APPLICATION OF THE OPTION MARKET PARADIGM TO THE SOLUTION OF INSURANCE PROBLEMS

MICHAEL G. WACEK

Abstract

The Black–Scholes option pricing formula from finance theory is consistent with the assumption that the market price of the underlying asset at any future date is lognormally distributed with time-dependent parameters and can be shown to be a special case of both a more general option model and a familiar actuarial function used in excess of loss applications. This insight leads to an understanding of the similarity between options and certain insurance concepts. Because insurance and finance have developed separately, different paradigms are used by the practitioners in each field. When these paradigms are shared, a new perspective on risk management, product development, and pricing, especially of insurance and reinsurance, emerges.

1. RELATIONSHIP OF THE BLACK–SCHOLES FORMULA AND THE ACTUARIAL EXCESS OF LOSS FUNCTION

In 1973, Fischer Black and Myron Scholes published their now classic paper entitled "The Pricing of Options and Corporate Liabilities," [1] in which they derived the option pricing formula that bears their name. Gerber and Shiu [2] described that paper as "perhaps the most important development in the theory of financial economics in the past two decades." The advent of the modern derivatives market is generally traced back to the introduction of exchange-traded equity options in the U.S. (1973) and the development of the Black–Scholes model [3].

Black and Scholes showed that under certain conditions the current pure premium,[1] $c_t(S)$, for a "call option" to buy a particular asset for price S, at, and only at, time t (where t is the time to expiry) is

$$c_t(S) = P_0 \cdot N(d_1) - Se^{-rt} \cdot N(d_2), \qquad \text{where}$$

$$d_1 = \frac{\ln(P_0/S) + (r + 0.5\sigma^2)t}{\sigma\sqrt{t}}, \tag{1.1}$$

$$d_2 = \frac{\ln(P_0/S) + (r - 0.5\sigma^2)t}{\sigma\sqrt{t}},$$

and where P_0 is the current market price, r is the risk-free force of interest, σ is a measure of annualized price volatility, and N is the cumulative distribution function of the standard normal distribution.

This is a daunting formula, and in this form it provides little insight into the underlying options pricing problem. One of the key points of this paper is that Formula 1.1, the Black–Scholes formula, is actually a special case of a familiar actuarial function written in an unfamiliar form. This will lead us to some important insights about both options and insurance.

Consider that the pure premium of a call option exercisable only on the expiry date (a "European" option) depends on the market's current opinion about the probability distribution of the market price of the underlying asset on the expiry date. If the option exercise price is S, the option will only be exercised in the event the market price at expiry exceeds S. Its value in these circumstances will be the amount by which the market price exceeds S. In other circumstances, the optionholder will let the

[1]Financial economists use the term "price" or "premium." However, to make clear to actuarial readers that there is no embedded charge for risk or expenses in the Black–Scholes valuation, we shall use the actuarial term "pure premium."

option expire unexercised and, if he wants to own the asset, buy the asset at the market price. The option to buy the asset at a higher than market price will be worthless. The value of the option at expiry is the probability-weighted average of all possible expiry scenarios.

Suppose the probability distribution of market prices at expiry is represented by the random variate x. Then the expected value of the option at expiry is

$$\text{Future Value } [c_t(S)] = \int_S^{\infty} (x - S) \cdot f(x) dx. \tag{1.2}$$

The expression on the right hand side of Formula 1.2 is the *future value* of the option pure premium, since x is defined for the expiry date, which is in the future. Its present value, discounted at the risk-free interest rate,[2] is

$$c_t(S) = e^{-rt} \int_S^{\infty} (x - S) \cdot f(x) dx. \tag{1.3}$$

Now compare Formulas 1.1 and 1.3. Formula 1.1, the Black–Scholes formula, depends on the assumption that market prices are lognormally distributed. Formula 1.3 is more general and has no embedded distributional assumption. However, if the variate x in Formula 1.3 is assumed to be lognormal and the correct

[2]This is justified on the basis that using any other rate would open the door to risk-free arbitrage profits. It is possible to create a riskless portfolio by hedging a long position in the underlying asset by selling short an appropriate number of call options on the underlying asset. Because it is riskless, this hedged portfolio must earn the risk-free rate of return. However, for this to be true (and it must be true to avoid risk-free arbitrage profit opportunities), it turns out that the interest rate for discounting the expected value of the call option at expiry must also be the risk-free rate. The finance literature refers to this phenomenon as "risk-neutral valuation" and it applies to valuation of all financial derivatives of assets where suitable conditions for hedging exist. For further discussion of risk-neutral valuation and risk-free discounting, see Hull [7].

