234	5,237	3	(3)	5,234	-
2007	6,470	6,479	9	(6)	6,473	3
2008	7,848	7,867	19	23	7,890	41
2009	7,020	7,046	26	20	7,066	46
2010	7,291	7,341	50	13	7,355	64
2011	8,134	8,225	91	12	8,237	103
2012	10,800	11,085	285	(61)	11,023	224
2013	7,522	7,810	288	46	7,856	335
2014	7,968	8,703	735	17	8,720	752
2015	9,309	12,788	3,478	263	13,051	3,742
Totals	77,596	82,580	4,984	324	82,905	5,308

prudent to make adjustments to the best estimate of the unpaid by shifting the results as noted earlier in this section. For example, since all of the models estimated the unpaid for 2012 to be less than the case reserves, if other studies show that the case reserves are not likely to be redundant then the actuary may decide to shift the unpaid for 2012 so that it is at least 285.

6.1. Additional Useful Output

Three rows of percentile numbers for the normal, lognormal, and gamma distributions, which have been fitted to the total unpaid-claim distribution, may be seen at the bottom of the table in Figure 6.5. The fitted mean, standard deviation, and selected percentiles are in their respective columns; the smoothed results can be used to assess the quality of fit, parameterize a DFA model, or used to smooth the estimate of extreme values,[63] among other applications.

Four rows of numbers indicating the Tail Value at Risk (TVaR), defined as the average of all of the simulated values equal to or greater than the percentile value, may also be seen at the bottom of Figure 6.5. For example, in this table, the 99th percentile value for the total unpaid claims for all accident years combined is 8,675, while the average of all simulated values that are greater than or equal to 8,675 is 10,091. The Normal TVaR, Lognormal TVaR, and Gamma TVaR rows are calculated similarly, except that they use the respective fitted distributions in the calculations rather than actual simulated values from the model.

An analysis of the TVaR values is likely to help clarify a critical issue: if the actual outcome exceeds the X percentile value, by how much will it exceed that value on average? This type of assessment can have important implications related to risk-based capital calculations and other technical aspects of enterprise risk management. But it is worth noting that the purpose of the normal, lognormal, and gamma TVaR numbers is to provide "smoothed" values—that is, that some of the random statistical noise is essentially prevented from distorting the calculations.

6.2. Estimated Cash Flow Results

A model's output may also be reviewed by calendar year (or by future diagonal), as shown in the table in Figure 6.7. A comparison of the values in Figures 6.5 and 6.7 indicates that the total rows are identical, because summing the future payments horizontally or diagonally will produce the same total. Similar diagnostic issues (as discussed in Section 5) may be reviewed in the table in Figure 6.7, with the exception of the relative values of the standard errors and coefficients of variation moving in opposite directions for calendar years compared to accident years. This phenomenon makes sense on an intuitive level when one considers that "final" payments, projected to the furthest point in the future, should actually be the smallest, yet relatively most uncertain.

[63] A random instance of an extreme percentile can be quite erratic compared to the same percentile of a distribution fitted to the simulated distribution. This random noise for extreme percentiles could be cause for increasing the number of iterations, but if the same percentiles for the fitted distributions are stable perhaps they can be used in lieu of more iterations. Of course the use of the extreme values assumes that the models are reliable.

Figure 6.7. Estimated Cash Flow (weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Calendar Year Unpaid
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	3,475	754	21.7%	1,297	8,420	3,414	3,797	4,730	5,948
2017	865	208	24.0%	293	2,148	843	982	1,224	1,483
2018	403	118	29.4%	115	1,298	387	467	614	740
2019	204	67	32.7%	56	654	194	240	325	412
2020	140	50	35.9%	40	539	132	165	233	297
2021	90	43	47.4%	12	611	82	112	169	229
2022	70	44	63.2%	6	409	60	91	152	215
2023	51	58	112.2%	-	735	36	75	151	253
2024	10	15	146.5%	-	199	4	15	41	67
Totals	5,308	1,044	19.7%	2,116	12,445	5,224	5,758	7,074	8,675

Figure 6.8. Estimated Loss Ratio (weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Ultimate Loss Ratios
Best Estimate (Weighted)

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	67.7%	28.5%	42.1%	0.4%	220.8%	66.1%	71.1%	130.9%	158.2%
2007	79.3%	30.2%	38.1%	8.2%	262.2%	77.8%	83.1%	145.5%	178.5%
2008	90.5%	31.2%	34.5%	16.9%	261.3%	89.0%	94.6%	159.9%	188.9%
2009	72.8%	26.8%	36.7%	10.2%	215.6%	71.4%	76.1%	131.7%	180.4%
2010	65.3%	23.3%	35.7%	10.2%	225.0%	63.8%	68.0%	116.1%	139.7%
2011	64.1%	21.2%	33.1%	13.0%	190.0%	63.2%	67.0%	111.8%	130.5%
2012	80.5%	24.0%	29.9%	25.0%	234.6%	79.0%	83.7%	132.9%	154.6%
2013	54.7%	18.8%	34.4%	9.9%	157.7%	53.9%	57.4%	96.2%	115.1%
2014	58.0%	19.2%	33.0%	13.0%	164.8%	57.1%	60.6%	99.8%	118.8%
2015	88.2%	21.5%	24.4%	30.9%	232.5%	85.5%	92.5%	127.9%	158.7%
Totals	71.3%	7.4%	10.4%	46.6%	112.7%	70.8%	75.7%	84.4%	91.7%

6.3. Estimated Ultimate Loss Ratio Results

Another output table, Figure 6.8, shows the estimated ultimate loss ratios by accident year. Unlike the estimated unpaid and estimated cash-flow tables, the values in this table are calculated using all simulated values, not just the values beyond the end of the historical triangle. Because the simulated sample triangles represent additional possibilities of what could have happened in the past, even as the "squaring of the triangle" and process variance represent what could happen as those same past values are played out into the future, we are in possession of sufficient information to enable us to estimate the variability in the loss ratio from day one until all claims are completely paid and settled for each accident year.[64]

Reviewing the simulated values indicates that the standard errors in Figure 6.8 should be proportionate to the means, while the coefficients of variation should be relatively constant by accident year. In terms of diagnostics, any increases in standard error and coefficient of variation for the most recent years would be consistent with the reasons

[64] If we are only interested in the "remaining" volatility in the loss ratio, then the values in the estimated unpaid table (Figure 6.5) can be added to the cumulative paid values by year and divided by the premiums.

Figure 6.9. Estimated Unpaid Claim Runoff (weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Calendar Year Unpaid Claim Runoff
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	5,308	1,044	19.7%	2,116	12,445	5,224	5,758	7,074	8,675
2016	1,834	365	19.9%	746	4,128	1,797	2,030	2,459	2,957
2017	969	218	22.5%	336	2,316	946	1,088	1,353	1,627
2018	566	146	25.8%	159	1,393	548	647	828	1,004
2019	362	114	31.5%	79	1,171	347	424	565	718
2020	222	92	41.4%	35	956	207	269	386	524
2021	132	76	57.6%	6	863	117	166	268	394
2022	62	59	96.3%	(0)	745	46	84	166	269
2023	10	15	146.5%	(0)	199	4	15	41	67

previously cited in Section 5.4 for the estimated unpaid tables. Risk management-wise, the loss ratio distributions have important implications for projecting pricing risk—the mean loss ratios can be used to view any underwriting cycles and help inform the projected mean for the next few years, while the coefficients of variation can be used to select a standard deviation for the next few years.[65]

6.4. Estimated Unpaid Claim Runoff Results

Figure 6.9, shows the runoff of the total unpaid claim distribution by future calendar year. Like the estimated unpaid and estimated cash-flow tables, the values in this table are calculated using only future simulated values, except that future diagonal results are sequentially removed so that we are left with the remaining unpaid claims at the end of future calendar periods. These results are quite useful for calculating the runoff of the unpaid claim distribution when calculating risk margins using the cost of capital method.

6.5. Distribution Graphs

A final model output to consider is a histogram of the estimated unpaid amounts for the total of all accident years combined, as shown in the graph in Figure 6.10. The histogram is created by counting the number of outcomes within each of 100 "buckets" of equal size spread between the minimum and maximum outcome. To smooth the histogram a kernel density function is often used, which is the green bars in Figure 6.10.

Another useful strategy for graphing the total unpaid distribution may be accomplished by creating a summary of the eight model distributions used to determine the weighted "best estimate" and distribution. An example of this graph using the kernel density functions is shown in Figure 6.11 and dots for the mean estimates, which would represent a traditional range,[66] are also included.

[65] The coefficients of variation measure the variability of the loss ratios, given the movements by year. Without this information, it is common to base the future standard deviation on the standard deviation of the historical mean loss ratios, but this is not ideal since the variability of the mean loss ratios is not the same as the possible variation in the actual outcomes given movements in the means.

[66] A traditional range would use deterministic point estimates instead of means of the distributions, but the intent is consistent. While the points would technically have an infinitesimal probability and should therefore sit on the x-axis, they are elevated above the zero probability level purely for illustration purposes.

Figure 6.10. Total Unpaid Claims Distribution

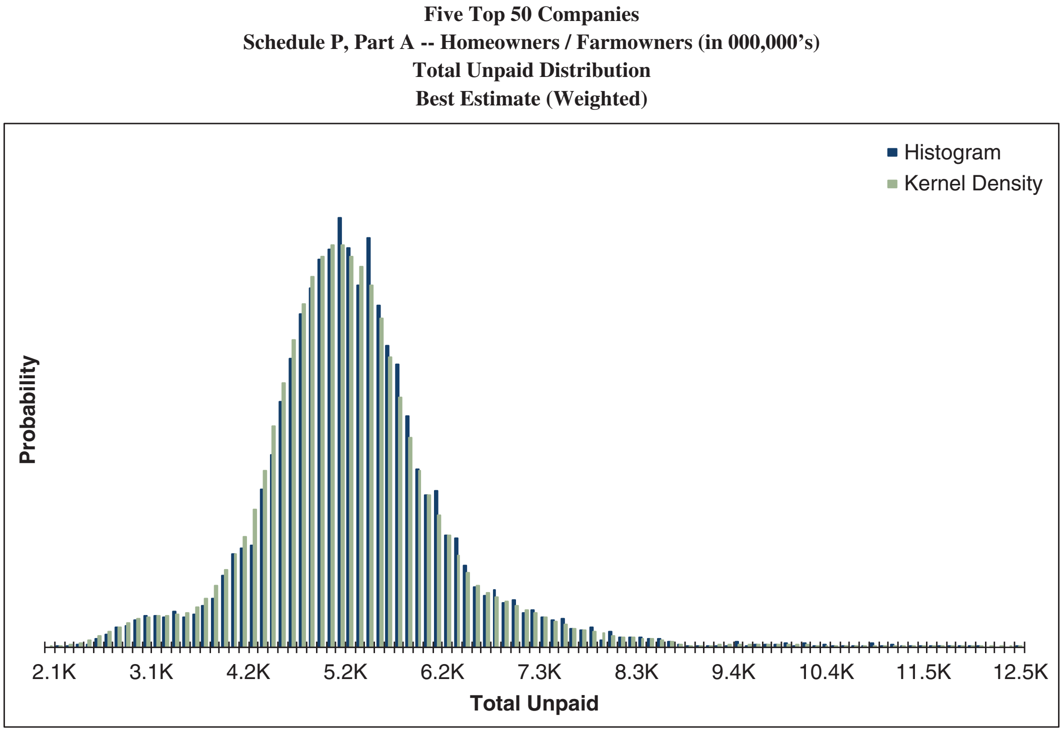
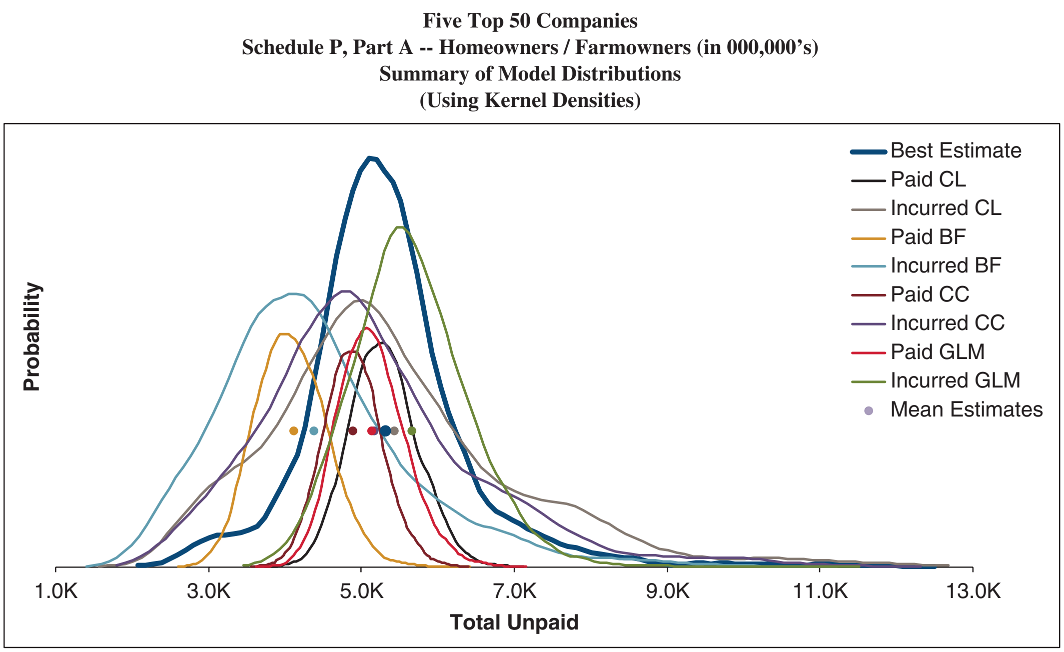


Figure 6.11. Summary of Model Distributions



The corresponding tables and graphs for the Part B and Part C results are shown in Appendices B and C, respectively.[67]

6.6. Correlation

Results for an entire business unit can be estimated, after each business segment has been analyzed and weighted into best estimates, using aggregation. This represents another area where caution is warranted. The procedure is not a simple matter of adding up the distributions for each segment. In order to estimate the distribution of possible outcomes for a company as a whole, a correlation of results between segments must be used.[68]

Simulating correlated variables is commonly accomplished with a multivariate distribution whose parameters and correlations have been previously specified. This type of simulation is most easily applied when distributions are uniformly identical and known in advance (for example, all derived from a multivariate normal distribution). Unfortunately, these conditions do not generally exist for the ODP bootstrap model (or other models), as quite often the modeling process does not allow us to know the characteristics of overall distributions in advance or combining distributions from different types of models is by definition not uniformly identical and known in advance. Indeed, as the shapes of different distributions are usually slightly different, another approach will be needed.[69]

Two useful correlation processes for the ODP bootstrap model are location mapping (or synchronized bootstrapping) and re-sorting.[70]

With location mapping, each iteration will include sampling residuals for the first segment and then going back to note the location in the original residual triangle of each sampled residual.[71] Each of the other segments is sampled using the residuals at the same locations for their respective residual triangles. Thus, the correlation of the original residuals is preserved in the sampling process.

The location-mapping process is easily implemented in Excel and does not require the need to estimate a correlation matrix. There are, however, two drawbacks to this process. First, it requires all of the business segments to use data triangles that are precisely the same size with no missing values or outliers when comparing each location of the residuals.[72] Second, the correlation of the original residuals is used in the model, and no other correlation assumptions can be used for stress testing the aggregate results.

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[67] For Part B and Part C, tail factors were used to illustrate the results when extrapolated beyond just squaring the triangle. This also flows through to the Aggregate results in Appendix D.

[68] This section assumes the reader is familiar with correlation.

[69] It is possible to use this process with a parametric ODP bootstrap model, as described in Section 4.10, but that is beyond the scope of the monograph.

[70] For a useful reference see Kirschner, et al. (2008).

[71] For example, in the "Bootstrap Models.xlsm" file the locations of the sampled residuals are shown in Step 15, which could be replicated iteration by iteration for each business segment.

[72] It is possible to fill in "missing" residuals in another segment using a randomly selected residual from elsewhere in the triangle, but in order to maintain the same amount of correlation the selection of the other residual would need to account for the correlation between the residuals, which complicates the process.

Figure 6.12. Estimated Correlation and P-values

Rank Correlation of Residuals after Hetero Adjustment - Paid

LOB	1	2	3
1	1.00	0.37	0.19
2	0.37	1.00	0.24
3	0.19	0.24	1.00

P-Values of Rank Correlation of Residuals after Hetero Adjustment - Paid

LOB	1	2	3
1	0.00	0.01	0.17
2	0.01	0.00	0.07
3	0.17	0.07	0.00

The second correlation process, re-sorting, can be accomplished with algorithms such as Iman-Conover[73] or Copulas, among others. The primary advantages of re-sorting include:

- The triangles for each segment may have different shapes and sizes,
- Different correlation assumptions may be employed, and
- Different correlation algorithms may also have other beneficial impacts on the aggregate distribution.

For example, using a *t*-distribution Copula with low degrees of freedom rather than a normal-distribution Copula, will effectively "strengthen" the focus of the correlation in the tail of the distribution, all else being equal. This type of consideration is important for risk-based capital and other risk modeling issues.

To induce correlation among different segments in the ODP bootstrap model, a calculation of the correlation matrix using Spearman's Rank Order and use of re-sorting based on the ranks of the total unpaid claims for all accident years combined may be done. The calculated correlations for Parts A, B, and C based on the paid residuals after hetero adjustments may be seen in the table in Figure 6.12. A second part of Figure 6.12 are the *P*-values for each correlation coefficient, which are an indication of whether a correlation coefficient is significantly different than zero as the *P*-value gets close to zero.[74]

By reviewing the correlation coefficients for each "pair" of segments, along with the *P*-values, from different sets of correlations matrices (e.g., from paid or incurred data before or after the hetero adjustment) judgment can be used to select a correlation matrix assumption. As noted above, caution is warranted as these calculated correlation matrices are limited to the data used in the calculation and the impact of other systemic issues, such as contagion, may also need to be considered.

Using these correlation coefficients, the "Aggregate Estimate.xlsm" file, and the simulation data for Parts A, B, and C, the aggregate results for the three lines of business

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[73] For a useful reference see Iman and Conover (1982) or Mildenhall (2006). In the "Aggregate Estimate.xlsm" file the Iman-Conover algorithm is used to "Generate Rank Values" on the Inputs sheet.

[74] While judgment is clearly appropriate, the typical threshold is a *P*-value of 5%—i.e., a *P*-value of 5% or less indicates the correlation is significantly different than zero, while a *P*-value greater than 5% indicates the correlation is not significantly different than zero.

Figure 6.13. Aggregate Estimated Unpaid

Five Top 50 Companies
Aggregate Three Lines of Business
Accident Year Unpaid

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	67	25	37.9%	0	186	66	83	110	130
2007	107	30	28.1%	25	295	105	126	158	185
2008	199	49	24.8%	67	622	194	226	285	342
2009	298	56	18.8%	123	800	293	331	395	457
2010	480	69	14.3%	248	959	475	522	599	668
2011	862	106	12.3%	503	1,561	860	923	1,041	1,135
2012	1,666	187	11.2%	383	2,555	1,662	1,771	1,985	2,148
2013	3,070	333	10.8%	1,808	6,522	3,066	3,249	3,649	3,928
2014	5,632	703	12.5%	2,435	8,555	5,632	6,075	6,801	7,326
2015	13,270	1,788	13.5%	5,217	22,660	13,262	14,348	16,180	18,011
Totals	25,650	2,080	8.1%	16,952	36,085	25,616	26,949	29,088	30,991
Normal Dist.	25,650	2,080	8.1%			25,650	27,053	29,072	30,490
logNormal Dist.	25,650	2,088	8.1%			25,566	27,006	29,222	30,885
Gamma Dist.	25,650	2,080	8.1%			25,594	27,021	29,165	30,736

were calculated and summarized in the table in Figure 6.13. A more complete set of tables for the aggregate results is shown in Appendix D.

Note that using residuals to correlate the lines of business (or other segments), as in the location mapping method, and measuring the correlation between residuals, as in the re-sorting method, both tend to create correlations that are close to zero. For reserve risk, the correlation that is desired is between the total unpaid amounts for two segments. The correlation that is being measured is the correlation between each incremental future loss amount, given the underlying model describing the overall trends in the data. This may or may not be a reasonable approximation.

While not the direct measure we are hoping for, keep in mind that some level of implied correlation between lines of business will naturally occur due to correlations between the model parameters—e.g., similarities in development parameters, so correlation based on the correlation between the remaining random movements in the incremental values given the model parameters (i.e., residuals) may be reasonable. However, an example of an issue not particularly well suited to measurement via residual correlation is contagion between lines of business—i.e., single events that result in claims in multiple lines of business. To account for this, and to add a bit of conservatism, the correlation assumption can be easily changed based on actuarial judgment.

Correlation is often thought of as being much stronger than "close to zero", but in this case the correlation being considered is typically the loss ratio movements by line of business. For pricing risk, the correlation that is desired is between the loss ratio movements by accident year between two segments. This correlation is not as likely to be close to zero, so correlation of loss ratios (e.g., for the data in Figure 6.7) is often done with a different correlation assumption compared to reserving risk.

7. Model Testing

Work on testing stochastic unpaid claim estimation models is still in its infancy. Most papers on stochastic models display results, and some even compare a few different models, but they tend to be void of any statistical evidence regarding how well the model in question predicts the underlying distribution. This is quite understandable since we don't know what the underlying distribution is, so with real data the best we can hope for is to retrospectively test a very old data set to see how well a model predicted the actual outcome.[75]

Testing a few old data sets is better than not, but ideally we would need many similar data sets to perform meaningful tests. One recent paper authored by the General Insurance Reserving Oversight Committee (GI ROC) in their papers for the General Insurance Research Organizing (GIRO) conference in 2007 titled "Best Estimates and Reserving Uncertainty" (ROC/GIRO 2007) and their updated paper in 2008 titled "Reserving Uncertainty" (ROC/GIRO 2008) took a first step in performing more meaningful statistical testing of a variety of models.

A large number of models were reviewed and tested in these studies, but one of the most interesting portions of the studies were done by comparing the unpaid liability distributions created by the Mack and ODP bootstrap model against the "true" artificially generated unpaid loss percentiles. To accomplish these tests, artificial datasets were constructed so that all of the Mack and ODP bootstrap assumptions, respectively, are satisfied. While the artificial datasets were recognized as not necessarily realistic, the "true" results are known so the Working Parties were able to test to see how well each model performed against datasets that could be considered "perfect."