In actuarial applications involving insurance claims (where hedging is not possible), it is sometimes implicitly recognized that the risk-free rate is not appropriate by discounting at the risk-free rate, and then adding a risk charge to the discounted result. This is equivalent to discounting at a rate less than the risk-free rate. We have deliberately chosen to characterize $c_t(S)$ as a "pure premium" to leave the door open to an additional risk charge where appropriate.

distribution parameters are chosen,[3] Formula 1.1 can be derived from Formula 1.3. In other words, the Black–Scholes formula is a special case of Formula 1.3. The proof of this is in Appendix A.

Formula 1.2, which differs from Formula 1.3 only by a present value factor, also defines a familiar actuarial function seen frequently in excess of loss insurance applications. For example, if x is a random variate representing the aggregate value of losses occurring during an annual period, then Formula 1.2 defines the expected value of losses in excess of an aggregate loss amount of S. This function is an important tool in pricing aggregate excess or stop-loss reinsurance covers.

A second example relates to the more common type of excess of loss coverage, where the excess attachment point S is defined in terms of individual losses, rather than in the aggregate over a period. In this context, if x is a random variate representing the loss severity distribution with mean M, then Formula 1.2 defines the expected portion of M attributable to losses in excess of S. If the result of Formula 1.2 equals C, then C/M is the excess pure premium factor. If N is the expected number of losses, then NC is the expected value of excess losses.

Let us summarize what we have established. Formula 1.2 defines an important element of excess of loss pricing. It differs only by a present value factor from Formula 1.3, which defines a general formula for European call option pricing. Formula 1.1, the Black–Scholes formula, is a special case of Formula 1.3.

The implication of this is that excess of loss insurance and call options are essentially the same concepts. The one deals with insurance claims and the other deals with asset prices, but the pricing mathematics is basically the same.

---

[3]Formulas 1.1 and 1.3 produce the same result if x is a lognormal variate with parameters $(\ln P_0 + rt - 0.5\sigma^2 t, \sigma\sqrt{t})$, where this characterization follows Hogg and Klugman [4], who define a lognormal distribution by reference to the μ and σ of the related normal distribution. See Appendix A for the proof of this.

This insight is potentially tremendously powerful. If excess of loss insurance and call options are essentially the same concept in different contexts, then it must be possible to translate ideas from one context into the other context. In the remainder of this paper, we will explore some of the potential applications of the options market paradigm to insurance problems.

2. IMPLICATIONS OF THE EQUIVALENCE OF OPTION AND ACTUARIAL EXCESS OF LOSS MODELS

The mathematical equivalence of finance theory's Black–Scholes formula and an important actuarial function used in excess of loss insurance applications has a number of important implications for the convergence of insurance and finance. In this paper we will explore a few of them.

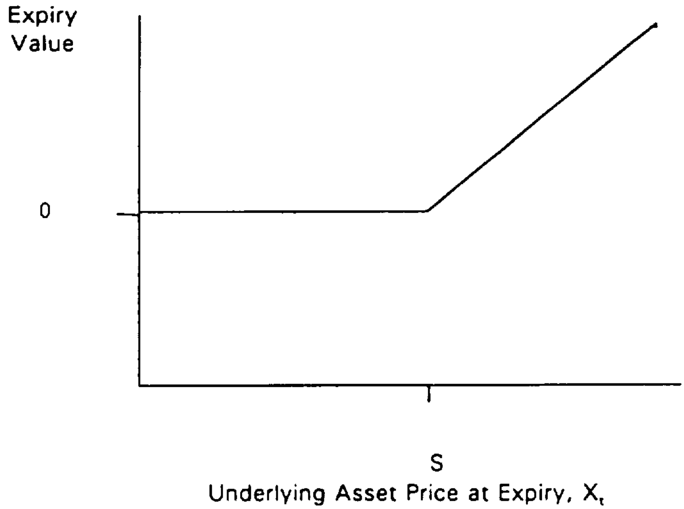
- Option market paradigms can be used to think about insurance problems; and this may well lead to new insurance or, perhaps more likely, reinsurance products.
- The more general actuarial excess of loss paradigm, which encompasses and frequently uses distributions other than the lognormal, can be used to think about the pricing of options on assets for which market prices are not lognormally distributed.
- Taking the two previous points together, it is possible to move beyond existing options and actuarial paradigms to spawn a new one that encompasses both. This, in turn, may lead to new product opportunities for insurers, investors, or both.

3. THE OPTION MARKET PARADIGM

The financial markets have been tremendously creative in devising products and techniques for managing financial risk. Most of this activity has occurred in what is loosely called the "derivatives market." Options are at the core of this market, and it is on

FIGURE 1

EXPIRY VALUE PROFILE: CALL OPTION, $c_t(S)$



this part of the derivatives market that we will focus our attention. Many derivative products are built around option features.

Basic Options

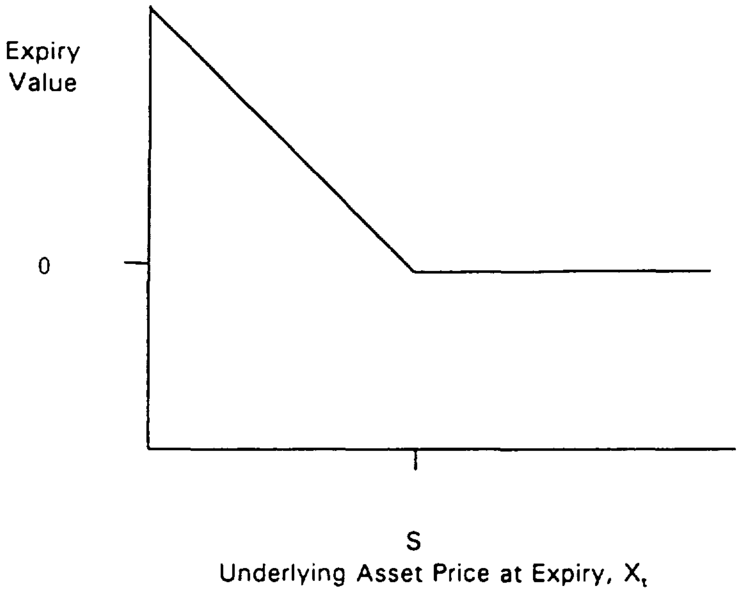
A "European" *call* option, $c_t(S)$, represents the right but not the obligation to buy the underlying asset at, and only at, time t at a price of S. Formula 1.3 describes the price of such a call option. Figure 1 shows its expiry value profile.

An "American" call option incorporates the right to exercise at any time up to and including time t. The Black–Scholes formula applies to the pricing of European calls. In this discussion our references will be to European-style options unless otherwise specified.

A "European" *put* option, $p_t(S)$, represents the right but not the obligation to sell the underlying asset at, and only at, time t

FIGURE 2

EXPIRY VALUE PROFILE: PUT OPTION, $p_t(S)$



at a price of S. Figure 2 shows the expiry value profile of a put option.

The general formula for the price of a put, $p_t(S)$ is

$$p_t(S) = e^{-rt} \int_0^S (S - x) \cdot f(x)dx. \qquad (3.1)$$

Spreads

The combination of two call options, one bought and one sold; e.g.,

$$c_t(S,T) = c_t(S) - c_t(T), \qquad \text{with} \quad T > S \qquad (3.2)$$

is known as a *call option spread.*

In insurance parlance, $c_t(S,T)$ refers to an *excess layer*. $c_t(S,T)$ is the pure premium for the layer of $T-S$ excess of S.