7.1. Bootstrap Model Results

To test the ODP bootstrap model, incremental losses were simulated for a 10 × 10 square of data based on the assumptions of the ODP bootstrap model. For the 30,000 datasets simulated, the upper triangles were used and the OPD bootstrap model from England and Verrall (1999; 2002) were used to estimate the expected results and various percentiles. The proportion of simulated scenarios in which the "true" outcome exceeded the 99th percentile of the ODP Bootstrap method's results was around 2.6–3.1%. For the Mack method, the "true" outcome exceeded the 99th percentile around 8–13%.

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[75] For example, data for accident years 1994 to 2004 could be completely settled and all results known as of 2014. Thus, we could use the triangle as it existed at year end 2004 to test how well a model predicted the final results.

Thus, the ODP bootstrap model performed better than the Mack model for "perfect" data, even though the results for both models were somewhat deficient in the sense that they both seem to under-predict the extremes of the "true" distribution. In fairness, it should be noted however, that the ODP bootstrap model that was tested did not include many of the "advancements" described in Section 3.2.

7.2. Future Testing

The testing done for GIRO was a significant improvement over simply looking at results for different models, without knowing anything about the "true" underlying distribution. The next step in the testing process will be to test models against "true" results for realistic data instead of "perfect" data. The CAS Loss Simulation Model Working Party (2011) has created a model that will create datasets from the claim transaction level up. The goal is to create thousands of datasets based on characteristics of real data that can be used for testing various models.

8. Future Research

With testing of stochastic models in its infancy, much work in the area of future research is needed. Only a few such areas are offered here.

- Expand testing of the ODP bootstrap model with realistic data using the CAS loss simulation model.
- Research on how the adjustments to the ODP bootstrap and GLM bootstrap suggested in this monograph perform relative to realistic data—i.e., is there a significant improvement in the predictive power of the model given the different model configurations and adjustments.
- Expand or change the ODP bootstrap model in other ways, for example use of the Munich chain ladder (Quarg and Mack 2008) or Berquist-Sherman (1977) method with an incurred/paid set of triangles, or the use of claim counts and average severities. Other examples could include the use of different residuals, such as deviance or Anscombe residuals noted in Section 3.2.
- Research the use of a Bayesian or other approach to selecting weights for different models by accident year to improve the process of combining multiple models discussed in Section 6.
- Research other risk analysis measures and how the ODP bootstrap model can be used for enterprise risk management.
- Research how the ODP bootstrap model can be used for Solvency II requirements in Europe and the International Accounting Standards.
- Research into the most difficult parameter to estimate: the correlation matrix.

9. Conclusions

While this monograph endeavored to show how the ODP bootstrap model can be used in a variety of practical ways, and to illustrate the diagnostic tools the actuary needs to assess whether the model is working well, it should not be assumed that the ODP bootstrap model is well suited for every data set. However, it is hoped that the ODP bootstrap and GLM bootstrap "toolsets" can become an integral part of the actuaries regular estimation of unpaid claim liabilities, rather than just a "black box" to be used only if necessary or after the deterministic methods have been used to select a point estimate. Finally, the modeling framework allows the actuary to "fit" the model to the data instead of simply accepting the model as is and essentially forcing the data to "fit" the model.

Acknowledgments

The author gratefully acknowledges the many authors listed in the References (and others not listed) that contributed to the foundation of the ODP bootstrap model, without which this research would not have been possible. He also wishes to thank the co-author of the predecessor paper, Jessica Leong, for all her support and the contributions that led to this revised monograph. He would also like to thank all the peer reviewers, Stephen Finch, Roger Hayne, Stephen Lienhard, John Major, Mark Mulvaney and Ben Zehnwirth, who helped to improve the quality of the monograph in a variety of ways. In particular, Stephen Finch is noteworthy for keeping his wits during an intoxicating discussion which led to the creation of the term "heteroecthesious" data. Finally, he is grateful to the CAS referees for their comments which also greatly improved the quality of the monograph.

Supplementary Materials

There are several companion files designed to give the reader a deeper understanding of the concepts discussed in the monograph. T he files are all in the "A Practitioners Guide. zip" file at https://www.casact.org/sites/default/files/2021-02/practitionerssuppl-shaplandmonograph04.zip The files are:

Model Instructions.pdf—this file contains a written description of how to use the primary bootstrap modeling files.

Primary bootstrap modeling iles:

Industry Data.xls—this file contains Schedule P data by line of business for the entire U.S.
industry and five of the top 50 companies, for each LOB that has 10 years of data.

Bootstrap Models.xlsm—this file contains the detailed model steps described in this monograph as well as various modeling options and diagnostic tests. Data can be entered and simulations run and saved for use in calculating a weighted best estimate.

Best Estimate.xlsm—this file can be used to weight the results from eight different models to get a "best estimate" of the distribution of possible outcomes.

Aggregate Estimate.xlsm—this file can be used to correlate the best estimate results from 3 LOBs/segments.

Correlation Ranks.xlsm—this file contains examples of ranks used to correlate results by LOB/segment.

Simple example calculation files:

GLM Framework.xlsm—this file illustrates the calculation of the GLM bootstrap model (framework) and the corresponding ODP bootstrap model for a simple 3×3 triangle using (3.8).

GLM Framework C.xlsm—this file illustrates the calculation of the GLM bootstrap model (framework) and the corresponding ODP bootstrap model for a simple 3×3 triangle using (3.7).

GLM Framework 6.xlsm—this file illustrates the calculation of the GLM bootstrap model (framework) and the corresponding ODP bootstrap model for a simple 6×6 triangle using (3.8).

GLM Framework 6C.xlsm—this file illustrates the calculation of the GLM bootstrap model (framework) and the corresponding ODP bootstrap model for a simple 6×6 triangle using (3.7).

GLM Bootstrap 6 with Outlier.xlsm—this file illustrates how the calculation of the GLM bootstrap for a simple 6 × 6 triangle is adjusted for an outlier. It includes different options for adjusting the ODP bootstrap model to remove an outlier.

GLM Bootstrap 6 with 3yr avg.xlsm—this file illustrates how the calculation of the GLM bootstrap for a simple 6 × 6 triangle is adjusted to only use the equivalent of a three-year average (i.e., the last four diagonals).

GLM Bootstrap 6 with 1 Acc Yr Parameter.xlsm—this file illustrates the calculation of the GLM bootstrap using only one accident year (level) parameter, a development year trend parameter for every year and no calendar year trend parameter for a simple 6 × 6 triangle.

GLM Bootstrap 6 with 1 Dev Yr Parameter.xlsm—this file illustrates the calculation of the GLM bootstrap using only one development year trend parameter, an accident year (level) parameter for every year and no calendar year trend parameter for a simple 6 × 6 triangle.

GLM Bootstrap 6 with 1 Acc Yr & 1 Dev Yr Parameter.xlsm—this file illustrates the calculation of the GLM bootstrap using only one accident year (level) parameter, one development year trend parameter and no calendar year trend parameter for a simple 6 × 6 triangle.

GLM 6 Bootstrap with 1 Acc Yr 1 Dev Yr & 1 Cal Yr Parameter.xlsm—this file illustrates the calculation of the GLM bootstrap using only one accident year (level) parameter, one development year trend parameter and one calendar year trend parameter for a simple 6 × 6 triangle.

Appendices

Appendix A—Schedule P, Part A Results

In this appendix the results for Schedule P, Part A (Homeowners/Farmowners) are shown.

Figure A.1. Estimated Unpaid Model Results (Paid Chain Ladder)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Paid Chain Ladder Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	3	7	264.9%	-	81	0	2	17	33
2008	41	31	74.7%	-	204	35	59	100	131
2009	45	30	65.5%	7	209	38	61	104	137
2010	63	31	49.4%	15	213	56	80	118	161
2011	103	36	34.9%	36	286	96	122	170	213
2012	222	58	26.1%	93	497	216	258	328	376
2013	294	80	27.3%	126	671	285	342	440	513
2014	679	128	18.9%	366	1,190	675	758	894	1,003
2015	3,851	356	9.2%	2,675	5,051	3,831	4,075	4,496	4,790
Totals	5,300	447	8.4%	4,132	6,907	5,282	5,579	6,056	6,421
Normal Dist.	5,300	447	8.4%			5,300	5,602	6,036	6,341
logNormal Dist.	5,300	448	8.4%			5,282	5,591	6,067	6,426
Gamma Dist.	5,300	447	8.4%			5,288	5,595	6,057	6,396

Figure A.2. Total Unpaid Claims Distribution (Paid Chain Ladder)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Total Unpaid Distribution
Paid Chain Ladder Model

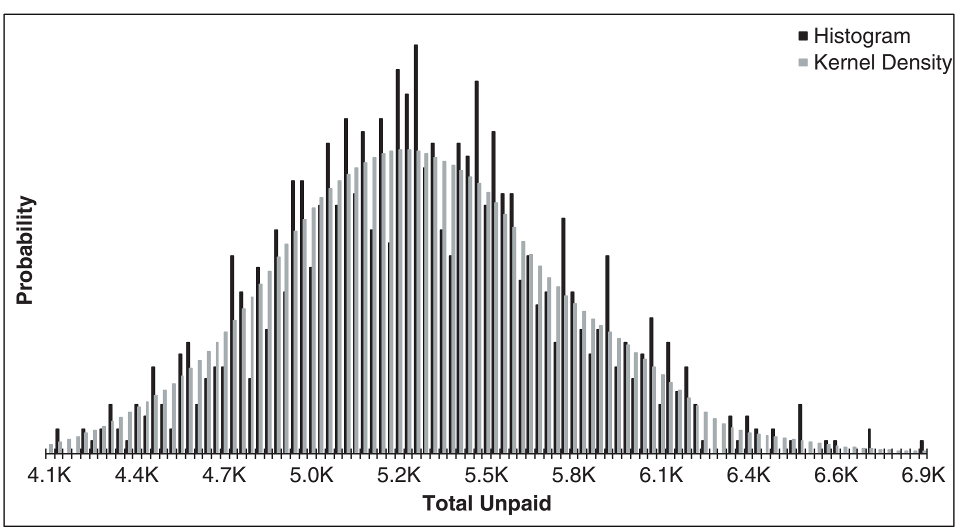


Figure A.3. Estimated Unpaid Model Results (Incurred Chain Ladder)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Incurred Chain Ladder Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	3	9	309.9%	-	93	0	1	17	48
2008	42	42	101.0%	-	306	30	56	126	189
2009	46	42	93.2%	1	325	33	57	135	205
2010	62	47	75.6%	4	355	52	83	149	253
2011	103	64	62.4%	12	473	89	129	231	338
2012	226	112	49.5%	43	984	202	276	435	587
2013	306	176	57.5%	36	1,449	271	384	621	860
2014	723	353	48.8%	109	2,452	664	884	1,418	1,842
2015	3,912	1,534	39.2%	1,306	10,236	3,694	4,523	6,708	9,175
Totals	5,422	1,575	29.0%	1,981	12,631	5,217	6,144	8,197	10,612
Normal Dist.	5,422	1,575	29.0%			5,422	6,485	8,013	9,086
logNormal Dist.	5,423	1,569	28.9%			5,209	6,307	8,305	10,076
Gamma Dist.	5,422	1,575	29.0%			5,271	6,386	8,246	9,741

Figure A.4. Total Unpaid Claims Distribution (Incurred Chain Ladder)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Total Unpaid Distribution
Incurred Chain Ladder Model

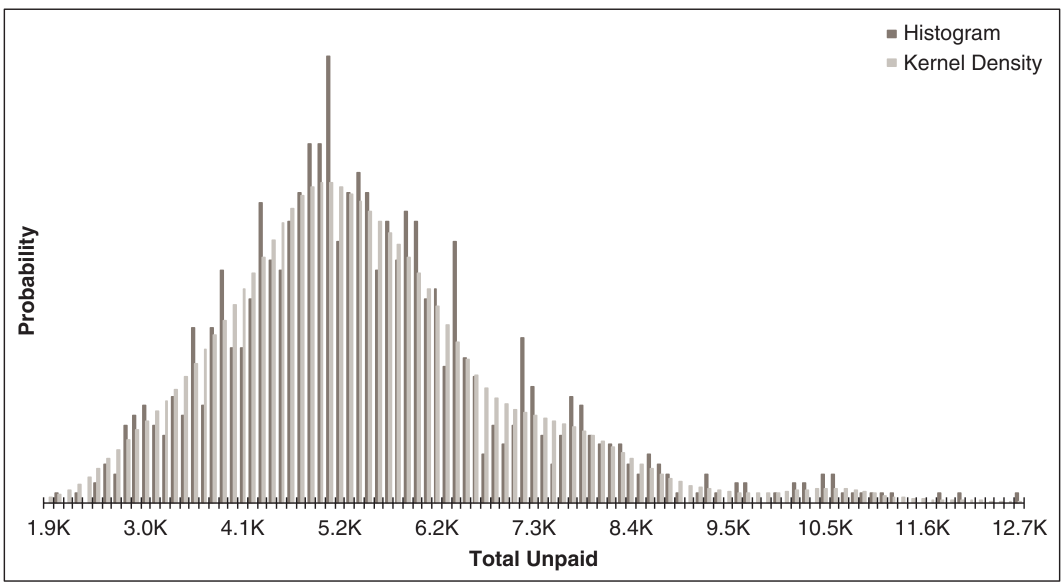


Figure A.5. Estimated Unpaid Model Results (Paid Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Paid Bornhuetter-Ferguson Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	2	6	310.2%	-	48	0	0	10	33
2008	28	25	89.2%	-	188	21	40	71	115
2009	37	26	69.7%	5	152	30	51	87	115
2010	60	31	52.2%	11	186	53	76	127	153
2011	96	34	35.7%	32	274	89	114	163	194
2012	169	53	31.3%	60	367	161	201	269	308
2013	327	88	26.9%	115	804	319	384	483	573
2014	722	157	21.8%	332	1,314	708	826	997	1,129
2015	2,660	383	14.4%	1,689	3,887	2,645	2,908	3,340	3,659
Totals	4,099	456	11.1%	2,835	5,789	4,096	4,392	4,849	5,218
Normal Dist.	4,099	456	11.1%			4,099	4,407	4,850	5,161
logNormal Dist.	4,099	458	11.2%			4,074	4,392	4,894	5,280
Gamma Dist.	4,099	456	11.1%			4,082	4,397	4,877	5,235

Figure A.6. Total Unpaid Claims Distribution (Paid Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Total Unpaid Distribution
Paid Bornhuetter-Ferguson Model

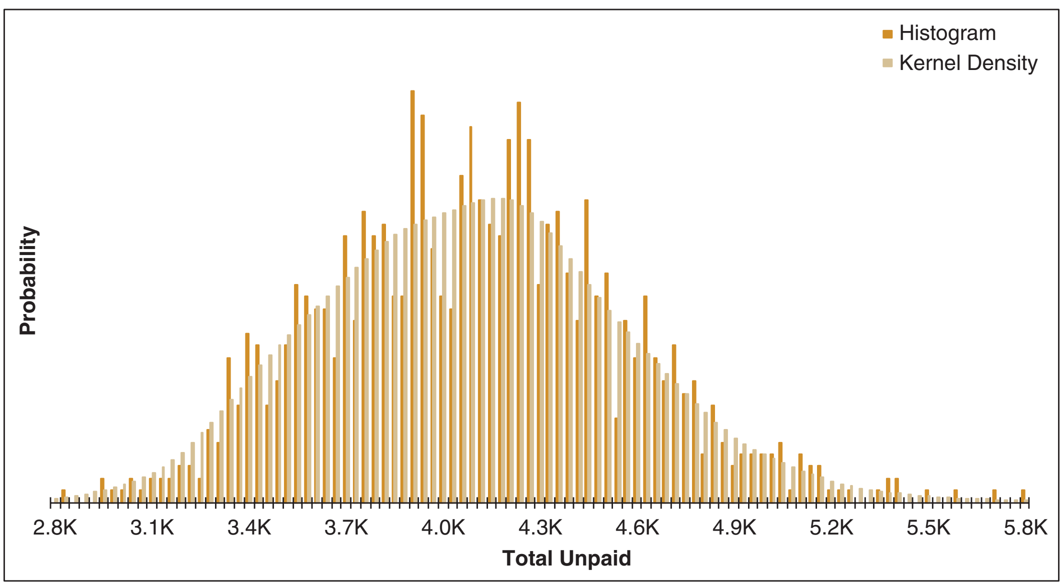


Figure A.7. Estimated Unpaid Model Results (Incurred Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Incurred Bornhuetter-Ferguson Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	2	7	318.6%	-	67	0	1	13	41
2008	27	30	109.3%	-	234	18	37	84	142
2009	39	36	93.5%	1	263	28	50	114	180
2010	59	46	78.0%	4	397	47	78	149	214
2011	98	63	64.6%	9	473	84	123	221	302
2012	168	86	51.4%	30	659	152	210	340	443
2013	334	198	59.3%	34	2,310	304	412	690	972
2014	753	384	51.0%	111	3,131	688	919	1,513	1,883
2015	2,885	1,168	40.5%	921	7,678	2,699	3,449	5,198	6,483
Totals	4,366	1,260	28.9%	1,873	9,804	4,224	5,048	6,860	8,182
Normal Dist.	4,366	1,260	28.9%			4,366	5,216	6,438	7,297
logNormal Dist.	4,367	1,272	29.1%			4,193	5,083	6,704	8,143
Gamma Dist.	4,366	1,260	28.9%			4,246	5,137	6,624	7,817

Figure A.8. Total Unpaid Claims Distribution (Incurred Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Total Unpaid Distribution
Incurred Bornhuetter-Ferguson Model

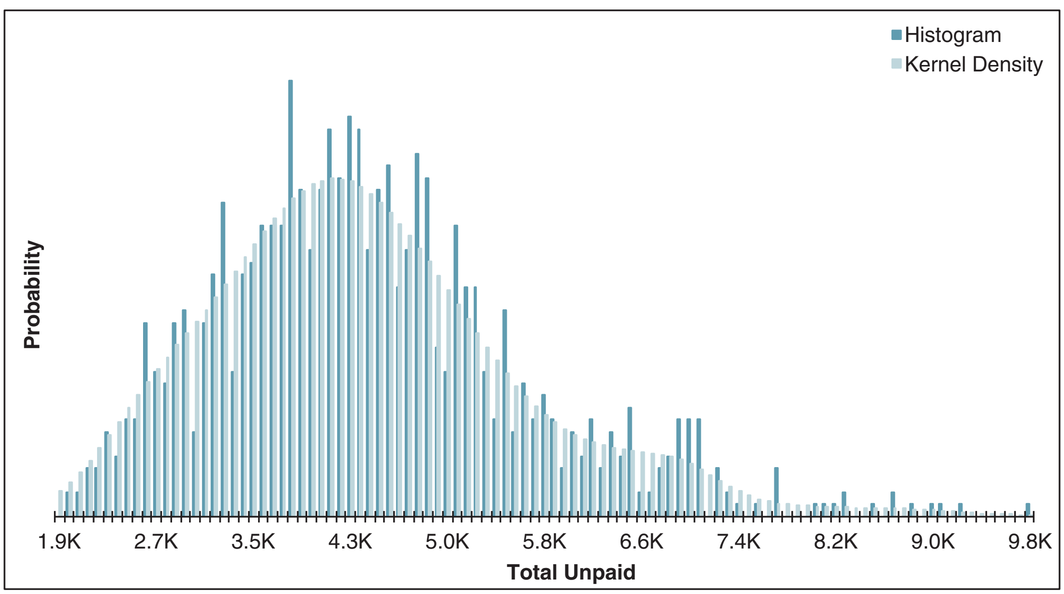


Figure A.9. Estimated Unpaid Model Results (Paid Cape Cod)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Paid Cape Cod Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	3	7	276.2%	-	59	0	1	17	38
2008	32	28	86.1%	-	178	25	45	89	125
2009	43	30	69.2%	6	259	36	59	97	137
2010	66	31	47.2%	16	225	59	85	122	166
2011	109	36	33.5%	43	283	102	130	176	213
2012	191	54	28.1%	74	401	184	226	288	337
2013	373	87	23.3%	156	719	366	424	525	600
2014	835	143	17.1%	407	1,520	832	921	1,082	1,192
2015	3,225	258	8.0%	2,384	4,098	3,227	3,389	3,659	3,855
Totals	4,878	384	7.9%	3,823	6,174	4,871	5,116	5,528	5,836
Normal Dist.	4,878	384	7.9%			4,878	5,137	5,510	5,772
logNormal Dist.	4,878	385	7.9%			4,863	5,128	5,536	5,841
Gamma Dist.	4,878	384	7.9%			4,868	5,132	5,527	5,816

Figure A.10. Total Unpaid Claims Distribution (Paid Cape Cod)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Total Unpaid Distribution
Paid Cape Cod Model

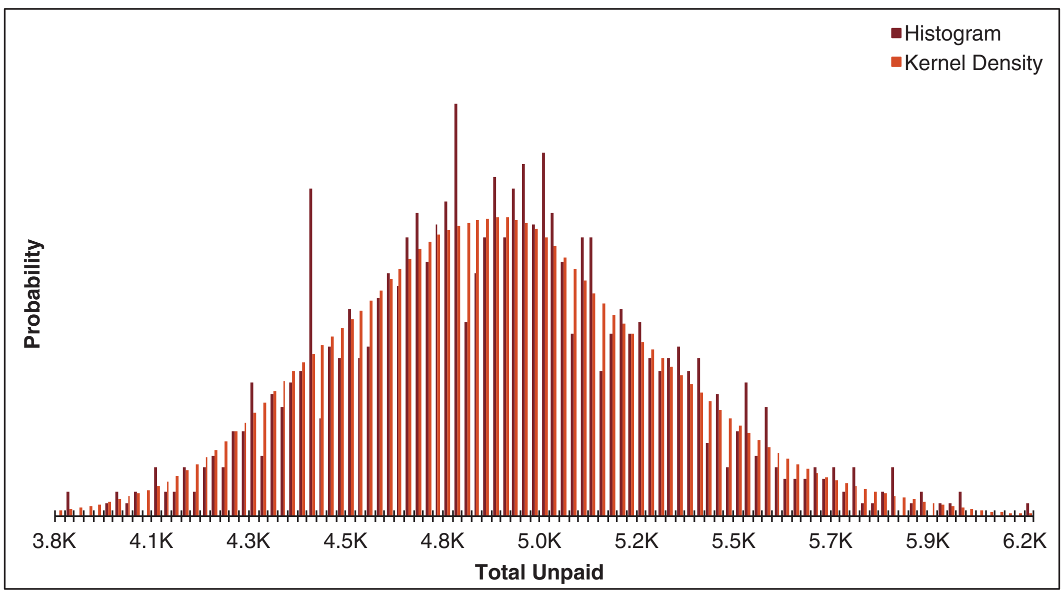


Figure A.11. Estimated Unpaid Model Results (Incurred Cape Cod)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Incurred Cape Cod Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	3	10	326.5%	-	117	0	1	17	50
2008	33	31	95.6%	-	213	24	46	91	148
2009	45	40	89.0%	1	317	33	61	122	184
2010	71	52	72.7%	3	375	58	91	174	251
2011	115	68	59.5%	16	512	102	146	242	366
2012	199	100	50.2%	31	933	181	252	388	499
2013	385	216	56.2%	46	1,629	343	477	812	1,081
2014	871	407	46.7%	132	3,029	802	1,049	1,658	2,191
2015	3,430	1,352	39.4%	1,074	9,190	3,253	3,977	5,946	7,972
Totals	5,151	1,417	27.5%	2,424	11,216	4,972	5,790	7,785	9,512
Normal Dist.	5,151	1,417	27.5%			5,151	6,107	7,482	8,448
logNormal Dist.	5,150	1,404	27.3%			4,969	5,953	7,719	9,264
Gamma Dist.	5,151	1,417	27.5%			5,022	6,023	7,682	9,007