Put option spreads can be defined in a similar way to call option spreads.[4]

Implications for Insurance Applications

Once we recognize that a call spread is the same thing as an excess layer, a new world opens up. In theory, every option and related derivative product must have an insurance analogue! Since the derivative markets have been enormously creative in developing new product ideas, it should be possible to mine that trove of ideas for potentially innovative insurance and reinsurance product concepts.

As an example of how this can be done, we will analyze the derivatives concept of a *cylinder*. Then we will reconstruct it as a reinsurance product.

In its extreme form, a zero cost cylinder is created by the simultaneous purchase of a call and sale of a put (or vice versa) of equal value, usually at different out-of-the-money exercise prices but having the same expiration date.[5] If the cylinder involves a long call (i.e., the purchase of a call) and a short put (i.e., the sale of a put), its value increases when the value of the underlying asset increases and decreases when the asset value decreases. This is a "bullish" position. If the cylinder involves a short call and a long put, its value increases when the value of the underlying asset decreases and declines when the underlying asset value increases. This is a "bearish" position.

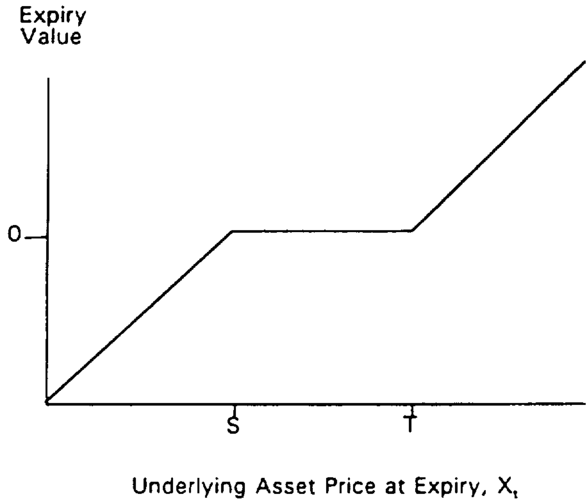
---

[4]For a detailed discussion of the mathematics of call, put, and cylinder spreads, see Appendix B. There are also a number of good reference books on financial derivatives, including Redhead [5] and Hull [7], that provide more comprehensive treatment of the subject. There is also a British paper, Kemp [8], which examines the subject from a more actuarial perspective, although it is not particularly oriented toward non-life issues.

[5]This is the extreme form. Note that a cylinder need not be "zero cost." For further discussion of cylinders and other option combinations, see [5].

FIGURE 3A

EXPIRY VALUE PROFILE: BULL CYLINDER OPTION $\text{cyl}_t(S,T)$



Bull and bear cylinders are defined as follows:

$$\text{cyl}_t(S,T) = c_t(T) - p_t(S), \qquad T > P_0 > S \quad \text{(bull)}$$

$$-\text{cyl}_t(S,T) = p_t(S) - c_t(T), \qquad T > P_0 > S \quad \text{(bear)}$$

and their expiry value profiles are shown in Figures 3A and 3B.

For an owner of the underlying asset, establishing a *bear* cylinder position partially hedges his asset position and reduces its volatility. Since in the case of a zero cost cylinder the values of the short call and long put are exactly offsetting, no money changes hands at inception of this position. At expiration, if the value of the underlying asset is X_t, the value of the cylinder position is

$$\begin{aligned} &-(X_t - T), && X_t \geq T; \\ &0, && T > X_t > S; \\ &S - X_t, && S \geq X_t. \end{aligned}$$