Figure A.12. Total Unpaid Claims Distribution (Incurred Cape Cod)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Total Unpaid Distribution
Incurred Cape Cod Model

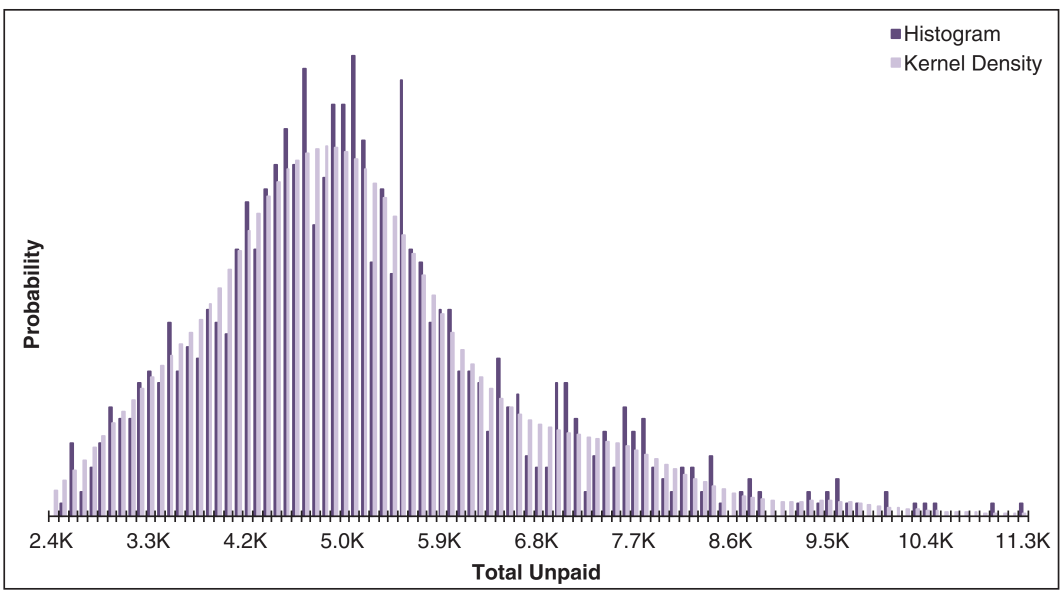


Figure A.13. Estimated Unpaid Model Results (Paid GLM)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Paid GLM Bootstrap Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	9	8	86.4%	0	53	7	13	24	32
2008	27	47	177.0%	0	436	12	24	109	253
2009	40	48	119.3%	2	537	27	44	117	270
2010	62	49	78.5%	11	525	51	69	136	287
2011	106	54	51.3%	31	559	94	117	202	347
2012	213	72	33.6%	79	731	201	242	333	455
2013	280	78	27.9%	100	707	271	325	418	507
2014	646	131	20.3%	337	1,368	634	730	871	979
2015	3,738	335	9.0%	2,696	4,939	3,731	3,953	4,307	4,583
Totals	5,120	447	8.7%	3,766	6,807	5,090	5,411	5,877	6,293
Normal Dist.	5,120	447	8.7%			5,120	5,422	5,856	6,161
logNormal Dist.	5,120	446	8.7%			5,101	5,409	5,886	6,246
Gamma Dist.	5,120	447	8.7%			5,107	5,415	5,878	6,218

Figure A.14. Total Unpaid Claims Distribution (Paid GLM)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Total Unpaid Distribution
Paid GLM Bootstrap Model

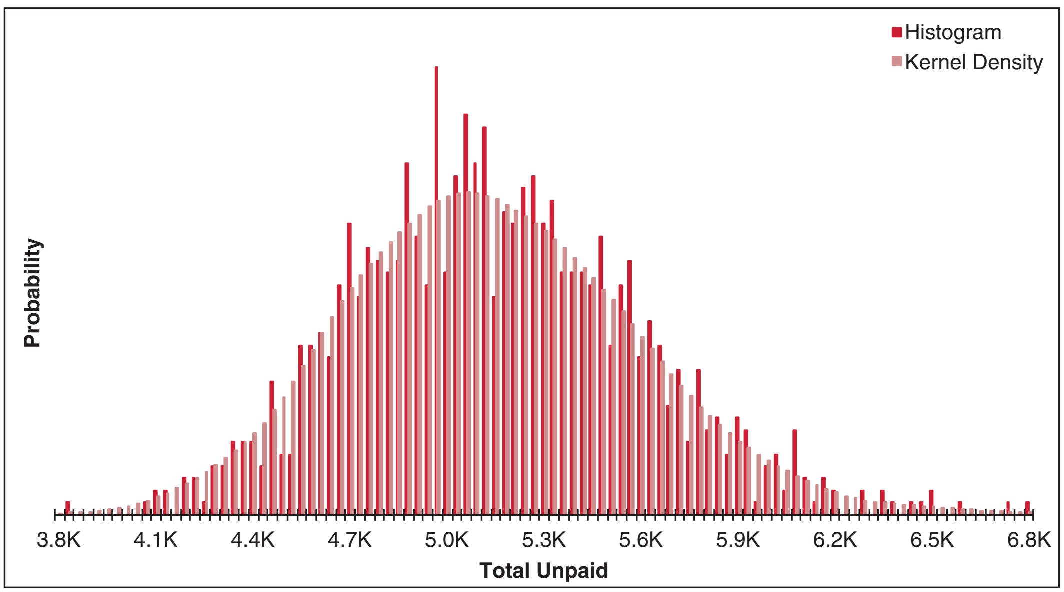


Figure A.15. Estimated Unpaid Model Results (Incurred GLM)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Incurred GLM Bootstrap Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	12	11	91.5%	0	66	8	16	34	48
2008	27	51	184.0%	0	520	12	25	111	262
2009	45	54	118.9%	3	678	31	52	117	268
2010	73	57	78.1%	11	892	59	85	150	301
2011	113	57	50.9%	30	771	101	128	215	360
2012	169	70	41.5%	53	712	153	198	288	415
2013	307	107	34.9%	93	1,550	293	362	491	615
2014	650	171	26.3%	280	2,713	630	743	928	1,057
2015	4,255	682	16.0%	2,581	6,888	4,216	4,670	5,413	6,295
Totals	5,650	751	13.3%	3,707	8,639	5,586	6,137	6,960	7,650
Normal Dist.	5,650	751	13.3%			5,650	6,157	6,886	7,398
logNormal Dist.	5,650	749	13.3%			5,601	6,123	6,960	7,616
Gamma Dist.	5,650	751	13.3%			5,617	6,137	6,940	7,543

Figure A.16. Total Unpaid Claims Distribution (Incurred GLM)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Total Unpaid Distribution
Incurred GLM Bootstrap Model

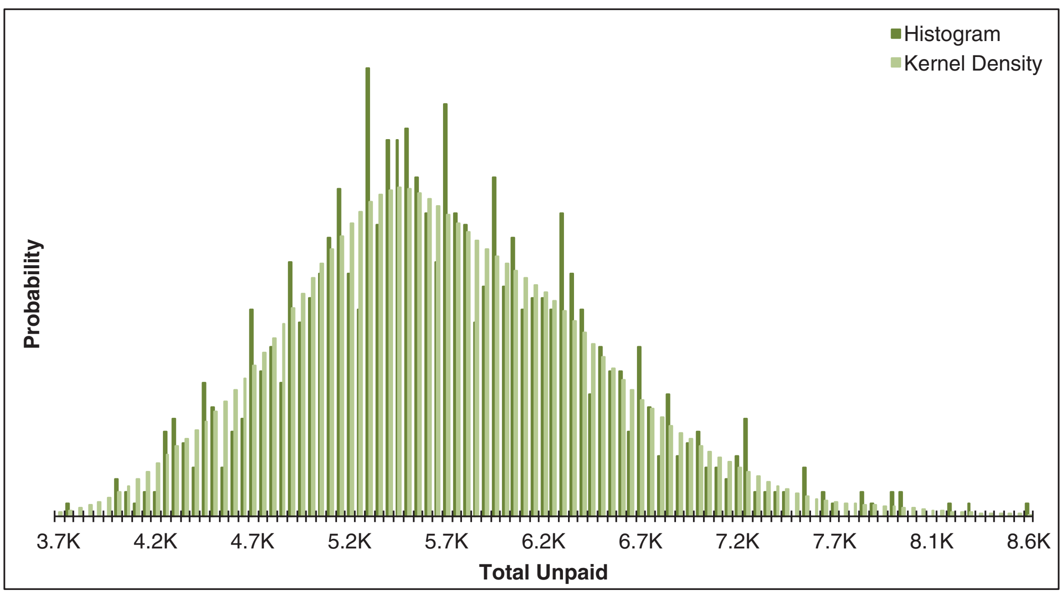


Figure A.17. Model Weights by Accident Year

Accident Year	Model Weights by Accident Year								
	Paid CL	Incd CL	Paid BF	Incd BF	Paid CC	Incd CC	Paid GLM	Incd GLM	TOTAL
2006	50.0%	50.0%							100.0%
2007	50.0%	50.0%							100.0%
2008	50.0%	50.0%							100.0%
2009	50.0%	50.0%							100.0%
2010	50.0%	50.0%							100.0%
2011	50.0%	50.0%							100.0%
2012	50.0%	50.0%							100.0%
2013	16.7%	16.7%	16.7%	16.7%	16.7%	16.7%			100.0%
2014	16.7%	16.7%	16.7%	16.7%	16.7%	16.7%			100.0%
2015	16.7%	16.7%			16.7%	16.7%	16.7%	16.7%	100.0%

Figure A.18. Estimated Mean Unpaid by Model

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Summary of Results by Model

Accident Year	Mean Estimated Unpaid								
	Chain Ladder		Bornhuetter-Ferguson		Cape Cod		GLM Bootstrap		Best Est.
	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred	(Weighted)
2006	-	-	-	-	-	-	-	-	-
2007	3	3	2	2	3	3	9	12	3
2008	41	42	28	27	32	33	27	27	41
2009	45	46	37	39	43	45	40	45	46
2010	63	62	60	59	66	71	62	73	64
2011	103	103	96	98	109	115	106	113	103
2012	222	226	169	168	191	199	213	169	224
2013	294	306	327	334	373	385	280	307	335
2014	679	723	722	753	835	871	646	650	752
2015	3,851	3,912	2,660	2,885	3,225	3,430	3,738	4,255	3,742
Totals	5,300	5,422	4,099	4,366	4,878	5,151	5,120	5,650	5,308

Figure A.19. Estimated Ranges

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Summary of Results by Model

Accident Year	Best Est. (Weighted)	Ranges			
		Weighted		Modeled	
		Minimum	Maximum	Mininum	Maximum
2006	-				
2007	3	3	3	2	12
2008	41	41	42	27	42
2009	46	45	46	37	46
2010	64	62	63	59	73
2011	103	103	103	96	115
2012	224	222	226	168	226
2013	335	294	385	280	385
2014	752	679	871	646	871
2015	3,742	3,225	4,255	2,660	4,255
Totals	5,308	4,674	5,992	4,099	5,650

Figure A.20. Reconciliation of Total Results (Weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Reconciliation of Total Results
Best Estimate (Weighted)

Accident Year	Paid To Date	Incurred To Date	Case Reserves	IBNR	Estimate of Ultimate	Estimate of Unpaid
2006	5,234	5,237	3	(3)	5,234	-
2007	6,470	6,479	9	(6)	6,473	3
2008	7,848	7,867	19	23	7,890	41
2009	7,020	7,046	26	20	7,066	46
2010	7,291	7,341	50	13	7,355	64
2011	8,134	8,225	91	12	8,237	103
2012	10,800	11,085	285	(61)	11,023	224
2013	7,522	7,810	288	46	7,856	335
2014	7,968	8,703	735	17	8,720	752
2015	9,309	12,788	3,478	263	13,051	3,742
Totals	77,596	82,580	4,984	324	82,905	5,308

Figure A.21. Estimated Unpaid Model Results (Weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Unpaid
Best Estimate (Weighted)

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	3	9	292.0%	-	173	0	1	17	42
2008	41	37	88.6%	-	391	32	57	111	168
2009	46	37	81.0%	1	522	36	60	114	175
2010	64	41	63.6%	4	537	55	81	139	205
2011	103	50	48.8%	10	636	94	125	193	276
2012	224	89	40.0%	36	917	211	266	382	529
2013	335	148	44.3%	25	1,460	315	401	594	865
2014	752	293	39.0%	106	2,881	725	873	1,265	1,789
2015	3,742	982	26.2%	1,094	10,700	3,654	4,118	5,392	7,059
Totals	5,308	1,044	19.7%	2,116	12,445	5,224	5,758	7,074	8,675
Normal Dist.	5,308	1,044	19.7%			5,308	6,013	7,026	7,738
logNormal Dist.	5,309	1,034	19.5%			5,211	5,935	7,158	8,164
Gamma Dist.	5,308	1,044	19.7%			5,240	5,971	7,135	8,035

Figure A.22. Estimated Cash Flow (Weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Calendar Year Unpaid
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	3,475	754	21.7%	1,297	8,420	3,414	3,797	4,730	5,948
2017	865	208	24.0%	293	2,148	843	982	1,224	1,483
2018	403	118	29.4%	115	1,298	387	467	614	740
2019	204	67	32.7%	56	654	194	240	325	412
2020	140	50	35.9%	40	539	132	165	233	297
2021	90	43	47.4%	12	611	82	112	169	229
2022	70	44	63.2%	6	409	60	91	152	215
2023	51	58	112.2%	-	735	36	75	151	253
2024	10	15	146.5%	-	199	4	15	41	67
Totals	5,308	1,044	19.7%	2,116	12,445	5,224	5,758	7,074	8,675

Figure A.23. Estimated Loss Ratio (Weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Ultimate Loss Ratios
Best Estimate (Weighted)

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	67.7%	28.5%	42.1%	0.4%	220.8%	66.1%	71.1%	130.9%	158.2%
2007	79.3%	30.2%	38.1%	8.2%	262.2%	77.8%	83.1%	145.5%	178.5%
2008	90.5%	31.2%	34.5%	16.9%	261.3%	89.0%	94.6%	159.9%	188.9%
2009	72.8%	26.8%	36.7%	10.2%	215.6%	71.4%	76.1%	131.7%	180.4%
2010	65.3%	23.3%	35.7%	10.2%	225.0%	63.8%	68.0%	116.1%	139.7%
2011	64.1%	21.2%	33.1%	13.0%	190.0%	63.2%	67.0%	111.8%	130.5%
2012	80.5%	24.0%	29.9%	25.0%	234.6%	79.0%	83.7%	132.9%	154.6%
2013	54.7%	18.8%	34.4%	9.9%	157.7%	53.9%	57.4%	96.2%	115.1%
2014	58.0%	19.2%	33.0%	13.0%	164.8%	57.1%	60.6%	99.8%	118.8%
2015	88.2%	21.5%	24.4%	30.9%	232.5%	85.5%	92.5%	127.9%	158.7%
Totals	71.3%	7.4%	10.4%	46.6%	112.7%	70.8%	75.7%	84.4%	91.7%

Figure A.24. Estimated Unpaid Claim Runoff (Weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Calendar Year Unpaid Claim Runoff
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	5,308	1,044	19.7%	2,116	12,445	5,224	5,758	7,074	8,675
2016	1,834	365	19.9%	746	4,128	1,797	2,030	2,459	2,957
2017	969	218	22.5%	336	2,316	946	1,088	1,353	1,627
2018	566	146	25.8%	159	1,393	548	647	828	1,004
2019	362	114	31.5%	79	1,171	347	424	565	718
2020	222	92	41.4%	35	956	207	269	386	524
2021	132	76	57.6%	6	863	117	166	268	394
2022	62	59	96.3%	(0)	745	46	84	166	269
2023	10	15	146.5%	(0)	199	4	15	41	67

Figure A.25. Mean Of Incremental Values (Weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Mean Values									
Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120 +**
2006	3,776	1,139	218	95	41	21	12	6	25	2
2007	4,635	1,398	268	115	51	25	15	7	31	3
2008	5,647	1,701	327	141	61	31	17	9	38	4
2009	5,065	1,525	294	126	56	28	16	8	34	3
2010	5,318	1,602	307	132	57	29	17	8	36	3
2011	5,882	1,774	340	145	64	32	18	9	40	4
2012	7,909	2,378	457	197	86	43	25	12	53	5
2013	5,589	1,683	323	156	68	35	20	10	42	4
2014	6,197	1,870	392	168	73	37	21	10	46	4
2015	9,615	2,744	521	222	92	53	33	20	47	10

Figure A.26. Standard Deviation of Incremental Values (Weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Standard Error Values									
Year	12	24	36	48	60	72	84	96	108	120 +
2006	1,597	502	119	64	33	11	7	2	23	4
2007	1,779	550	129	68	36	12	8	3	26	9
2008	1,960	603	147	77	38	13	8	3	35	10
2009	1,873	576	139	73	38	13	8	3	34	9
2010	1,906	596	143	75	38	13	9	3	34	9
2011	1,952	610	147	76	40	14	9	3	37	10
2012	2,375	733	173	92	49	17	11	4	44	11
2013	1,938	599	142	88	45	16	10	4	38	10
2014	2,054	639	173	90	47	16	10	4	41	11
2015	2,342	727	178	101	51	20	16	13	57	15

Figure A.27. Coefficient of Variation of Incremental Values (Weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Coefficients of Variation									
Year	12	24	36	48	60	72	84	96	108	120 +
2006	42.3%	44.1%	54.4%	67.8%	80.8%	52.6%	58.8%	43.2%	89.8%	157.5%
2007	38.4%	39.3%	48.2%	59.5%	71.1%	47.7%	52.9%	38.7%	82.4%	292.0%
2008	34.7%	35.5%	44.8%	54.5%	62.5%	43.0%	47.9%	34.9%	92.6%	266.2%
2009	37.0%	37.8%	47.3%	58.1%	68.6%	45.4%	50.3%	37.7%	98.4%	272.2%
2010	35.8%	37.2%	46.5%	56.8%	66.1%	44.8%	52.4%	36.5%	95.5%	279.7%
2011	33.2%	34.4%	43.1%	52.8%	62.4%	42.5%	49.3%	34.0%	92.6%	267.9%
2012	30.0%	30.8%	37.8%	46.6%	57.2%	38.4%	44.6%	30.6%	82.8%	234.2%
2013	34.7%	35.6%	43.9%	56.5%	66.6%	45.8%	51.2%	37.4%	91.1%	250.9%
2014	33.2%	34.2%	44.2%	53.4%	64.1%	43.4%	49.6%	36.0%	88.9%	253.1%
2015	24.4%	26.5%	34.2%	45.3%	55.8%	37.2%	47.4%	66.2%	120.2%	146.5%

Figure A.28. Total Unpaid Claims Distribution (Weighted)

Five Top 50 Companies
Schedule P, Part A -- Homeowners / Farmowners (in 000,000's)
Total Unpaid Distribution
Best Estimate (Weighted)

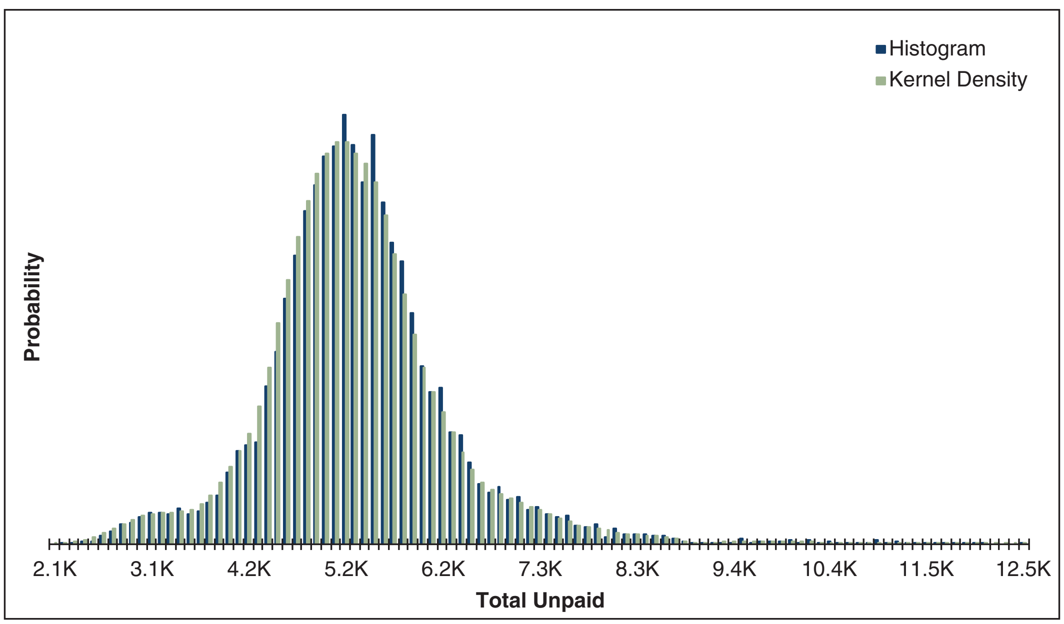
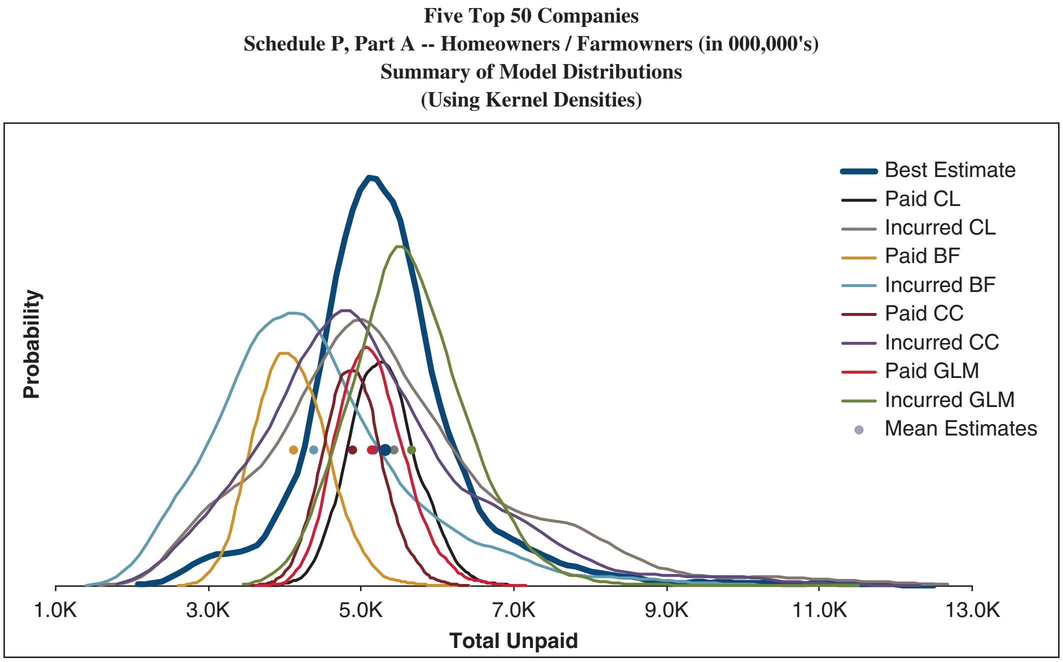


Figure A.29. Summary of Model Distributions



Appendix B—Schedule P, Part B Results

In this appendix the results for Schedule P, Part B (Private Passenger Auto Liability) are shown.

Figure B.1. Estimated Unpaid Model Results (Paid Chain Ladder)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Paid Chain Ladder Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	59	23	38.8%	-	125	58	75	97	112
2007	90	25	27.3%	26	164	90	107	131	147
2008	135	27	19.9%	64	217	134	153	178	196
2009	214	32	14.8%	128	322	213	237	265	289
2010	339	31	9.2%	252	443	340	361	390	413
2011	586	38	6.6%	459	707	585	610	651	687
2012	1,109	51	4.6%	949	1,281	1,108	1,144	1,191	1,226
2013	2,089	75	3.6%	1,868	2,329	2,090	2,140	2,211	2,252
2014	3,917	127	3.3%	3,457	4,357	3,919	4,002	4,129	4,203
2015	8,033	219	2.7%	7,335	8,667	8,042	8,175	8,399	8,532
Totals	16,573	385	2.3%	15,252	17,728	16,581	16,842	17,192	17,399
Normal Dist.	16,573	385	2.3%			16,573	16,833	17,207	17,469
logNormal Dist.	16,573	386	2.3%			16,569	16,831	17,216	17,491
Gamma Dist.	16,573	385	2.3%			16,570	16,831	17,212	17,482

Figure B.2. Total Unpaid Claims Distribution (Paid Chain Ladder)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Total Unpaid Distribution
Paid Chain Ladder Model

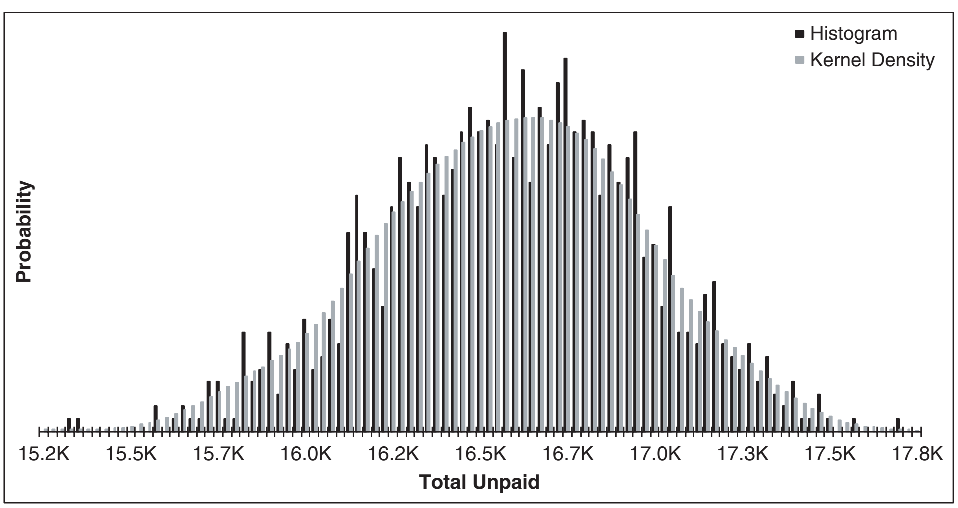


Figure B.3. Estimated Unpaid Model Results (Incurred Chain Ladder)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Incurred Chain Ladder Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	58	27	46.8%	-	156	56	77	103	131
2007	89	31	34.9%	17	212	87	108	146	170
2008	135	37	27.5%	48	278	133	159	196	226
2009	213	46	21.8%	106	397	210	246	290	326
2010	343	63	18.4%	178	560	342	387	445	492
2011	590	106	18.0%	304	886	590	661	764	823
2012	1,125	196	17.4%	610	2,320	1,133	1,265	1,439	1,502
2013	2,133	370	17.4%	1,167	3,115	2,165	2,404	2,722	2,846
2014	4,025	680	16.9%	2,324	5,470	4,078	4,514	5,076	5,298
2015	8,343	1,369	16.4%	4,886	12,352	8,502	9,290	10,413	10,940
Totals	17,054	1,620	9.5%	11,558	21,439	17,111	18,280	19,534	20,583
Normal Dist.	17,054	1,620	9.5%			17,054	18,147	19,719	20,824
logNormal Dist.	17,055	1,653	9.7%			16,976	18,120	19,902	21,257
Gamma Dist.	17,054	1,620	9.5%			17,003	18,117	19,804	21,048

Figure B.4. Total Unpaid Claims Distribution (Incurred Chain Ladder)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Total Unpaid Distribution
Incurred Chain Ladder Model

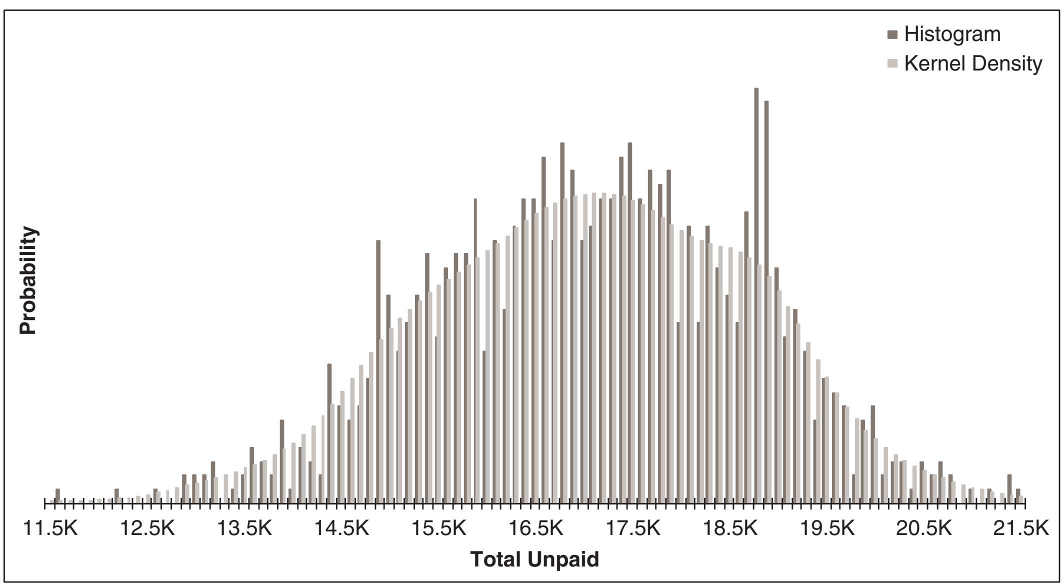


Figure B.5. Estimated Unpaid Model Results (Paid Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Paid Bornhuetter-Ferguson Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	54	22	40.2%	-	126	54	68	91	109
2007	76	22	28.7%	22	157	77	90	112	130
2008	112	24	21.2%	52	189	112	127	154	171
2009	188	30	16.0%	97	295	188	208	238	258
2010	343	36	10.4%	227	472	343	366	404	429
2011	625	50	8.0%	459	819	624	657	709	747
2012	1,162	77	6.7%	910	1,386	1,160	1,212	1,289	1,353
2013	2,217	134	6.1%	1,855	2,666	2,215	2,312	2,450	2,536
2014	3,942	218	5.5%	3,304	4,750	3,937	4,083	4,308	4,444
2015	7,990	441	5.5%	6,885	9,426	7,988	8,271	8,763	9,066
Totals	16,709	562	3.4%	15,239	18,369	16,701	17,096	17,695	18,035
Normal Dist.	16,709	562	3.4%			16,709	17,088	17,633	18,016
logNormal Dist.	16,709	561	3.4%			16,700	17,083	17,648	18,057
Gamma Dist.	16,709	562	3.4%			16,703	17,085	17,644	18,043

Figure B.6. Total Unpaid Claims Distribution (Paid Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Total Unpaid Distribution
Paid Bornhuetter-Ferguson Model

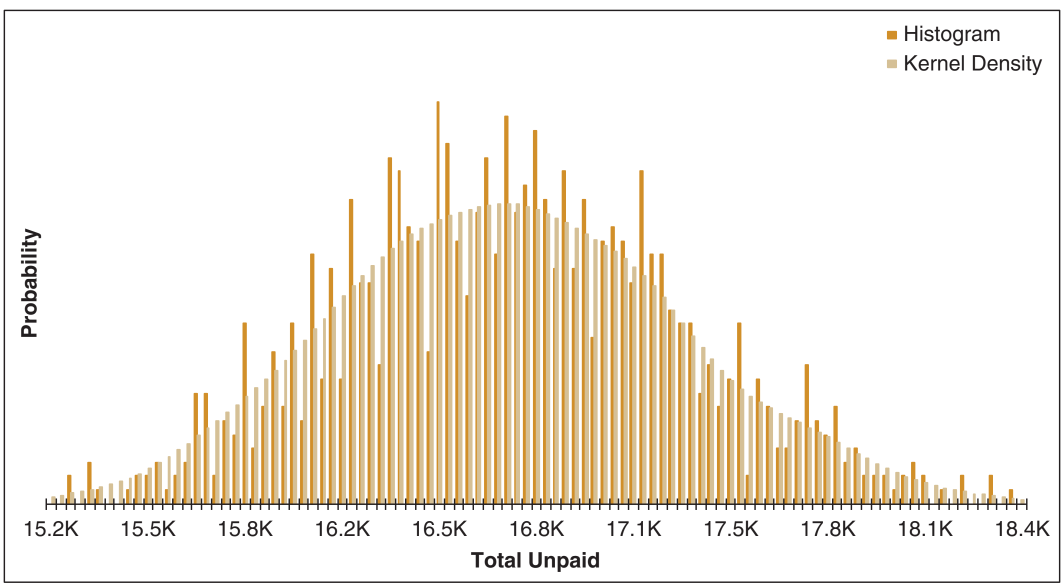


Figure B.7. Estimated Unpaid Model Results (Incurred Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Incurred Bornhuetter-Ferguson Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	54	24	45.4%	-	155	52	68	97	121
2007	76	25	33.1%	13	181	74	92	120	141
2008	111	30	27.2%	42	213	108	132	165	187
2009	188	42	22.5%	78	337	187	215	261	295
2010	344	68	19.7%	142	577	347	391	455	502
2011	627	116	18.5%	319	979	626	709	816	888
2012	1,167	217	18.6%	614	2,121	1,175	1,309	1,517	1,655
2013	2,234	420	18.8%	1,124	5,710	2,270	2,517	2,855	3,060
2014	3,997	689	17.2%	2,017	5,678	4,025	4,470	5,113	5,363
2015	8,289	1,370	16.5%	2,250	11,646	8,398	9,216	10,412	10,925
Totals	17,088	1,617	9.5%	10,942	22,273	17,177	18,198	19,785	20,539
Normal Dist.	17,088	1,617	9.5%			17,088	18,178	19,747	20,849
logNormal Dist.	17,089	1,648	9.6%			17,010	18,150	19,926	21,277
Gamma Dist.	17,088	1,617	9.5%			17,037	18,149	19,831	21,072

Figure B.8. Total Unpaid Claims Distribution (Incurred Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Total Unpaid Distribution
Incurred Bornhuetter-Ferguson Model

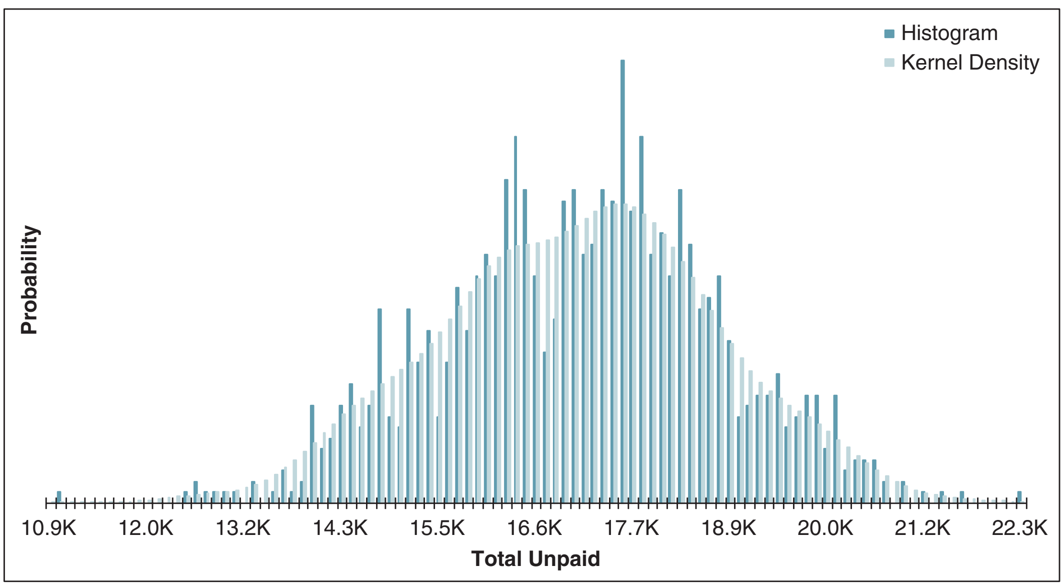


Figure B.9. Estimated Unpaid Model Results (Paid Cape Cod)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Paid Cape Cod Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	55	23	41.2%	-	136	55	70	94	108
2007	80	23	28.9%	23	161	79	95	118	133
2008	117	24	20.9%	57	205	117	134	159	175
2009	196	30	15.5%	116	305	195	216	247	270
2010	354	34	9.5%	263	459	353	377	410	436
2011	642	42	6.5%	513	773	642	670	710	738
2012	1,197	54	4.5%	1,042	1,365	1,198	1,234	1,288	1,331
2013	2,292	80	3.5%	2,045	2,553	2,294	2,345	2,424	2,474
2014	4,145	118	2.9%	3,761	4,502	4,145	4,219	4,345	4,439
2015	8,598	172	2.0%	8,057	9,073	8,596	8,711	8,894	8,987
Totals	17,676	376	2.1%	16,428	18,791	17,675	17,929	18,306	18,488
Normal Dist.	17,676	376	2.1%			17,676	17,930	18,295	18,551
logNormal Dist.	17,676	377	2.1%			17,672	17,928	18,302	18,570
Gamma Dist.	17,676	376	2.1%			17,674	17,929	18,300	18,563

Figure B.10. Total Unpaid Claims Distribution (Paid Cape Cod)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Total Unpaid Distribution
Paid Cape Cod Model

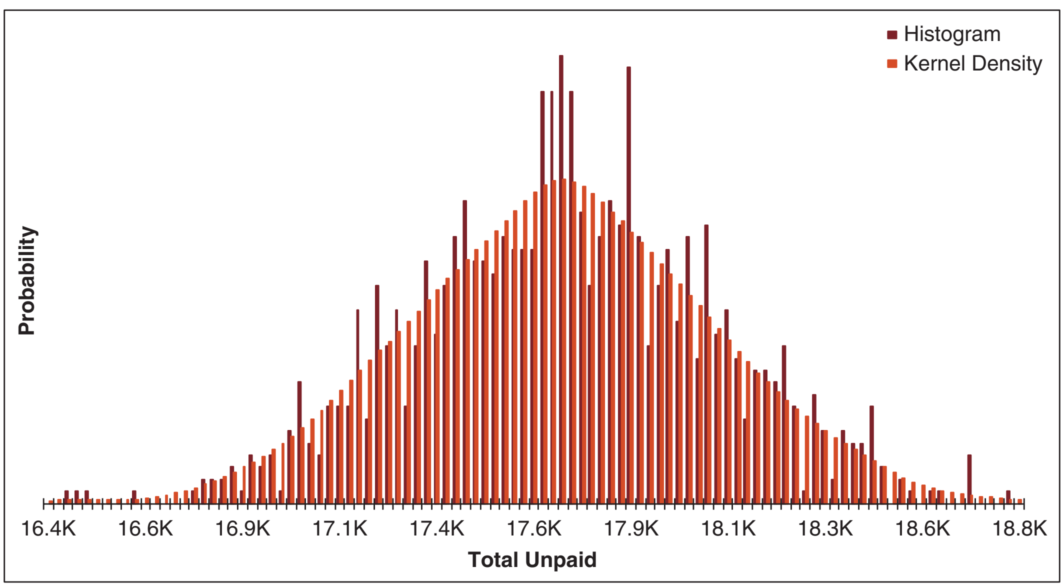


Figure B.11. Estimated Unpaid Model Results (Incurred Cape Cod)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Incurred Cape Cod Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	56	24	43.7%	-	133	54	71	96	114
2007	80	27	33.8%	18	175	79	98	126	147
2008	118	32	27.4%	35	230	116	138	175	203
2009	197	44	22.5%	90	351	194	228	271	309
2010	358	69	19.3%	184	544	358	407	474	509
2011	650	115	17.6%	324	953	647	730	843	900
2012	1,201	213	17.7%	675	1,697	1,224	1,352	1,534	1,632
2013	2,308	388	16.8%	1,247	3,598	2,335	2,579	2,939	3,074
2014	4,178	701	16.8%	2,248	5,709	4,247	4,697	5,271	5,516
2015	8,526	1,424	16.7%	4,605	11,643	8,725	9,508	10,707	11,151
Totals	17,672	1,677	9.5%	12,794	21,955	17,649	18,915	20,366	21,243
Normal Dist.	17,672	1,677	9.5%			17,672	18,803	20,430	21,573
logNormal Dist.	17,673	1,706	9.7%			17,591	18,772	20,610	22,008
Gamma Dist.	17,672	1,677	9.5%			17,619	18,772	20,517	21,805

Figure B.12. Total Unpaid Claims Distribution (Incurred Cape Cod)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Total Unpaid Distribution
Incurred Cape Cod Model

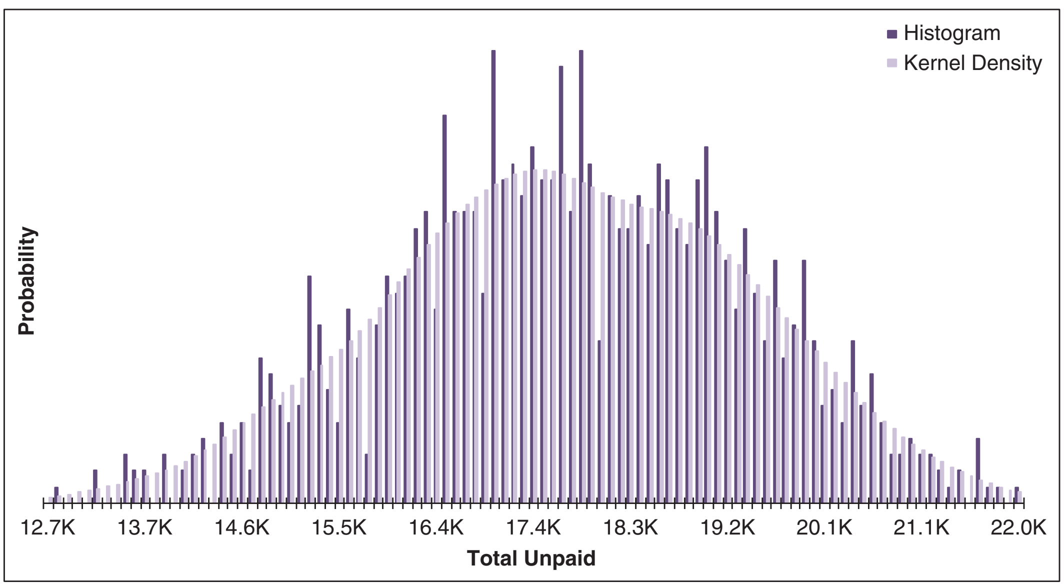


Figure B.13. Estimated Unpaid Model Results (Paid GLM)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Paid GLM Bootstrap Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	29	15	53.7%	2	106	26	37	58	79
2007	56	23	40.9%	7	158	53	69	98	120
2008	99	29	29.6%	29	223	96	116	151	179
2009	177	33	18.5%	99	317	173	198	233	260
2010	302	32	10.7%	200	450	299	324	356	377
2011	552	34	6.2%	465	740	550	573	613	643
2012	1,071	53	5.0%	914	1,288	1,067	1,107	1,162	1,197
2013	2,053	78	3.8%	1,831	2,295	2,052	2,106	2,180	2,244
2014	3,879	118	3.0%	3,525	4,361	3,875	3,955	4,080	4,177
2015	8,004	229	2.9%	7,329	8,746	7,999	8,165	8,380	8,509
Totals	16,222	369	2.3%	15,169	17,945	16,200	16,473	16,833	17,164
Normal Dist.	16,222	369	2.3%			16,222	16,471	16,829	17,080
logNormal Dist.	16,222	367	2.3%			16,218	16,468	16,834	17,096
Gamma Dist.	16,222	369	2.3%			16,220	16,469	16,833	17,092

Figure B.14. Total Unpaid Claims Distribution (Paid GLM)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Total Unpaid Distribution
Paid GLM Bootstrap Model