FIGURE 3B

EXPIRY VALUE PROFILE: BEAR CYLINDER OPTION $\mathrm{cyl}_t(S,T)$

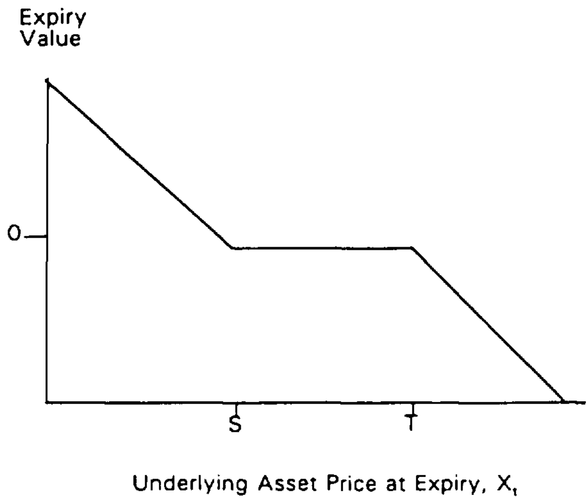


TABLE 1

Expiry Asset Price	Value of Cylinder	Expiry Value Asset + Cylinder
$X_t \geq T$	$-(X_t - T)$	T
$T > X_t > S$	0	X_t
$S \geq X_t$	$S - X_t$	S

The holder of this position gains $S - X_t$ for small values of X_t and loses $X_t - T$ for large values of X_t. For middle values of X_t he gains or loses nothing. His hedged position at expiry of the cylinder is summarized in Table 1.

In words, this implies that the hedged position yields the returns of the underlying asset (i.e., $X_t - P_0$), but subject to a maximum loss of $P_0 - S$ and a maximum gain of $T - P_0$.

If, rather than owning the underlying asset, an investor has a short position in it (i.e., it is a liability), he can partially hedge that position with a *bull* cylinder.

Suppose the underlying asset is the right to recover insurance claims. To an insured, this is an asset (a "long" position). To an insurer, it is a liability (a "short" position). Therefore, an insurer could use a bull cylinder to partially hedge his exposure.

If there were an established derivatives market trading options on insurance claims, as there is for a number of other financial assets, an insurer would be able to hedge its exposure by buying any of a variety of products; e.g., call options, call spreads, bull cylinders, or bull cylinder spreads. At present, there is only a limited derivatives market for options on insurance claims (namely, the excess of loss reinsurance market) and, broadly speaking, it offers only one product: the call spread.[6] One of the key themes of this paper is that conceptually there is no reason why the reinsurance market could not offer similar products to those found in the broader derivatives market.

Now let us consider how the cylinder concept, which has the advantage of lower initial cost to the buyer compared to a simple call option, might be translated into a reinsurance product. To illustrate one way this might work, first imagine a high level excess of loss layer with a retention of T_1 and a limit of $T_2 - T_1$. The market premium, ignoring all expenses, for conventional coverage is $c_t(T_1, T_2)$.

To create the cylinder type structure, we need to introduce a feature equivalent to the sale of a put. Consider a second, unreinsured, layer of $S_1 - S_2$ excess of S_2 *within* the company's reinsurance retention, which will form the basis of the required put spread. Let $p_t(S_1, S_2)$ denote the value of this put spread.

[6] At the time this paper was written, the Chicago Board of Trade's efforts to create a market for options on U.S. catastrophe losses had not yet produced significant capacity.

A reinsurance cylinder spread can be created by the purchase by a ceding company of the high level excess of loss layer at a cost of $c_t(T_1,T_2)$ and the equivalent of the sale of a put spread on the lower layer at a price of $p_t(S_1,S_2)$. (This is not necessarily a zero cost cylinder.) The premium outlay of the ceding company at the beginning of the contract would then be $c_t(T_1,T_2) - p_t(S_1,S_2)$. Since the reinsurer may require a minimum initial premium of $M \geq 0$, it may be necessary to allow the ratio of puts to calls to be different from one. If this ratio is represented by Q, the initial premium is given by

$$M = c_t(T_1,T_2) - Qp_t(S_1,S_2).$$

Under this structure, the premium of $c_t(T_1,T_2)$ buys exactly the same excess protection against large claims as the conventional reinsurance provides. The premium credit of $Qp_t(S_1,S_2)$ embedded in the initial premium represents the sale of a put spread on the lower layer by the ceding company to the reinsurer, the final value of which will be settled as an additional premium of $\min(Q(S_1 - X_t), Q(S_1 - S_2))$ when claim experience is known.

Let us now put some numbers to it. Let

$$\begin{aligned} c_t(T_1,T_2) &= \$2{,}500{,}000, \\ p_t(S_1,S_2) &= \$3{,}889{,}000, \\ Q &= 45\%, \\ S_1 &= \$15{,}000{,}000, \qquad \text{and} \\ S_1 - S_2 &= \$5{,}000{,}000. \end{aligned}$$

Then the initial premium is calculated as follows:

$$\begin{aligned} M &= c_t(T_1,T_2) - Q \cdot p_t(S_1,S_2) \\ &= \$2{,}500{,}000 - (.45)(\$3{,}889{,}000) \\ &= \$750{,}000. \end{aligned}$$

TABLE 2

Claims X_t	Initial Premium	Additional Premium	Total Premium
$X_t < S_2$	\$750	\$2,250	\$3,000
$S_2 \leq X_t \leq S_1$	\$750	(45%)(\$15,000 − X_t)	Slides \$750 to \$3,000
$S_1 < X_t$	\$750	0	\$750

Note: Premium figures in thousands.

At expiry of the contract (or at such time as agreed), an additional premium, A, equal to the expiry value of the "put spread" is due:

$$A = \min[Q(S_1 - X_t), Q(S_1 - S_2)]$$
$$= \text{lesser of: } (.45)(\$15{,}000{,}000 - X_t) \text{ and } (.45)(\$5{,}000{,}000).$$

The total premium under a "cylinder" reinsurance structure depends on the final cost of claims, X_t, as shown in Table 2.

This compares to the fixed premium of \$2,500,000 under the conventional contract and is shown graphically on Figure 4. In the cylinder structure, the ceding company pays a higher premium for its coverage of $T_2 - T_1$ excess of T_1 when the claim experience in the retained sublayer of $S_1 - S_2$ excess of S_2 is good (up to \$3,000,000 versus \$2,500,000). It pays a lower premium when claim experience in that layer is bad (\$750,000 versus \$2,500,000). In other words, the company pays more when its net claims experience is relatively good and it can afford higher reinsurance premiums, and less when its net is poor and it can least afford the burden of even normal reinsurance premiums. This is illustrated graphically in Figure 5 in terms of the effect on underwriting profit. This premium structure is more effective in reducing the volatility of a ceding company's net underwriting result than the conventional structure. Because of this stability, it might appeal to reinsurance buyers who use excess of loss coverage to reduce underwriting volatility.

FIGURE 4

ILLUSTRATION OF "CYLINDER" REINSURANCE PREMIUM STRUCTURE

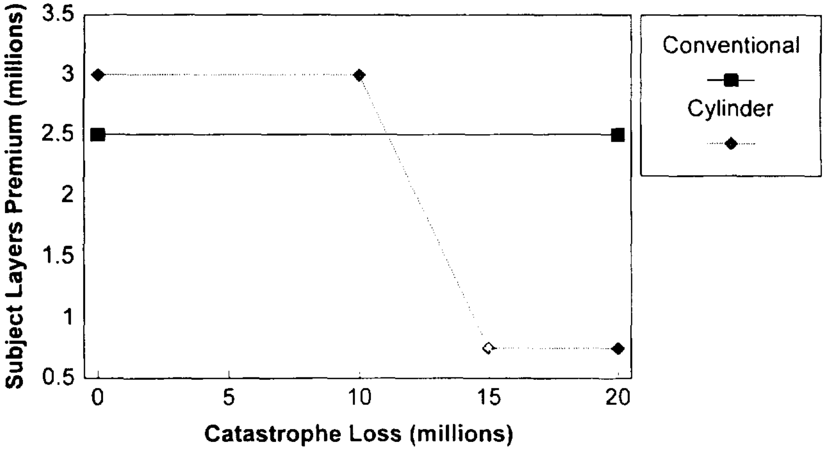
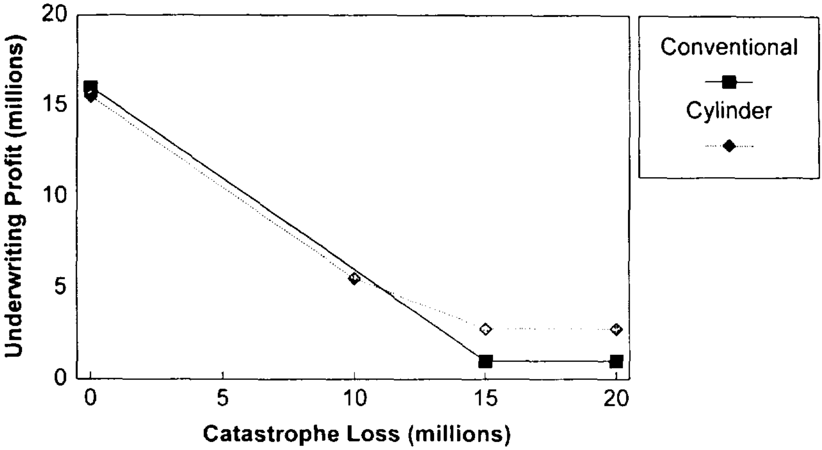