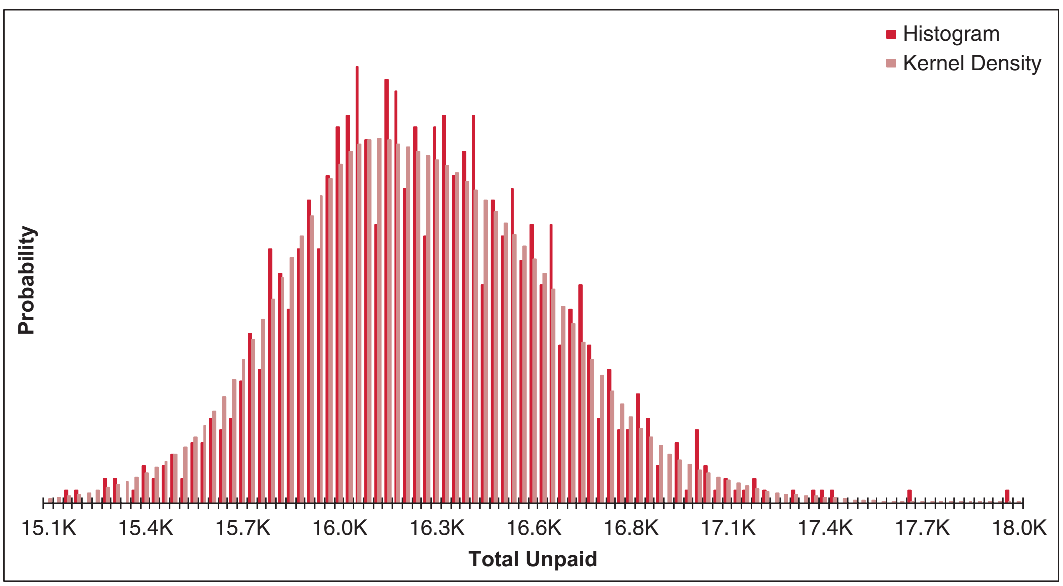


Figure B.15. Estimated Unpaid Model Results (Incurred GLM)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Incurred GLM Bootstrap Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	28	15	55.2%	3	110	25	35	58	76
2007	56	24	42.7%	7	178	53	69	102	138
2008	107	33	30.8%	43	298	101	127	168	200
2009	172	34	19.6%	91	301	169	191	235	263
2010	295	36	12.4%	204	419	290	316	361	394
2011	568	49	8.6%	434	764	565	597	652	702
2012	1,130	90	8.0%	857	1,422	1,126	1,189	1,285	1,332
2013	2,193	168	7.7%	1,738	2,884	2,193	2,307	2,468	2,605
2014	4,058	319	7.9%	3,096	5,040	4,063	4,294	4,573	4,764
2015	8,390	723	8.6%	5,922	10,670	8,375	8,917	9,524	9,986
Totals	16,996	985	5.8%	13,965	19,871	16,965	17,696	18,619	19,079
Normal Dist.	16,996	985	5.8%			16,996	17,660	18,616	19,287
logNormal Dist.	16,996	989	5.8%			16,967	17,645	18,669	19,424
Gamma Dist.	16,996	985	5.8%			16,977	17,649	18,647	19,371

Figure B.16. Total Unpaid Claims Distribution (Incurred GLM)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Total Unpaid Distribution
Incurred GLM Bootstrap Model

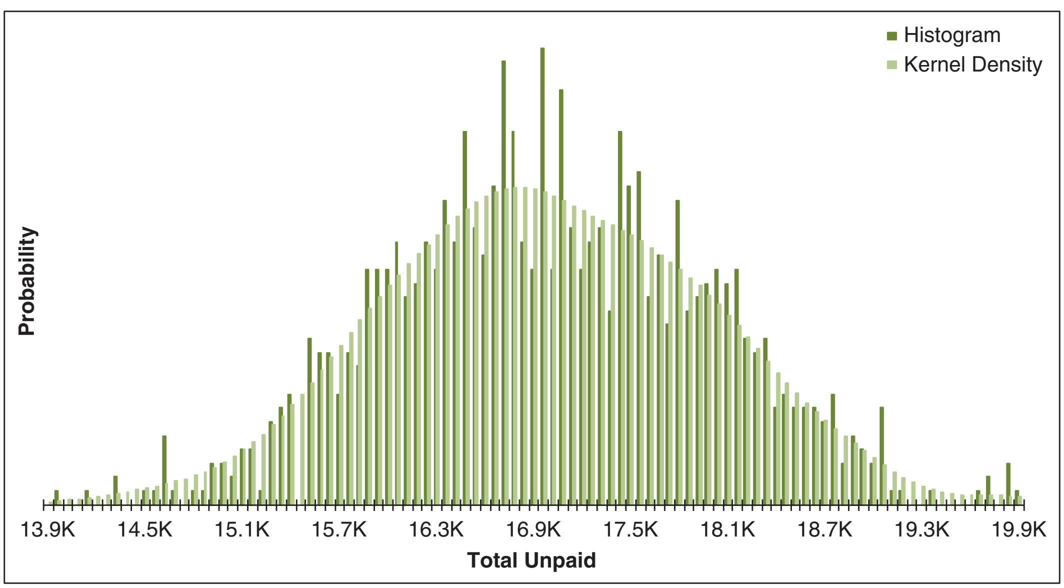


Figure B.17. Model Weights by Accident Year

Accident Year	Model Weights by Accident Year								
	Paid CL	Incd CL	Paid BF	Incd BF	Paid CC	Incd CC	Paid GLM	Incd GLM	TOTAL
2006	50.0%	50.0%							100.0%
2007	50.0%	50.0%							100.0%
2008	50.0%	50.0%							100.0%
2009	50.0%	50.0%							100.0%
2010			25.0%	25.0%	25.0%	25.0%			100.0%
2011			25.0%	25.0%	25.0%	25.0%			100.0%
2012			25.0%	25.0%	25.0%	25.0%			100.0%
2013			25.0%	25.0%	25.0%	25.0%			100.0%
2014		25.0%		25.0%		25.0%		25.0%	100.0%
2015		25.0%		25.0%		25.0%		25.0%	100.0%

Figure B.18. Estimated Mean Unpaid by Model

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Summary of Results by Model

Accident Year	Mean Estimated Unpaid								
	Chain Ladder		Bornhuetter-Ferguson		Cape Cod		GLM Bootstrap		Best Est.
	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred	(Weighted)
2006	59	58	54	54	55	56	29	28	59
2007	90	89	76	76	80	80	56	56	90
2008	135	135	112	111	117	118	99	107	134
2009	214	213	188	188	196	197	177	172	214
2010	339	343	343	344	354	358	302	295	351
2011	586	590	625	627	642	650	552	568	636
2012	1,109	1,125	1,162	1,167	1,197	1,201	1,071	1,130	1,184
2013	2,089	2,133	2,217	2,234	2,292	2,308	2,053	2,193	2,255
2014	3,917	4,025	3,942	3,997	4,145	4,178	3,879	4,058	4,077
2015	8,033	8,343	7,990	8,289	8,598	8,526	8,004	8,390	8,394
Totals	16,573	17,054	16,709	17,088	17,676	17,672	16,222	16,996	17,395

Figure B.19. Estimated Ranges

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Summary of Results by Model

Accident Year	Best Est. (Weighted)	Ranges			
		Weighted		Modeled	
		Minimum	Maximum	Mininum	Maximum
2006	59	58	59	28	59
2007	90	89	90	56	90
2008	134	135	135	107	135
2009	214	213	214	172	214
2010	351	343	358	295	358
2011	636	625	650	568	650
2012	1,184	1,162	1,201	1,109	1,201
2013	2,255	2,217	2,308	2,089	2,308
2014	4,077	3,997	4,178	3,917	4,178
2015	8,394	8,289	8,526	7,990	8,598
Totals	17,395	17,127	17,720	16,573	17,676

Figure B.20. Reconciliation of Total Results (Weighted)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Reconciliation of Total Results
Best Estimate (Weighted)

Accident Year	Paid To Date	Incurred To Date	Case Reserves	IBNR	Estimate of Ultimate	Estimate of Unpaid
2006	11,816	11,863	47	12	11,875	59
2007	12,679	12,752	72	18	12,770	90
2008	13,631	13,743	112	22	13,765	134
2009	14,472	14,687	216	(1)	14,686	214
2010	13,717	14,079	362	(11)	14,068	351
2011	13,090	13,691	600	36	13,726	636
2012	12,490	13,683	1,193	(9)	13,674	1,184
2013	11,598	13,912	2,313	(58)	13,854	2,255
2014	10,306	14,625	4,319	(243)	14,383	4,077
2015	6,357	15,188	8,830	(437)	14,751	8,394
Totals	120,157	138,223	18,066	(671)	137,551	17,395

Figure B.21. Estimated Unpaid Model Results (Weighted)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Unpaid
Best Estimate (Weighted)

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	59	25	42.2%	-	178	58	75	102	122
2007	90	28	30.8%	17	221	89	109	137	161
2008	134	32	24.0%	41	297	133	156	189	215
2009	214	41	18.9%	73	401	213	240	284	321
2010	351	55	15.6%	160	600	350	383	444	492
2011	636	91	14.2%	314	1,020	636	684	794	867
2012	1,184	157	13.3%	(27)	1,857	1,188	1,260	1,465	1,597
2013	2,255	293	13.0%	1,073	5,710	2,267	2,389	2,781	2,982
2014	4,077	616	15.1%	833	6,049	4,097	4,460	5,120	5,398
2015	8,394	1,234	14.7%	980	12,352	8,468	9,175	10,444	10,911
Totals	17,395	1,428	8.2%	10,057	23,150	17,439	18,375	19,729	20,525
Normal Dist.	17,395	1,428	8.2%			17,395	18,358	19,744	20,717
logNormal Dist.	17,395	1,451	8.3%			17,335	18,336	19,879	21,040
Gamma Dist.	17,395	1,428	8.2%			17,356	18,336	19,809	20,889

Figure B.22. Estimated Cash Flow (Weighted)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Calendar Year Unpaid
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	8,275	715	8.6%	4,501	10,746	8,299	8,761	9,426	9,838
2017	4,072	340	8.4%	2,450	5,608	4,079	4,304	4,621	4,845
2018	2,266	198	8.7%	1,319	3,149	2,267	2,397	2,590	2,718
2019	1,210	109	9.0%	699	1,574	1,210	1,285	1,389	1,461
2020	638	58	9.1%	405	885	638	677	735	778
2021	358	35	9.8%	203	511	358	381	416	439
2022	217	30	13.7%	95	351	216	237	267	291
2023	144	25	17.2%	57	258	144	161	186	205
2024	99	23	23.4%	16	214	98	114	139	157
2025	67	22	33.1%	-	157	66	81	106	124
2026	32	13	40.8%	-	91	32	41	55	66
2027	16	9	57.3%	-	57	15	22	31	38
Totals	17,395	1,428	8.2%	10,057	23,150	17,439	18,375	19,729	20,525

Figure B.23. Estimated Loss Ratio (Weighted)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Ultimate Loss Ratios
Best Estimate (Weighted)

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	75.6%	9.7%	12.9%	38.9%	104.5%	75.8%	77.8%	94.6%	99.4%
2007	82.2%	10.3%	12.5%	43.9%	114.0%	82.4%	84.6%	102.2%	107.6%
2008	83.9%	10.2%	12.1%	45.1%	114.4%	83.9%	86.3%	103.7%	108.6%
2009	79.6%	9.2%	11.6%	45.2%	108.3%	79.7%	81.8%	97.8%	102.7%
2010	69.3%	8.2%	11.9%	37.9%	94.6%	69.1%	71.1%	85.3%	90.1%
2011	66.0%	8.1%	12.3%	35.2%	89.7%	66.0%	67.9%	81.7%	85.9%
2012	66.9%	8.1%	12.1%	-1.5%	94.6%	66.9%	68.8%	82.7%	86.8%
2013	66.9%	8.1%	12.2%	35.2%	186.1%	66.9%	68.9%	82.4%	86.3%
2014	71.9%	10.6%	14.7%	14.4%	101.9%	72.7%	78.5%	89.0%	93.5%
2015	73.0%	10.6%	14.5%	8.4%	110.0%	73.9%	79.8%	90.3%	94.2%
Totals	72.9%	3.0%	4.1%	61.6%	90.9%	73.0%	75.0%	77.7%	79.5%

Figure B.24. Estimated Unpaid Claim Runoff (Weighted)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Calendar Year Unpaid Claim Runoff
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	17,395	1,428	8.2%	10,057	23,150	17,439	18,375	19,729	20,525
2016	9,120	739	8.1%	5,556	12,446	9,136	9,623	10,325	10,767
2017	5,048	419	8.3%	3,106	6,838	5,054	5,330	5,738	6,000
2018	2,782	243	8.7%	1,709	3,689	2,781	2,945	3,184	3,360
2019	1,572	157	10.0%	902	2,165	1,570	1,675	1,838	1,951
2020	934	117	12.6%	494	1,387	930	1,011	1,131	1,224
2021	576	94	16.3%	247	988	573	638	733	807
2022	359	75	21.0%	104	687	356	408	488	546
2023	214	59	27.6%	30	467	211	252	317	365
2024	115	41	36.0%	(0)	283	112	142	188	222
2025	48	20	42.4%	(0)	137	47	62	84	101
2026	16	9	57.3%	(0)	57	15	22	31	38
2027	(0)	0	-10502.4%	(0)	0	(0)	0	0	0

Figure B.25. Mean of Incremental Values (Weighted)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Mean Values												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156 +
2006	5,232	3,354	1,456	842	457	224	113	58	32	25	30	15	15
2007	5,631	3,608	1,566	907	491	241	121	62	34	27	32	16	16
2008	6,082	3,902	1,691	981	530	261	131	67	37	29	34	17	17
2009	6,480	4,155	1,802	1,043	565	278	139	71	39	31	36	18	18
2010	6,225	3,992	1,732	1,002	543	267	138	71	39	31	36	18	18
2011	6,043	3,876	1,681	974	527	280	141	72	40	31	36	18	18
2012	6,008	3,851	1,671	968	560	274	138	71	39	31	36	18	18
2013	6,046	3,876	1,681	1,051	569	279	140	72	40	31	36	18	18
2014	6,453	4,138	1,821	1,055	572	281	141	72	41	30	32	17	16
2015	6,549	4,261	1,847	1,070	579	284	143	73	41	31	32	17	16

Figure B.26. Standard Deviation of Incremental Values (Weighted)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Standard Error Values												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156 +
2006	677	440	199	115	65	35	15	12	4	3	12	6	6
2007	708	460	207	120	68	36	16	13	4	3	13	7	7
2008	742	484	217	126	70	38	16	13	5	4	14	7	7
2009	756	493	220	129	72	39	17	16	5	4	15	8	8
2010	745	485	218	127	71	38	18	16	5	4	15	8	8
2011	747	486	218	128	71	44	19	16	5	4	15	8	8
2012	729	475	213	124	78	42	18	16	5	4	15	8	8
2013	741	483	218	142	79	43	19	16	5	4	15	8	8
2014	955	618	282	165	92	47	22	18	8	7	17	9	9
2015	966	634	280	166	92	48	22	18	9	7	17	9	9

Figure B.27. Coefficient of Variation of Incremental Values (Weighted)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Coefficients of Variation												
Year	12	24	36	48	60	72	84	96	108	120	132	144	156 +
2006	12.9%	13.1%	13.6%	13.7%	14.2%	15.5%	13.4%	21.2%	12.9%	12.9%	42.1%	42.2%	42.3%
2007	12.6%	12.7%	13.2%	13.3%	13.8%	15.0%	13.0%	20.6%	12.5%	12.6%	42.0%	42.1%	42.2%
2008	12.2%	12.4%	12.8%	12.8%	13.3%	14.5%	12.6%	19.9%	12.2%	12.3%	42.3%	42.4%	42.5%
2009	11.7%	11.9%	12.2%	12.4%	12.7%	13.9%	12.1%	22.0%	11.7%	11.7%	42.0%	42.1%	42.2%
2010	12.0%	12.2%	12.6%	12.6%	13.1%	14.3%	13.1%	22.5%	12.5%	12.6%	42.5%	42.6%	42.7%
2011	12.4%	12.5%	13.0%	13.1%	13.5%	15.6%	13.4%	22.5%	12.9%	12.9%	42.5%	42.7%	42.8%
2012	12.1%	12.3%	12.8%	12.9%	13.9%	15.3%	13.2%	22.4%	12.6%	12.7%	42.3%	42.5%	42.5%
2013	12.3%	12.5%	13.0%	13.5%	13.9%	15.4%	13.2%	22.3%	12.7%	12.7%	42.0%	42.2%	42.2%
2014	14.8%	14.9%	15.5%	15.7%	16.1%	16.8%	15.4%	24.5%	20.8%	23.7%	52.6%	51.2%	57.5%
2015	14.7%	14.9%	15.2%	15.5%	15.9%	16.7%	15.2%	24.4%	20.6%	23.4%	52.1%	51.1%	57.3%

Figure B.28. Total Unpaid Claims Distribution (Weighted)

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Total Unpaid Distribution
Best Estimate (Weighted)

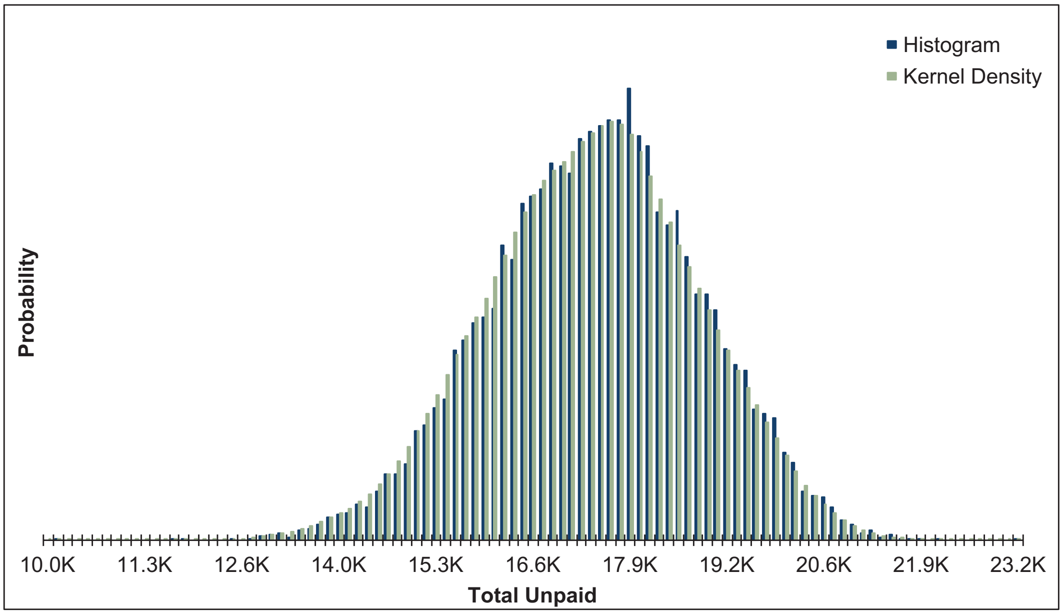
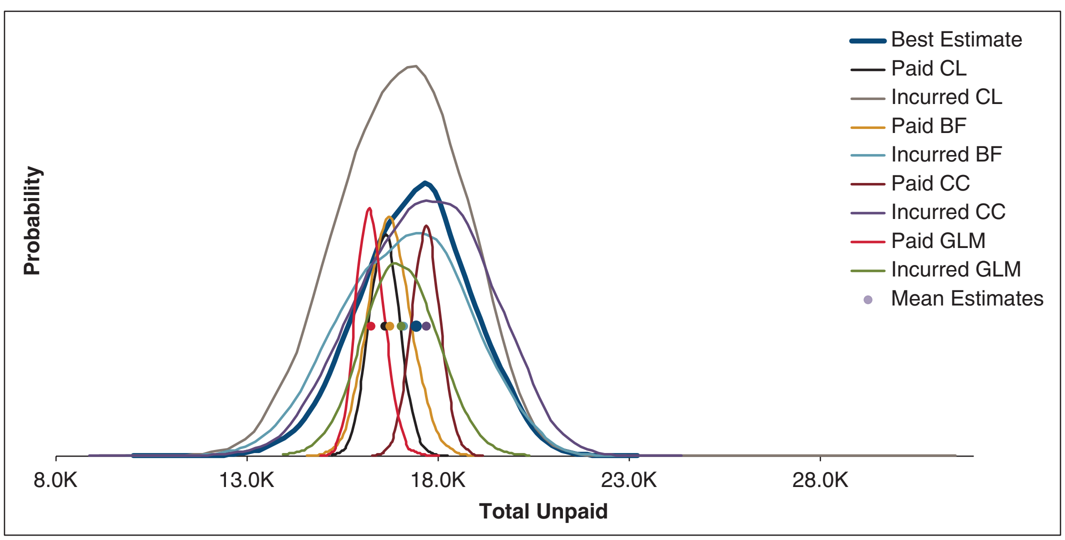


Figure B.29. Summary of Model Distributions

Five Top 50 Companies
Schedule P, Part B -- Private Passenger Auto Liability (in 000,000's)
Summary of Model Distributions
(Using Kernel Densities)



Appendix C—Schedule P, Part C Results

In this appendix the results for Schedule P, Part C (Commercial Auto Liability) are shown.

Figure C.1. Estimated Unpaid Model Results (Paid Chain Ladder)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Paid Chain Ladder Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	8	4	50.6%	-	22	8	10	15	19
2007	11	4	39.9%	(0)	28	10	13	18	22
2008	21	5	24.3%	7	43	21	24	29	34
2009	35	6	18.3%	18	66	34	39	46	51
2010	61	10	16.6%	34	97	60	67	80	87
2011	110	22	20.0%	57	195	107	124	150	173
2012	216	33	15.4%	111	359	215	237	273	296
2013	410	39	9.4%	294	550	408	434	474	513
2014	773	52	6.7%	610	946	770	806	863	901
2015	1,103	75	6.8%	872	1,345	1,100	1,152	1,232	1,285
Totals	2,746	122	4.4%	2,357	3,171	2,741	2,830	2,951	3,019
Normal Dist.	2,746	122	4.4%			2,746	2,828	2,946	3,029
logNormal Dist.	2,746	122	4.4%			2,743	2,827	2,951	3,041
Gamma Dist.	2,746	122	4.4%			2,744	2,827	2,949	3,037

Figure C.2. Total Unpaid Claims Distribution (Paid Chain Ladder)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Paid Chain Ladder Model

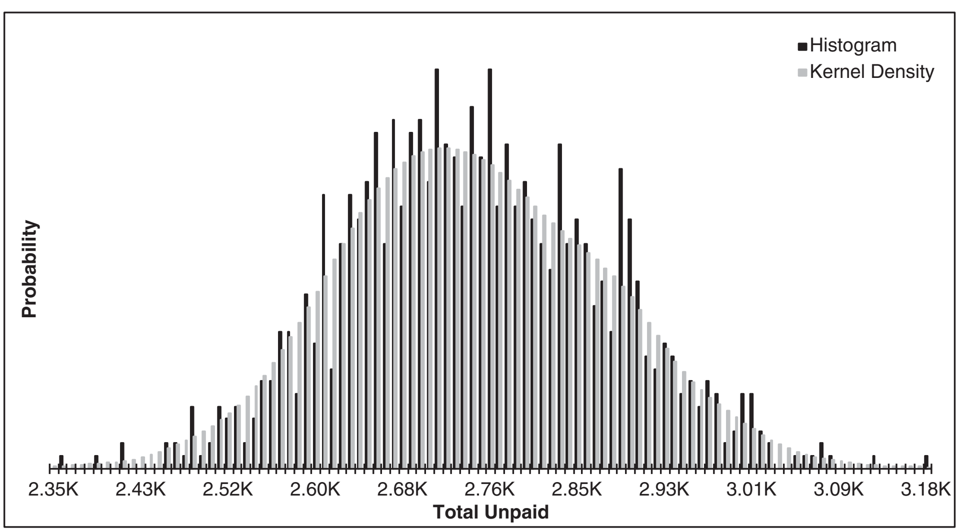


Figure C.3. Estimated Unpaid Model Results (Incurred Chain Ladder)

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Incurred Chain Ladder Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	11	12	108.0%	-	74	7	16	35	48
2007	15	16	110.1%	0	157	9	22	46	66
2008	31	33	105.0%	-	354	23	47	91	127
2009	53	54	102.3%	-	533	41	86	144	200
2010	92	103	111.1%	-	1,654	69	145	258	369
2011	168	176	104.5%	-	1,625	127	264	498	681
2012	328	372	113.3%	-	4,031	217	528	963	1,307
2013	623	615	98.7%	-	3,767	484	1,049	1,782	2,238
2014	1,223	1,415	115.7%	-	21,802	1,019	2,010	3,319	4,335
2015	1,513	1,618	107.0%	-	13,830	1,062	2,546	4,356	5,798
Totals	4,056	2,421	59.7%	146	30,092	3,725	5,273	7,786	10,983
Normal Dist.	4,056	2,421	59.7%			4,056	5,689	8,038	9,687
logNormal Dist.	4,168	2,899	69.6%			3,422	5,227	9,616	14,755
Gamma Dist.	4,056	2,421	59.7%			3,586	5,328	8,677	11,670

Figure C.4. Total Unpaid Claims Distribution (Incurred Chain Ladder)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Incurred Chain Ladder Model

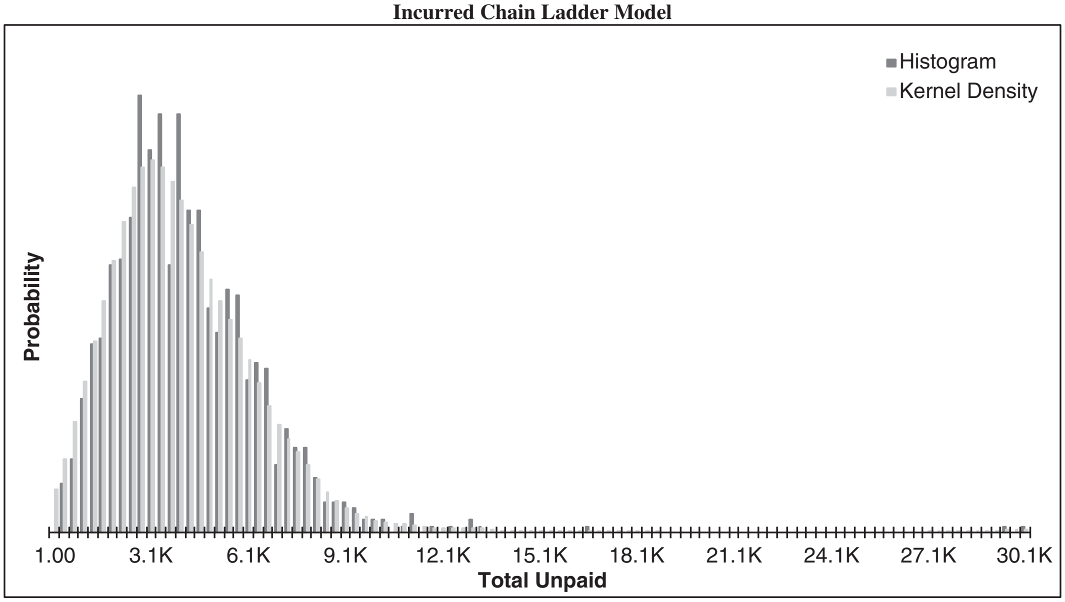


Figure C.5. Estimated Unpaid Model Results (Paid Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Paid Bornhuetter-Ferguson Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	5	3	54.4%	-	16	5	7	10	13
2007	8	3	42.3%	0	22	8	10	14	17
2008	17	4	26.7%	5	32	17	20	25	29
2009	35	7	19.3%	13	64	34	39	46	52
2010	65	11	17.1%	38	110	65	73	84	94
2011	123	25	20.5%	44	211	121	140	167	197
2012	259	40	15.6%	145	420	256	287	327	353
2013	481	52	10.8%	315	658	477	517	565	607
2014	812	76	9.3%	590	1,078	811	860	936	996
2015	1,132	100	8.9%	857	1,480	1,127	1,198	1,300	1,369
Totals	2,936	153	5.2%	2,472	3,474	2,939	3,040	3,180	3,313
Normal Dist.	2,936	153	5.2%			2,936	3,040	3,188	3,293
logNormal Dist.	2,936	154	5.2%			2,932	3,038	3,196	3,312
Gamma Dist.	2,936	153	5.2%			2,934	3,038	3,193	3,305

Figure C.6. Total Unpaid Claims Distribution (Paid Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Paid Bornhuetter-Ferguson Model

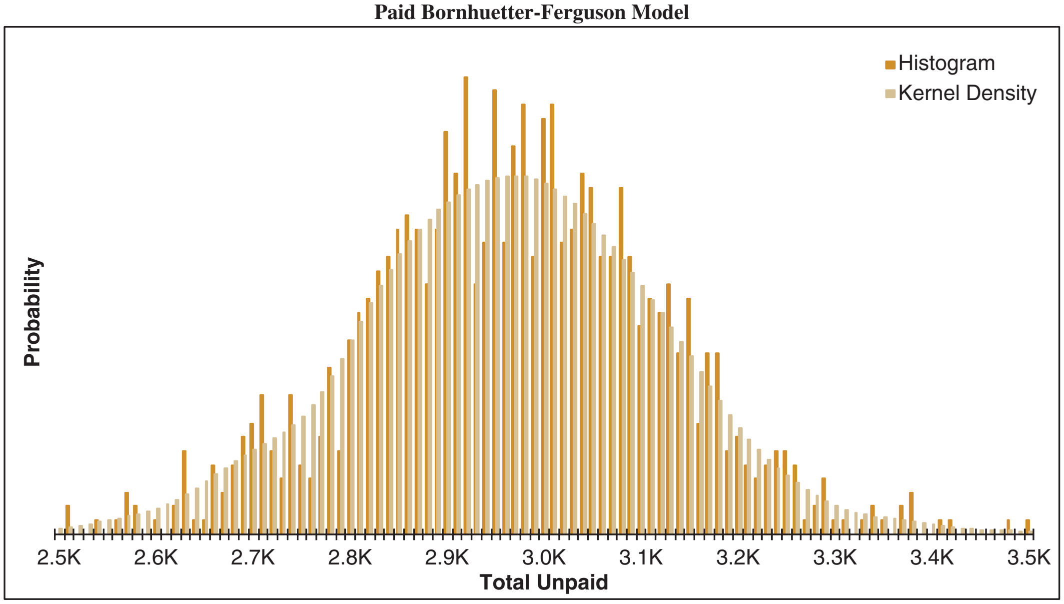


Figure C.7. Estimated Unpaid Model Results (Incurred Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Incurred Bornhuetter-Ferguson Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	7	8	116.0%	-	48	4	10	24	34
2007	11	12	110.5%	-	61	7	15	36	52
2008	24	23	96.0%	-	124	18	37	68	93
2009	49	45	92.9%	-	216	38	80	139	165
2010	99	88	88.8%	0	375	82	162	265	318
2011	176	164	93.3%	0	821	134	279	505	630
2012	362	338	93.5%	0	1,547	296	584	1,005	1,228
2013	642	597	93.1%	1	2,344	502	1,066	1,792	2,119
2014	1,118	996	89.1%	0	4,243	980	1,862	2,919	3,447
2015	1,554	1,409	90.6%	0	5,956	1,371	2,626	4,146	4,729
Totals	4,040	1,873	46.4%	387	10,575	3,901	5,304	7,418	8,445
Normal Dist.	4,040	1,873	46.4%			4,040	5,303	7,120	8,397
logNormal Dist.	4,116	2,390	58.1%			3,560	5,120	8,638	12,472
Gamma Dist.	4,040	1,873	46.4%			3,755	5,099	7,530	9,612

Figure C.8. Total Unpaid Claims Distribution (Incurred Bornhuetter-Ferguson)

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Incurred Bornhuetter-Ferguson Model

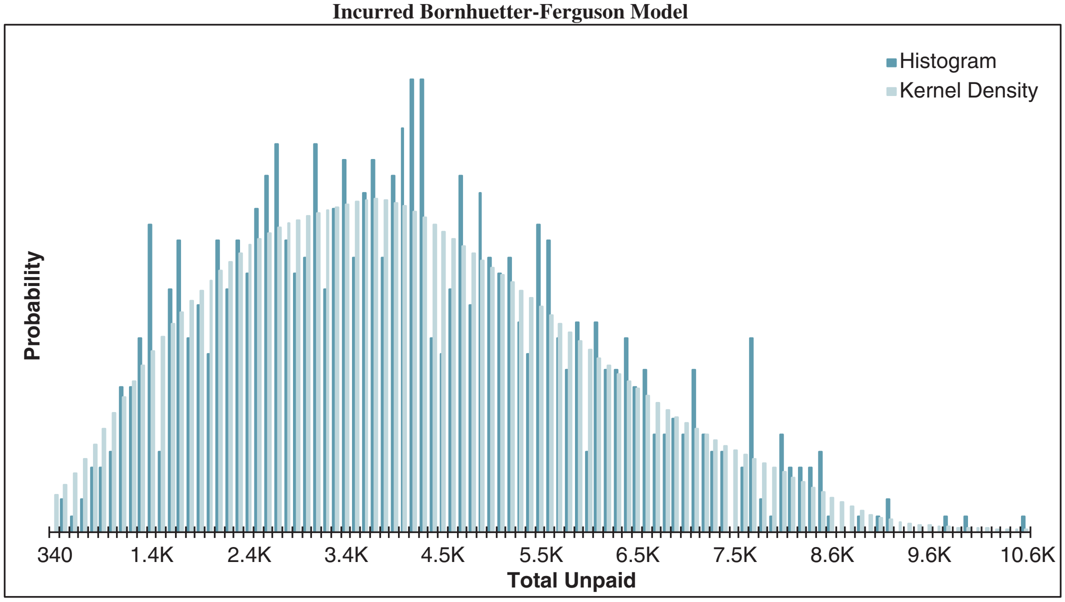


Figure C.9. Estimated Unpaid Model Results (Paid Cape Cod)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Paid Cape Cod Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	6	3	52.3%	-	17	6	8	12	14
2007	9	4	41.0%	0	26	9	11	15	19
2008	18	5	26.1%	7	34	18	22	27	31
2009	36	7	17.9%	20	59	36	41	48	52
2010	67	11	16.1%	39	101	66	74	86	94
2011	124	23	18.8%	67	245	122	138	163	192
2012	258	38	14.8%	166	416	255	283	323	359
2013	481	40	8.4%	363	629	478	509	548	583
2014	827	50	6.0%	684	975	827	858	915	948
2015	1,178	53	4.5%	990	1,348	1,176	1,212	1,268	1,308
Totals	3,004	122	4.0%	2,559	3,428	3,001	3,088	3,204	3,297
Normal Dist.	3,004	122	4.0%			3,004	3,086	3,204	3,286
logNormal Dist.	3,004	121	4.0%			3,001	3,084	3,208	3,297
Gamma Dist.	3,004	122	4.0%			3,002	3,085	3,206	3,294

Figure C.10. Total Unpaid Claims Distribution (Paid Cape Cod)

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Paid Cape Cod Model

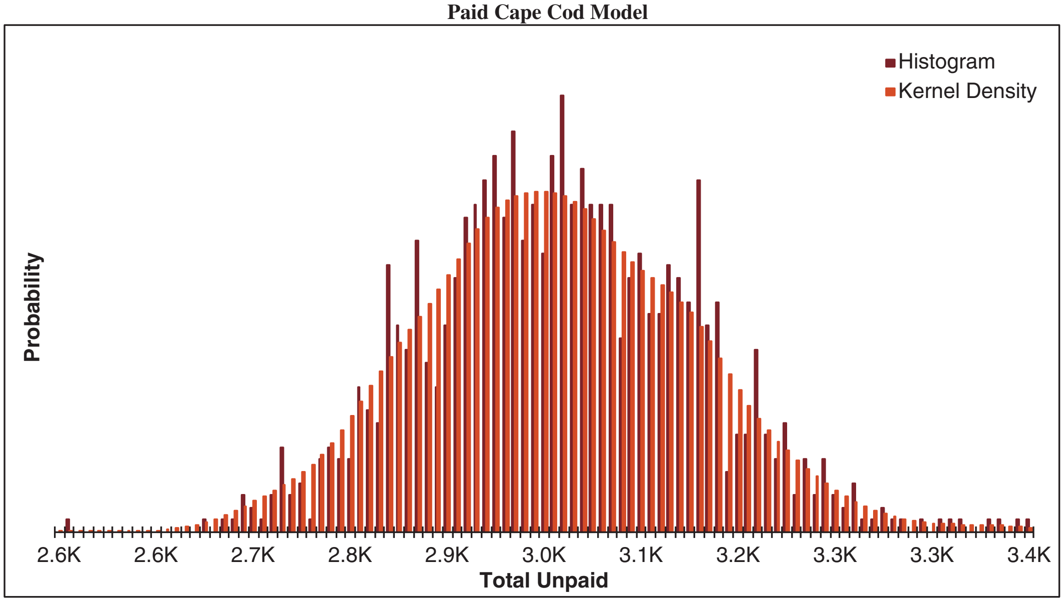


Figure C.11. Estimated Unpaid Model Results (Incurred Cape Cod)

Five Top 50 Companies
Schedule P, Part C -- Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Incurred Cape Cod Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	8	9	110.7%	-	62	5	12	25	36
2007	13	14	108.2%	-	98	9	19	43	60
2008	25	25	98.6%	0	185	18	40	76	99
2009	52	51	98.2%	0	481	37	82	145	201
2010	101	98	97.9%	0	1,082	81	160	267	339
2011	183	199	108.7%	0	3,031	140	282	515	644
2012	403	410	101.7%	0	4,350	320	637	1,106	1,514
2013	696	747	107.4%	0	11,739	577	1,110	1,930	2,405
2014	1,287	1,239	96.3%	0	20,322	1,121	2,045	3,306	4,162
2015	1,647	1,748	106.1%	1	31,078	1,408	2,694	4,317	5,401
Totals	4,415	3,174	71.9%	372	72,036	4,089	5,685	8,357	11,920
Normal Dist.	4,415	3,174	71.9%			4,415	6,555	9,635	11,797
logNormal Dist.	4,465	2,906	65.1%			3,743	5,588	9,947	14,914
Gamma Dist.	4,415	3,174	71.9%			3,682	5,956	10,581	14,867

Figure C.12. Total Unpaid Claims Distribution (Incurred Cape Cod)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Incurred Cape Cod Model

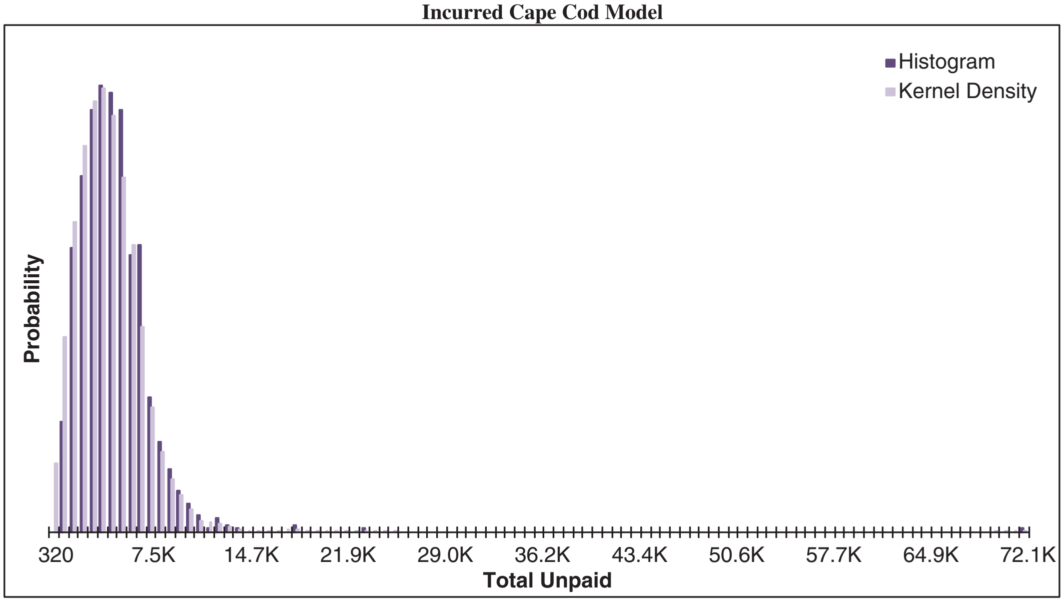


Figure C.13. Estimated Unpaid Model Results (Paid GLM)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Paid GLM Bootstrap Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	8	5	63.7%	(5)	33	7	10	17	23
2007	14	7	52.9%	(3)	52	12	18	27	33
2008	23	9	39.9%	(1)	72	22	29	39	49
2009	38	12	30.2%	8	90	38	45	58	70
2010	64	13	20.8%	27	112	64	73	88	100
2011	123	17	13.8%	81	178	122	135	152	162
2012	244	25	10.4%	169	331	243	261	286	305
2013	457	37	8.1%	361	577	455	480	520	543
2014	747	53	7.1%	597	926	749	784	831	870
2015	1,063	77	7.3%	851	1,346	1,060	1,112	1,192	1,259
Totals	2,781	188	6.8%	2,234	3,480	2,775	2,904	3,097	3,251
Normal Dist.	2,781	188	6.8%			2,781	2,907	3,090	3,218
logNormal Dist.	2,781	188	6.8%			2,774	2,903	3,100	3,246
Gamma Dist.	2,781	188	6.8%			2,776	2,905	3,097	3,237

Figure C.14. Total Unpaid Claims Distribution (Paid GLM)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Paid GLM Bootstrap Model

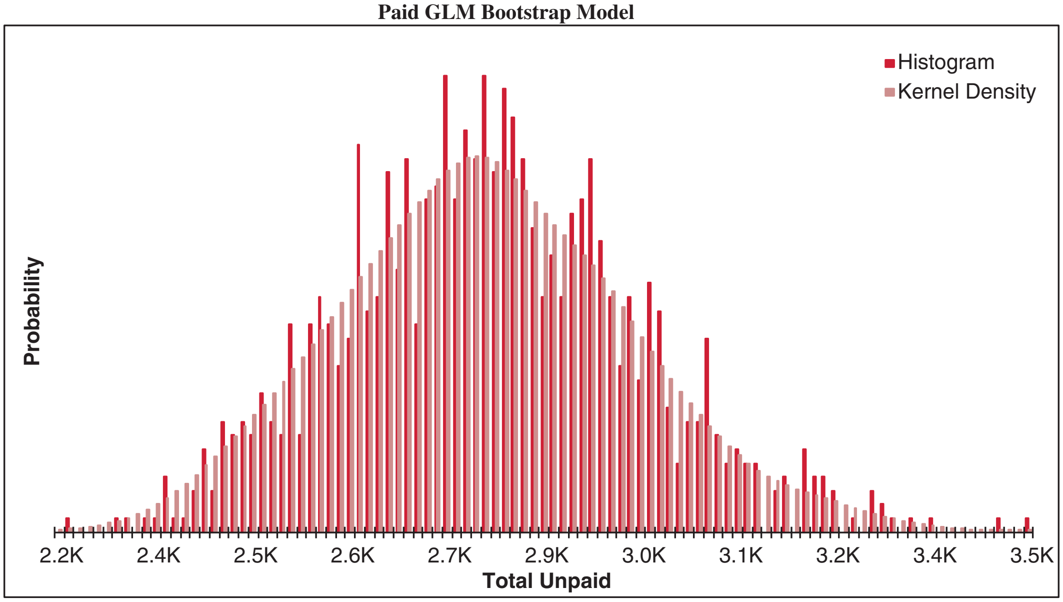


Figure C.15. Estimated Unpaid Model Results (Incurred GLM)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Incurred GLM Bootstrap Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	10	8	81.9%	(9)	57	8	13	25	39
2007	17	11	62.4%	(5)	65	15	22	39	53
2008	31	17	53.2%	(0)	104	29	40	63	82
2009	54	23	43.3%	7	177	50	67	97	119
2010	92	35	38.2%	17	251	88	113	153	184
2011	174	63	36.1%	23	378	171	217	278	333
2012	363	119	32.8%	76	773	360	443	572	648
2013	682	224	32.9%	100	1,490	666	833	1,078	1,211
2014	1,097	366	33.3%	267	2,346	1,084	1,334	1,716	2,055
2015	1,567	555	35.4%	452	4,027	1,515	1,899	2,536	3,071
Totals	4,087	760	18.6%	2,190	6,754	4,018	4,584	5,485	6,034
Normal Dist.	4,087	760	18.6%			4,087	4,599	5,336	5,854
logNormal Dist.	4,087	769	18.8%			4,017	4,555	5,460	6,200
Gamma Dist.	4,087	760	18.6%			4,040	4,570	5,411	6,058

Figure C.16. Total Unpaid Claims Distribution (Incurred GLM)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Incurred GLM Bootstrap Model

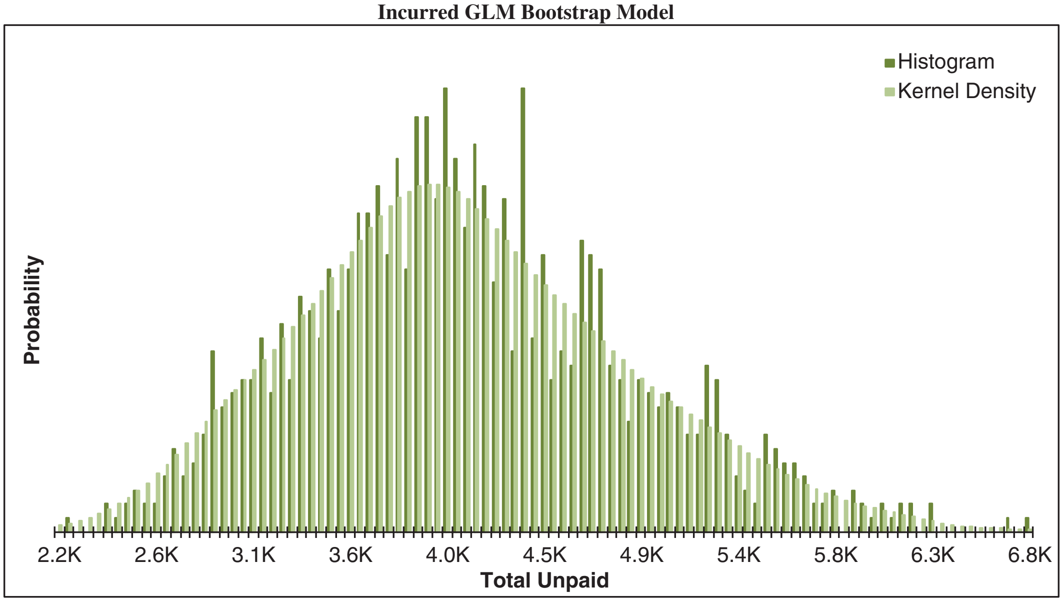


Figure C.17. Model Weights By Accident Year

Accident	Model Weights by Accident Year								
Year	Paid CL	Incd CL	Paid BF	Incd BF	Paid CC	Incd CC	Paid GLM	Incd GLM	TOTAL
2006							100.0%		100.0%
2007							100.0%		100.0%
2008							100.0%		100.0%
2009							100.0%		100.0%
2010			33.3%		33.3%		33.3%		100.0%
2011			33.3%		33.3%		33.3%		100.0%
2012			50.0%		50.0%				100.0%
2013			50.0%		50.0%				100.0%
2014	33.3%		33.3%		33.3%				100.0%
2015	33.3%		33.3%		33.3%				100.0%

Figure C.18. Estimated Mean Unpaid By Model

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Summary of Results by Model

	Mean Estimated Unpaid								
Accident	Chain Ladder		Bornhuetter-Ferguson		Cape Cod		GLM Bootstrap		Best Est.
Year	Paid	Incurred	Paid	Incurred	Paid	Incurred	Paid	Incurred	(Weighted)
2006	8	11	5	7	6	8	8	10	8
2007	11	15	8	11	9	13	14	17	13
2008	21	31	17	24	18	25	23	31	23
2009	35	53	35	49	36	52	38	54	38
2010	61	92	65	99	67	101	64	92	66
2011	110	168	123	176	124	183	123	174	124
2012	216	328	259	362	258	403	244	363	258
2013	410	623	481	642	481	696	457	682	480
2014	773	1,223	812	1,118	827	1,287	747	1,097	803
2015	1,103	1,513	1,132	1,554	1,178	1,647	1,063	1,567	1,134
Totals	2,746	4,056	2,936	4,040	3,004	4,415	2,781	4,087	2,947

Figure C.19. Estimated Ranges

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Summary of Results by Model

		Ranges			
Accident	Best Est.	Weighted		Modeled	
Year	(Weighted)	Minimum	Maximum	Mininum	Maximum
2006	8	8	8	5	8
2007	13	14	14	8	14
2008	23	23	23	17	23
2009	38	38	38	35	38
2010	66	64	67	61	67
2011	124	123	124	110	124
2012	258	258	259	216	259
2013	480	481	481	410	481
2014	803	773	827	747	827
2015	1,134	1,103	1,178	1,063	1,178
Totals	2,947	2,884	3,018	2,746	3,004

Figure C.20. Reconciliation of Total Results (Weighted)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Reconciliation of Total Results
Best Estimate (Weighted)

Accident Year	Paid To Date	Incurred To Date	Case Reserves	IBNR	Estimate of Ultimate	Estimate of Unpaid
2006	1,563	1,577	14	(6)	1,571	8
2007	1,469	1,505	36	(23)	1,482	13
2008	1,387	1,436	49	(26)	1,410	23
2009	1,350	1,417	67	(29)	1,388	38
2010	1,342	1,445	102	(37)	1,408	66
2011	1,198	1,345	147	(24)	1,321	124
2012	1,061	1,339	278	(20)	1,318	258
2013	853	1,327	474	6	1,333	480
2014	645	1,442	797	6	1,448	803
2015	294	1,422	1,128	6	1,428	1,134
Totals	11,162	14,255	3,093	(146)	14,109	2,947

Figure C.21. Estimated Unpaid Model Results (Weighted)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Unpaid
Best Estimate (Weighted)

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	8	5	65.8%	(8)	35	7	11	18	24
2007	13	7	51.3%	(7)	52	13	17	26	33
2008	23	9	39.4%	(5)	72	22	28	39	48
2009	38	11	28.9%	7	92	37	45	58	68
2010	66	12	17.8%	30	130	65	73	86	96
2011	124	22	17.6%	59	247	122	137	161	182
2012	258	40	15.4%	140	485	255	284	326	359
2013	480	47	9.8%	311	737	478	509	559	604
2014	803	65	8.1%	580	1,151	802	845	912	967
2015	1,134	83	7.3%	800	1,569	1,138	1,189	1,266	1,327
Totals	2,947	132	4.5%	2,471	3,532	2,947	3,036	3,162	3,257
Normal Dist.	2,947	132	4.5%			2,947	3,036	3,164	3,254
logNormal Dist.	2,947	132	4.5%			2,944	3,035	3,170	3,268
Gamma Dist.	2,947	132	4.5%			2,945	3,035	3,168	3,263

Figure C.22. Estimated Cash Flow (Weighted)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Calendar Year Unpaid
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	1,156	58	5.0%	937	1,378	1,155	1,194	1,254	1,299
2017	796	53	6.7%	611	993	795	832	886	927
2018	475	42	8.9%	332	668	474	503	547	580
2019	248	38	15.3%	129	410	246	273	315	342
2020	125	23	18.6%	60	260	123	139	165	187
2021	64	11	16.6%	25	110	63	71	82	91
2022	37	6	17.2%	15	71	37	41	48	53
2023	22	5	23.7%	5	52	21	25	30	35
2024	11	4	35.5%	(1)	31	10	13	17	21
2025	7	3	43.3%	-	28	7	9	13	16
2026	4	2	53.2%	-	17	3	5	7	9
2027	2	1	69.8%	-	11	2	2	4	6
2028	1	1	95.7%	-	9	1	1	3	4
Totals	2,947	132	4.5%	2,471	3,532	2,947	3,036	3,162	3,257

Figure C.23. Estimated Loss Ratio (Weighted)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Ultimate Loss Ratios
Best Estimate (Weighted)

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	88.5%	2.7%	3.0%	79.6%	98.3%	88.5%	90.3%	92.9%	94.7%
2007	82.9%	2.5%	3.0%	73.9%	92.3%	82.9%	84.6%	87.0%	88.6%
2008	74.9%	2.3%	3.1%	65.8%	83.0%	74.9%	76.5%	78.7%	80.3%
2009	60.3%	1.9%	3.2%	52.4%	67.3%	60.4%	61.7%	63.5%	64.7%
2010	55.0%	2.1%	3.9%	47.5%	62.5%	55.0%	56.5%	58.4%	59.7%
2011	54.3%	1.9%	3.5%	46.8%	62.7%	54.4%	55.7%	57.5%	58.7%
2012	51.8%	2.0%	3.9%	44.0%	61.8%	51.8%	53.1%	55.2%	56.7%
2013	54.1%	2.3%	4.2%	46.9%	64.8%	54.1%	55.6%	57.9%	59.8%
2014	58.3%	2.8%	4.9%	48.6%	72.0%	58.3%	60.1%	63.0%	65.1%
2015	59.9%	3.6%	6.1%	45.7%	77.7%	60.1%	62.4%	65.7%	68.3%
Totals	62.4%	0.8%	1.3%	59.1%	65.8%	62.4%	63.0%	63.7%	64.3%

Figure C.24. Estimated Unpaid Claim Runoff (Weighted)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Calendar Year Unpaid Claim Runoff
Best Estimate (Weighted)

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	2,947	132	4.5%	2,471	3,532	2,947	3,036	3,162	3,257
2016	1,791	101	5.6%	1,449	2,233	1,790	1,859	1,959	2,027
2017	995	73	7.3%	739	1,286	993	1,043	1,117	1,170
2018	520	52	10.0%	345	712	518	554	608	649
2019	271	31	11.5%	161	415	270	291	325	352
2020	147	18	12.6%	65	246	146	159	178	193
2021	83	13	16.0%	31	156	82	91	106	116
2022	46	10	22.2%	11	97	45	53	63	71
2023	24	7	30.7%	1	65	24	29	37	44
2024	14	5	37.9%	(0)	42	13	17	23	27
2025	6	3	45.2%	(0)	24	6	8	11	14
2026	3	2	60.2%	(0)	13	2	4	6	8
2027	1	1	95.7%	(0)	9	1	1	3	4
2028	0	0	19936.0%	(0)	0	0	0	0	0

Figure C.25. Mean of Incremental Values (Weighted)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Mean Values													
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168 +
2006	326	384	355	237	124	55	27	16	10	6	4	2	1	1
2007	328	388	326	218	114	61	27	16	9	6	3	2	1	1
2008	331	356	299	200	125	60	27	16	9	6	3	2	1	1
2009	303	327	274	219	124	60	27	16	9	6	3	2	1	1
2010	290	328	306	218	121	58	27	16	11	4	4	2	1	1
2011	269	323	291	207	115	60	27	15	10	4	4	2	1	1
2012	269	312	281	198	130	62	28	16	11	3	4	2	1	1
2013	266	308	278	229	126	60	27	16	11	3	4	2	1	1
2014	299	346	325	228	126	60	27	15	11	3	4	2	1	1
2015	294	351	317	223	123	59	26	15	11	3	4	2	1	1

Figure C.26. Standard Deviation of Incremental Values (Weighted)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Standard Error Values													
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168 +
2006	21	23	22	18	13	9	6	5	4	3	3	2	2	1
2007	21	23	21	17	13	9	6	5	4	3	3	2	2	1
2008	21	22	20	17	13	9	6	5	4	3	3	2	2	1
2009	21	21	19	17	13	9	6	5	4	3	3	2	2	1
2010	18	26	23	14	22	15	8	4	4	3	2	2	1	1
2011	18	17	22	15	22	18	8	4	4	3	2	2	1	1
2012	13	14	22	11	30	21	8	4	3	2	2	1	1	1
2013	13	14	22	18	31	21	8	4	3	2	2	1	1	1
2014	13	14	33	18	30	21	8	4	3	2	2	1	1	1
2015	13	26	32	19	30	21	8	4	3	2	2	1	1	1

Figure C.27. Coefficient of Variation of Incremental Values (Weighted)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Accident Year Incremental Values by Development Period
Best Estimate (Weighted)

Accident	Coefficients of Variation													
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168 +
2006	6.5%	6.0%	6.2%	7.6%	10.4%	15.9%	22.6%	29.2%	37.6%	49.0%	74.6%	95.5%	122.0%	156.9%
2007	6.4%	5.9%	6.5%	8.0%	11.0%	15.0%	22.4%	28.9%	38.1%	56.2%	73.4%	94.7%	123.1%	157.6%
2008	6.5%	6.2%	6.8%	8.3%	10.4%	15.0%	22.6%	29.3%	41.7%	56.2%	73.4%	95.5%	122.4%	160.6%
2009	6.8%	6.4%	7.1%	7.9%	10.6%	15.2%	22.7%	31.5%	42.2%	56.9%	74.2%	96.8%	121.7%	162.3%
2010	6.1%	7.9%	7.3%	6.3%	18.2%	25.8%	28.6%	26.8%	35.0%	65.1%	63.8%	81.4%	111.5%	113.7%
2011	6.6%	5.4%	7.6%	7.2%	18.7%	30.1%	29.0%	27.1%	34.8%	66.8%	64.0%	82.4%	113.2%	115.9%
2012	4.8%	4.5%	7.8%	5.6%	23.4%	34.1%	30.0%	24.4%	30.1%	60.7%	57.6%	71.7%	93.4%	94.2%
2013	4.8%	4.4%	7.9%	7.9%	24.2%	34.5%	30.4%	24.4%	30.2%	61.4%	58.2%	72.2%	94.5%	94.4%
2014	4.5%	4.2%	10.0%	8.1%	23.6%	34.6%	30.2%	24.7%	30.3%	62.0%	59.4%	73.2%	94.7%	95.6%
2015	4.6%	7.4%	10.0%	8.3%	24.3%	35.0%	31.0%	24.6%	30.6%	61.4%	59.9%	73.5%	97.0%	95.7%

Figure C.28. Total Unpaid Claims Distribution (Weighted)

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Total Unpaid Distribution
Best Estimate (Weighted)

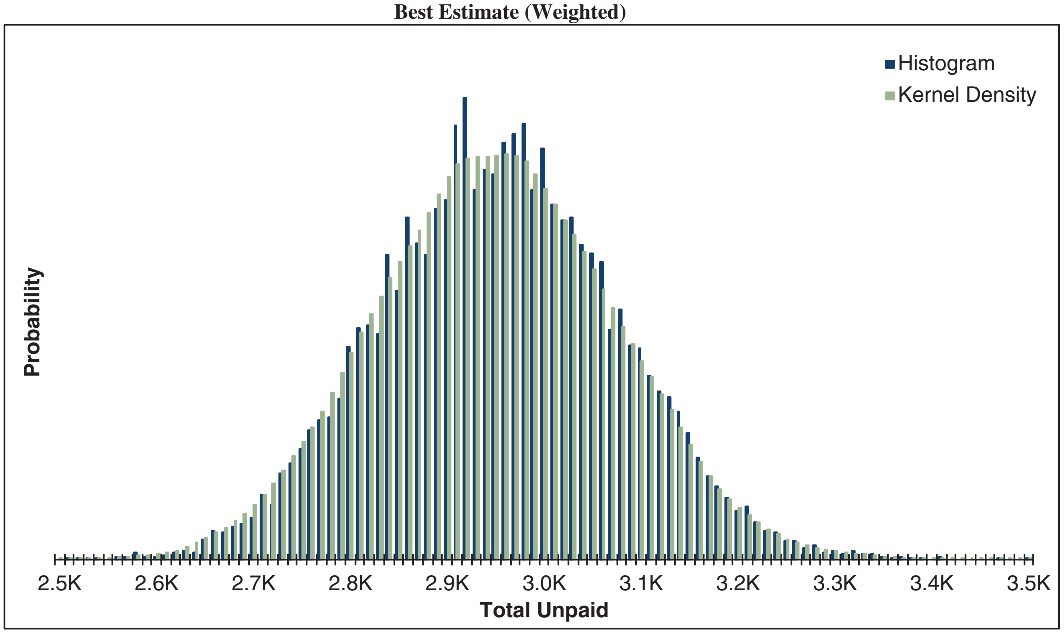
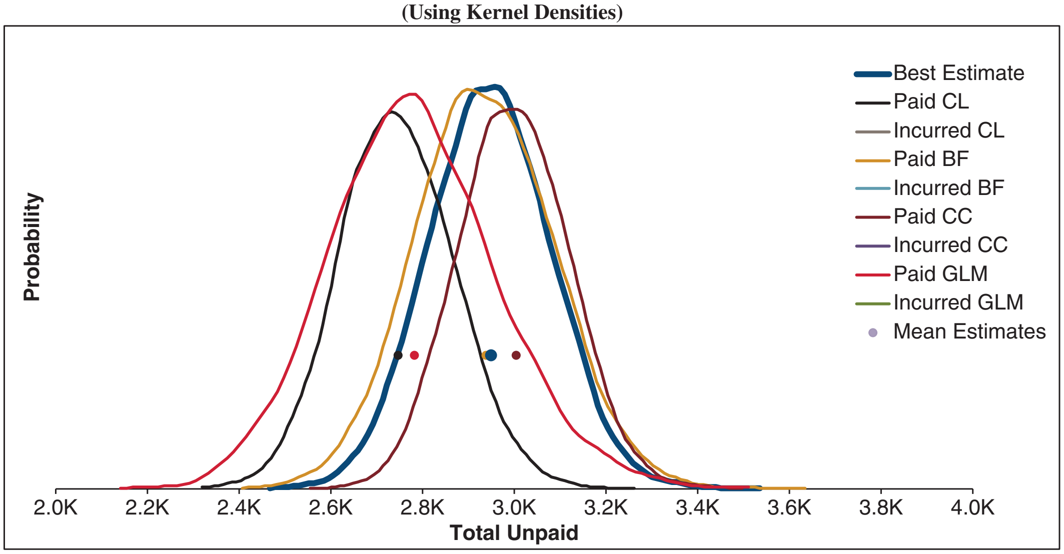


Figure C.29. Summary of Model Distributions

Five Top 50 Companies
Schedule P, Part C – Commercial Auto Liability (in 000,000's)
Summary of Model Distributions
(Using Kernel Densities)



Appendix D—Aggregate Results

In this appendix the results for the correlated aggregate of the three Schedule P lines of business (Parts A, B, and C) are shown, using the correlation calculated from the paid data after adjustment for heteroscedasticity.

Figure D.1. Estimated Unpaid Model Results

Five Top 50 Companies
Aggregate Three Lines of Business
Accident Year Unpaid

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	67	25	37.9%	0	186	66	83	110	130
2007	107	30	28.1%	25	295	105	126	158	185
2008	199	49	24.8%	67	622	194	226	285	342
2009	298	56	18.8%	123	800	293	331	395	457
2010	480	69	14.3%	248	959	475	522	599	668
2011	862	106	12.3%	503	1,561	860	923	1,041	1,135
2012	1,666	187	11.2%	383	2,555	1,662	1,771	1,985	2,148
2013	3,070	333	10.8%	1,808	6,522	3,066	3,249	3,649	3,928
2014	5,632	703	12.5%	2,435	8,555	5,632	6,075	6,801	7,326
2015	13,270	1,788	13.5%	5,217	22,660	13,262	14,348	16,180	18,011
Totals	25,650	2,080	8.1%	16,952	36,085	25,616	26,949	29,088	30,991
Normal Dist.	25,650	2,080	8.1%			25,650	27,053	29,072	30,490
logNormal Dist.	25,650	2,088	8.1%			25,566	27,006	29,222	30,885
Gamma Dist.	25,650	2,080	8.1%			25,594	27,021	29,165	30,736

Figure D.2. Estimated Cash Flow

Five Top 50 Companies
Aggregate Three Lines of Business
Calendar Year Unpaid

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	12,906	1,209	9.4%	8,242	19,475	12,869	13,611	14,897	16,182
2017	5,733	453	7.9%	3,991	7,589	5,727	6,024	6,488	6,836
2018	3,144	257	8.2%	2,132	4,373	3,137	3,310	3,573	3,781
2019	1,663	144	8.6%	1,163	2,415	1,657	1,757	1,906	2,018
2020	903	86	9.5%	617	1,331	900	958	1,050	1,122
2021	512	59	11.5%	319	1,064	508	546	613	678
2022	324	55	16.9%	140	699	317	353	423	484
2023	217	64	29.4%	86	931	205	245	328	431
2024	120	28	23.7%	21	308	118	137	170	197
2025	74	22	30.1%	7	165	73	89	113	131
2026	36	13	37.2%	2	94	35	45	59	70
2027	18	9	51.9%	0	58	17	24	33	41
2028	1	1	95.7%	-	9	1	1	3	4
Totals	25,650	2,080	8.1%	16,952	36,085	25,616	26,949	29,088	30,991

Figure D.3. Estimated Loss Ratio

Five Top 50 Companies
Aggregate Three Lines of Business
Accident Year Ultimate Loss Ratios

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	74.0%	10.7%	14.5%	33.5%	132.5%	73.7%	77.5%	93.7%	109.6%
2007	81.3%	11.5%	14.2%	38.3%	147.1%	81.0%	85.0%	102.0%	121.0%
2008	85.4%	11.8%	13.8%	39.5%	153.1%	85.0%	89.2%	107.7%	123.9%
2009	76.0%	10.2%	13.4%	36.8%	131.0%	75.6%	79.4%	94.7%	111.2%
2010	66.9%	9.3%	13.9%	31.0%	119.9%	66.3%	70.1%	84.1%	97.9%
2011	64.5%	8.9%	13.8%	30.1%	117.2%	64.2%	67.5%	81.1%	91.4%
2012	71.0%	10.1%	14.3%	31.6%	129.3%	70.5%	74.0%	90.5%	104.6%
2013	61.4%	8.5%	13.9%	29.3%	125.5%	61.1%	64.2%	77.3%	88.8%
2014	65.4%	9.7%	14.8%	31.3%	115.9%	65.2%	70.3%	82.2%	94.9%
2015	78.2%	11.5%	14.7%	39.0%	143.2%	77.8%	83.8%	97.6%	113.8%
Totals	71.6%	3.3%	4.6%	59.5%	88.1%	71.5%	73.7%	77.3%	80.3%

Figure D.4. Estimated Unpaid Claim Runoff

Five Top 50 Companies
Aggregate Three Lines of Business
Calendar Year Unpaid Claim Runoff

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	25,650	2,080	8.1%	16,952	36,085	25,616	26,949	29,088	30,991
2016	12,744	944	7.4%	8,710	17,043	12,733	13,373	14,296	15,047
2017	7,012	536	7.6%	4,664	9,551	7,000	7,368	7,905	8,324
2018	3,868	319	8.2%	2,512	5,388	3,861	4,075	4,406	4,671
2019	2,205	213	9.7%	1,348	3,259	2,196	2,340	2,567	2,762
2020	1,302	158	12.1%	730	2,266	1,292	1,400	1,574	1,733
2021	790	126	15.9%	401	1,697	781	864	1,003	1,145
2022	466	99	21.2%	166	1,272	458	524	636	746
2023	249	62	24.9%	45	533	245	289	359	403
2024	129	42	32.4%	13	294	126	156	202	236
2025	55	21	37.9%	3	141	53	68	90	107
2026	19	9	49.6%	0	60	18	25	34	42
2027	1	1	95.7%	(0)	9	1	1	3	4

Figure D.5. Mean of Incremental Values

Five Top 50 Companies
Aggregate Three Lines of Business
Accident Year Incremental Values by Development Period

Accident	Mean Values													
Year	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168 +**
2006	9,334	4,878	2,029	1,175	621	300	151	79	67	33	33	17	16	1
2007	10,595	5,394	2,159	1,239	655	327	163	85	75	35	35	18	17	1
2008	12,060	5,959	2,317	1,321	716	352	175	91	84	38	38	19	18	1
2009	11,848	6,007	2,371	1,389	745	365	182	95	83	40	40	20	20	1
2010	11,834	5,923	2,345	1,351	721	354	182	95	85	38	40	20	19	1
2011	12,195	5,972	2,312	1,326	707	372	185	96	90	39	40	20	19	1
2012	14,186	6,541	2,409	1,362	775	380	191	99	103	39	40	20	19	1
2013	11,901	5,868	2,282	1,436	763	374	187	97	93	38	40	20	19	1
2014	12,949	6,354	2,538	1,451	771	378	189	98	98	38	36	19	17	1
2015	16,458	7,356	2,685	1,515	794	395	202	108	99	44	36	19	17	1

Figure D.6. Standard Deviation of Incremental Values

Five Top 50 Companies
Aggregate Three Lines of Business
Accident Year Incremental Values by Development Period

Accident	Standard Deviation Values													
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168 +
2006	1,735	668	233	134	74	37	18	13	23	6	13	7	6	6
2007	1,909	712	244	140	77	39	18	14	26	10	14	7	7	10
2008	2,085	768	264	147	81	41	20	14	35	11	15	8	7	11
2009	2,010	754	260	149	82	41	19	17	34	10	15	8	8	10
2010	2,059	775	264	148	84	43	22	17	35	11	15	8	8	11
2011	2,085	777	261	150	84	49	22	17	37	11	16	8	8	11
2012	2,492	875	277	155	98	50	23	17	44	12	15	8	8	12
2013	2,078	767	261	169	97	51	23	17	39	11	15	8	8	11
2014	2,300	907	341	192	109	55	26	19	42	13	17	9	9	13
2015	2,728	1,087	365	210	116	59	30	23	58	17	17	9	9	17

Figure D.7. Coefficient of Variation of Incremental Values

Five Top 50 Companies
Aggregate Three Lines of Business
Accident Year Incremental Values by Development Period

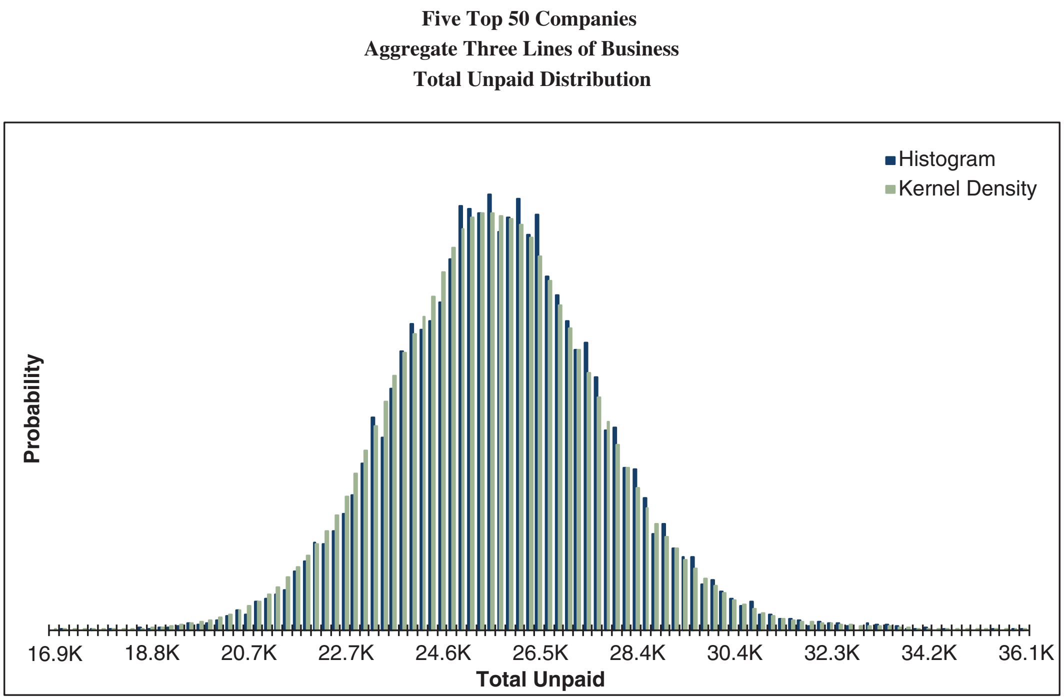
Accident	Coefficients of Variation													
Year	12	24	36	48	60	72	84	96	108	120	132	144	156	168 +
2006	18.6%	13.7%	11.5%	11.4%	11.9%	12.5%	11.7%	16.8%	35.0%	17.0%	38.4%	38.5%	40.0%	702.6%
2007	18.0%	13.2%	11.3%	11.3%	11.8%	12.0%	11.4%	16.3%	35.1%	27.8%	38.5%	38.6%	40.0%	1216.6%
2008	17.3%	12.9%	11.4%	11.1%	11.3%	11.7%	11.2%	15.7%	42.1%	28.0%	39.1%	39.2%	40.5%	1356.5%
2009	17.0%	12.6%	11.0%	10.7%	11.0%	11.3%	10.7%	17.7%	41.4%	26.2%	38.8%	39.0%	40.2%	1287.1%
2010	17.4%	13.1%	11.3%	11.0%	11.6%	12.2%	11.9%	17.7%	40.5%	27.6%	39.0%	39.4%	40.8%	1164.5%
2011	17.1%	13.0%	11.3%	11.3%	11.9%	13.3%	11.9%	17.5%	41.5%	28.2%	39.2%	39.5%	40.9%	1219.1%
2012	17.6%	13.4%	11.5%	11.4%	12.6%	13.2%	12.0%	16.9%	42.8%	31.6%	38.6%	39.0%	40.6%	1268.9%
2013	17.5%	13.1%	11.4%	11.8%	12.6%	13.6%	12.1%	17.3%	41.9%	29.1%	38.5%	38.9%	40.4%	1214.5%
2014	17.8%	14.3%	13.5%	13.3%	14.2%	14.5%	13.8%	19.0%	43.0%	34.8%	47.6%	46.7%	54.6%	1429.6%
2015	16.6%	14.8%	13.6%	13.8%	14.6%	15.0%	14.9%	21.5%	58.1%	38.8%	47.3%	46.7%	54.4%	1901.0%

Figure D.8. Calculation of Risk Based Capital

Five Top 50 Companies
Aggregate Three Lines of Business
Indicated Unpaid Claim Risk Portion of Required Capital

LOB / Segment	Earned Premium	Mean Unpaid	99.0% Unpaid	Value at Risk Capital	Allocated Capital	Unpaid Ratio	Premium Ratio
Schedule P, Part A	15,148	5,308	8,675	3,367	2,642	49.8%	17.4%
Schedule P, Part B	20,467	17,395	20,525	3,130	2,456	14.1%	12.0%
Schedule P, Part C	2,383	2,947	3,257	310	243	8.3%	10.2%
Total	37,997	25,650	32,457	6,807			
Aggregate	37,997	25,650	30,991	5,341	5,341	20.8%	14.1%

Figure D.9. Total Unpaid Claims Distribution



Appendix E—GLM Bootstrap Results

In this appendix the results for the GLM Bootstrap model, as illustrated in Figures 5.9 through 5.12 using the Taylor and Ashe (1983) data, are shown.

Figure E.1. Estimated Unpaid Model Results

Taylor and Ashe Data
Accident Year Unpaid
Paid GLM Bootstrap Model

Accident Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	-	-		-	-	-	-	-	-
2007	201,062	86,944	43.2%	13,857	542,484	186,940	254,238	361,288	438,224
2008	438,222	193,377	44.1%	48,640	1,570,379	405,070	547,131	798,395	996,074
2009	701,223	229,176	32.7%	192,462	1,747,698	679,682	831,657	1,122,868	1,320,964
2010	1,024,913	264,752	25.8%	405,036	2,286,536	1,009,377	1,186,714	1,467,758	1,825,411
2011	1,452,650	315,901	21.7%	619,534	2,544,116	1,424,030	1,660,714	1,996,927	2,261,272
2012	2,181,115	481,962	22.1%	916,307	4,248,064	2,136,166	2,480,213	3,027,607	3,396,995
2013	3,468,030	603,268	17.4%	1,751,033	5,598,537	3,424,738	3,862,292	4,553,992	4,965,982
2014	4,568,990	695,194	15.2%	2,331,572	6,824,685	4,526,036	5,039,460	5,731,706	6,408,694
2015	5,672,877	744,661	13.1%	3,681,244	8,333,062	5,657,952	6,171,074	6,954,411	7,414,615
Totals	19,709,081	2,176,864	11.0%	13,360,401	27,429,908	19,594,207	21,069,822	23,354,466	24,752,422
Normal Dist.	19,709,081	2,176,864	11.0%			19,709,081	21,177,353	23,289,703	24,773,224
logNormal Dist.	19,709,844	2,194,514	11.1%			19,588,799	21,111,651	23,512,537	25,360,134
Gamma Dist.	19,709,081	2,176,864	11.0%			19,628,994	21,130,455	23,421,097	25,123,713

Figure E.2. Estimated Cash Flow

Taylor and Ashe Data
Calendar Year Unpaid
Paid GLM Bootstrap Model

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2016	5,367,217	639,639	11.9%	3,363,863	7,428,225	5,343,203	5,770,597	6,447,544	6,986,539
2017	4,312,360	599,300	13.9%	2,363,704	6,455,658	4,279,059	4,673,264	5,338,534	5,922,511
2018	3,310,498	539,509	16.3%	1,993,107	5,419,760	3,288,209	3,657,889	4,209,239	4,690,515
2019	2,245,627	417,764	18.6%	1,078,000	4,088,770	2,221,086	2,510,176	2,948,019	3,475,039
2020	1,676,436	369,916	22.1%	619,943	3,157,564	1,644,779	1,921,249	2,318,054	2,614,635
2021	1,224,109	326,624	26.7%	444,913	2,352,525	1,202,484	1,436,029	1,782,066	2,085,204
2022	838,442	264,751	31.6%	226,969	2,477,444	803,316	991,076	1,302,125	1,532,640
2023	507,334	211,762	41.7%	104,873	1,268,302	480,233	635,243	889,537	1,135,405
2024	227,058	93,270	41.1%	32,667	711,619	213,471	277,710	403,483	498,676
2025	-	-		-	-	-	-	-	-
Totals	19,709,081	2,176,864	11.0%	13,360,401	27,429,908	19,594,207	21,069,822	23,354,466	24,752,422

Figure E.3. Estimated Loss Ratio

Taylor and Ashe Data
Accident Year Ultimate Loss Ratios
Paid GLM Bootstrap Model

Accident Year	Mean Loss Ratio	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2006	54.8%	6.4%	11.7%	38.1%	74.0%	54.7%	59.0%	65.7%	70.4%
2007	65.0%	6.4%	9.8%	48.1%	84.1%	65.0%	68.9%	75.7%	80.7%
2008	63.1%	6.4%	10.1%	42.6%	82.0%	63.1%	67.3%	73.4%	78.6%
2009	56.0%	6.2%	11.0%	38.0%	76.4%	55.9%	60.0%	66.2%	71.6%
2010	53.1%	5.9%	11.0%	34.7%	74.7%	52.8%	57.1%	63.1%	66.6%
2011	50.5%	5.6%	11.1%	33.9%	70.0%	50.2%	54.2%	60.0%	63.9%
2012	53.8%	7.6%	14.2%	31.3%	81.3%	53.1%	59.1%	66.8%	72.8%
2013	55.3%	6.9%	12.5%	34.6%	78.4%	55.1%	59.5%	66.9%	73.4%
2014	52.9%	6.8%	12.8%	31.7%	74.5%	52.4%	57.2%	64.5%	70.1%
2015	50.7%	6.5%	12.8%	33.0%	72.4%	50.6%	55.1%	61.6%	65.8%
Totals	55.1%	2.9%	5.2%	46.9%	63.6%	55.1%	57.1%	60.0%	61.7%

Figure E.4. Estimated Unpaid Claim Runoff

Taylor and Ashe Data
Calendar Year Unpaid Claim Runoff
Paid GLM Bootstrap Model

Calendar Year	Mean Unpaid	Standard Error	Coefficient of Variation	Minimum	Maximum	50.0% Percentile	75.0% Percentile	95.0% Percentile	99.0% Percentile
2015	19,709,081	2,176,864	11.0%	13,360,401	27,429,908	19,594,207	21,069,822	23,354,466	24,752,422
2016	14,341,864	1,839,659	12.8%	8,990,374	21,139,070	14,231,008	15,525,987	17,412,102	19,106,264
2017	10,029,504	1,499,062	14.9%	5,923,686	15,623,104	9,926,619	10,979,472	12,605,655	13,627,923
2018	6,719,006	1,188,158	17.7%	3,317,118	11,201,515	6,612,903	7,438,758	8,841,160	9,734,081
2019	4,473,380	922,335	20.6%	1,884,408	7,436,971	4,366,371	5,040,244	6,143,079	6,968,601
2020	2,796,943	678,192	24.2%	1,137,743	5,050,304	2,740,868	3,192,138	4,018,580	4,623,373
2021	1,572,834	443,756	28.2%	595,162	3,523,942	1,524,022	1,852,397	2,369,545	2,820,528
2022	734,392	257,467	35.1%	204,545	1,654,724	708,577	888,204	1,167,670	1,463,534
2023	227,058	93,270	41.1%	32,667	711,619	213,471	277,710	403,483	498,676
2024	0	0	4017.8%	(0)	0	-	0	0	0

Figure E.5. Mean of Incremental Values

Taylor and Ashe Data
Accident Year Incremental Values by Development Period
Paid GLM Bootstrap Model

Accident	Mean Values									
Year	12	24	36	48	60	72	84	96	108	120+
2006	260,293	698,693	688,850	704,606	388,809	311,880	258,794	214,532	169,749	142,707
2007	353,111	978,505	972,391	972,627	539,441	447,302	359,572	300,611	234,076	201,062
2008	355,598	975,396	989,087	971,633	541,986	440,002	357,470	297,335	237,981	200,241
2009	343,575	914,108	911,442	913,681	502,676	421,801	335,888	282,854	231,129	187,240
2010	341,295	923,102	914,709	919,809	500,195	420,057	337,719	275,372	224,883	186,939
2011	336,529	924,119	917,372	913,328	503,784	409,092	338,662	284,360	234,436	186,099
2012	381,818	1,028,561	1,036,624	1,025,187	578,558	451,767	374,253	312,453	251,461	212,623
2013	402,258	1,107,072	1,108,427	1,111,762	614,292	501,712	410,170	332,713	265,392	231,989
2014	408,511	1,104,124	1,109,649	1,096,598	616,324	491,960	408,977	338,285	274,715	232,482
2015	406,207	1,098,540	1,104,298	1,121,727	609,668	497,186	407,810	331,738	274,852	227,058

Figure E.6. Standard Deviation of Incremental Values

Taylor and Ashe Data
Accident Year Incremental Values by Development Period
Paid GLM Bootstrap Model

Accident	Standard Error Values									
Year	12	24	36	48	60	72	84	96	108	120+
2006	108,496	120,663	181,091	248,062	129,788	119,862	108,476	67,654	126,590	56,408
2007	131,381	142,390	209,358	306,437	159,961	138,486	133,743	80,122	152,143	86,944
2008	127,448	146,072	215,044	306,874	152,841	142,207	122,350	78,132	159,664	86,683
2009	125,340	137,368	201,409	295,530	154,057	138,440	121,788	90,057	156,660	80,901
2010	127,558	139,764	193,891	297,664	152,539	136,999	129,441	88,860	139,515	78,776
2011	125,839	139,522	196,494	285,649	156,869	139,339	128,102	92,988	160,838	81,152
2012	137,400	150,449	208,476	321,223	187,435	156,077	150,736	103,377	165,336	93,178
2013	137,189	150,565	221,025	338,863	195,056	173,394	151,060	103,542	169,356	96,165
2014	132,459	159,062	254,892	329,254	195,407	162,115	149,531	106,739	171,923	94,787
2015	135,172	183,619	247,413	336,959	177,810	163,745	147,122	102,400	167,873	93,270

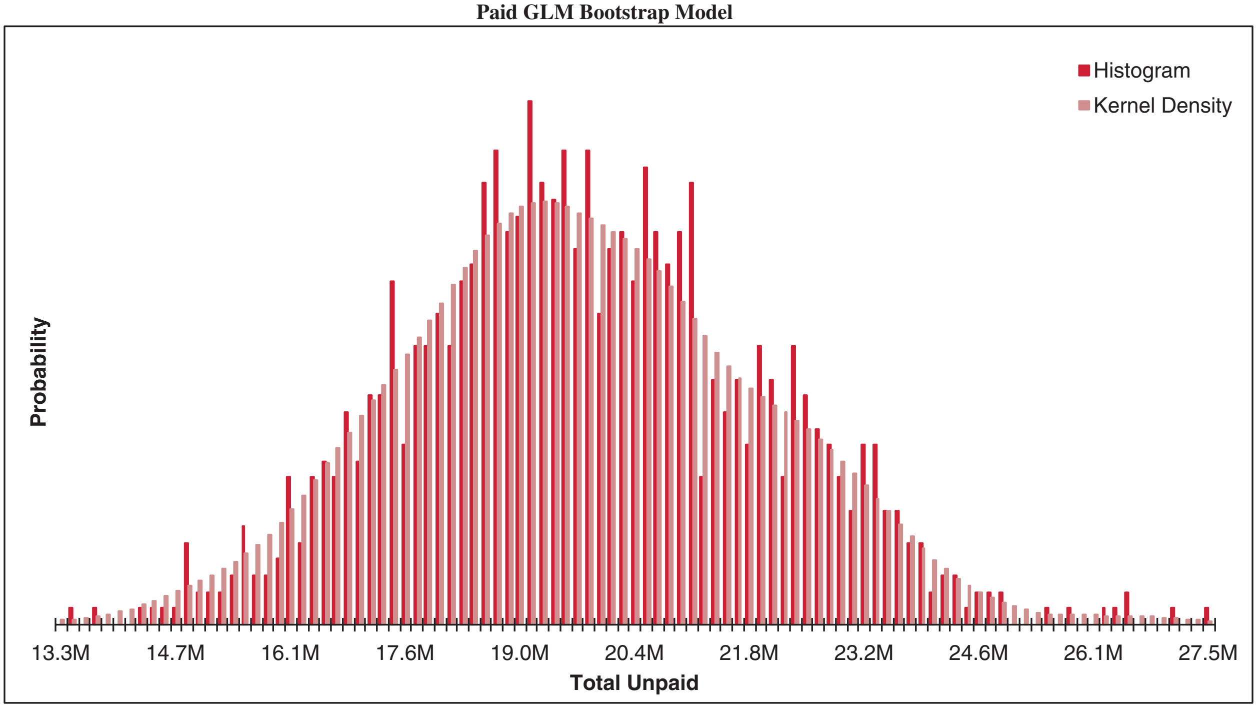
Figure E.7. Coefficient of Variation of Incremental Values

Taylor and Ashe Data
Accident Year Incremental Values by Development Period
Paid GLM Bootstrap Model

Accident	Coefficient of Variation Values									
Year	12	24	36	48	60	72	84	96	108	120+
2006	41.7%	17.3%	26.3%	35.2%	33.4%	38.4%	41.9%	31.5%	74.6%	39.5%
2007	37.2%	14.6%	21.5%	31.5%	29.7%	31.0%	37.2%	26.7%	65.0%	43.2%
2008	35.8%	15.0%	21.7%	31.6%	28.2%	32.3%	34.2%	26.3%	67.1%	43.3%
2009	36.5%	15.0%	22.1%	32.3%	30.6%	32.8%	36.3%	31.8%	67.8%	43.2%
2010	37.4%	15.1%	21.2%	32.4%	30.5%	32.6%	38.3%	32.3%	62.0%	42.1%
2011	37.4%	15.1%	21.4%	31.3%	31.1%	34.1%	37.8%	32.7%	68.6%	43.6%
2012	36.0%	14.6%	20.1%	31.3%	32.4%	34.5%	40.3%	33.1%	65.8%	43.8%
2013	34.1%	13.6%	19.9%	30.5%	31.8%	34.6%	36.8%	31.1%	63.8%	41.5%
2014	32.4%	14.4%	23.0%	30.0%	31.7%	33.0%	36.6%	31.6%	62.6%	40.8%
2015	33.3%	16.7%	22.4%	30.0%	29.2%	32.9%	36.1%	30.9%	61.1%	41.1%

Figure E.8. Total Unpaid Claims Distribution

Taylor and Ashe Data
Total Unpaid Distribution
Paid GLM Bootstrap Model



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Abbreviations and Notations

AIC: Akaike Information Criterion
APD: Automobile Physical Damage
BIC: Bayesian Information Criterion
BF: Bornhuetter-Ferguson
CC: Cape Cod
CL: Chain Ladder
CoV: Coefficient of Variation
DFA: Dynamic Financial Analysis
ELR: Expected Loss Ratio
ERM: Enterprise Risk Management
GLM: Generalized Linear Models
MLE: Maximum Likelihood Estimate
ODP: Over-Dispersed Poisson
OLS: Ordinary Least Squares
RSS: Residual Sum Squared
SSE: Sum of Squared Errors

About the Author

Mark R. Shapland is Senior Consulting Actuary in Milliman's Dubai office where he is responsible for various reserving and pricing projects for a variety of clients and was previously the lead actuary for the Property & Casualty Insurance Software (PCIS) development team. He has a B.S. degree in Integrated Studies (Actuarial Science) from the University of Nebraska-Lincoln. He is a Fellow of the Casualty Actuarial Society, a Fellow of the Society of Actuaries and a Member of the American Academy of Actuaries. He was the leader of Section 3 of the Reserve Variability Working Party, the Chair of the CAS Committee on Reserves, co-chair of the Tail Factor Working Party, and co-chair of the Loss Simulation Model Working Party. He is also a co-developer and co-presenter of the CAS Reserve Variability Limited Attendance Seminar and has spoken frequently on this subject both within the CAS and internationally. He can be contacted at mark.shapland@milliman.com.

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