FIGURE 5

ILLUSTRATION OF "CYLINDER" REINSURANCE EFFECT ON UNDERWRITING PROFIT



The flip side of this is that the reinsurer's volatility is increased. Why would a reinsurer be willing to offer such a structure, which reduces premiums when claims are higher? The answer is that, in the context of a reinsurer's diversified portfolio, the incremental volatility will be small, while the extra benefit to the reinsurer's customer may well strengthen the overall reinsurance relationship. The reinsurance market has sometimes been criticized for selling "off the shelf" products that it wants to sell, rather than what ceding companies actually want to buy. In classes of reinsurance where reinsurers can sell as much off-the-shelf product as they want, there exists little or no pressure for them to introduce innovative structures like the foregoing example. However, to the extent some reinsurers want to pursue a more customer-focused strategy or simply feel competitive pressure, product innovation will increasingly begin to emerge. Indeed, the author is aware of at least one major reinsurer that has developed a product that has features similar to this example.

The cylinder is only one example. There are undoubtedly many other practical insurance and reinsurance products waiting to be discovered by exploring the derivatives product paradigm.

4. PRICING OPTIONS WHEN FUTURE PRICES ARE NOT LOGNORMAL

The Black–Scholes model relies on the assumption that market price changes over any finite time interval (expressed by the ratio P_n/P_{n-1}) are lognormally distributed. Since the product of lognormal variates is also lognormal, this assumption leads to the convenient conclusion that future market prices are also so distributed with predictable time-dependent parameters. The beauty of this is that the same framework can be used to determine the pure premium price for a one month, six month, or one year option, or one for any other time period.

Other stochastic price movement models have been described by others [2]. Like Black–Scholes, they support the pricing of options of any maturity. However, for assets subject to sudden or extreme price movements, or which are highly illiquid, a realistic stochastic price movement model may not exist. (Indeed, some analysts (e.g., Peters [6]) argue that *all* such models are flawed since they rely on too many assumptions that market experience has shown to be unrealistic.) This does not mean that options cannot be priced for such assets, but we need a different model.[7]

To price a call option exercisable at time t, we need an estimate of the probability distribution of the underlying asset price at time t as viewed from the vantage point of today. If it is possible to estimate this price distribution, it is possible to price an option. Pricing options of different maturities consistently is more difficult without a price movement model, because it requires separate estimates of the price distribution for each exercise date; but it can be done.

Formula 1.3, without the requirement that x be lognormal, can be used to price any option in this way. Of course, if the asset price at time t is not lognormal, the call option pure premium derived using Formula 1.3 is not equivalent to Black–Scholes. As with the estimation of loss distributions, determination of the price distribution of an asset may be made difficult by sparseness of data.

5. COMBINING THE OPTION AND ACTUARIAL PARADIGMS

Section 1 established that option pricing is analogous to excess of loss insurance pricing. Section 3 showed how new insurance innovations can be developed using the option market product paradigm. Section 4 discussed how to price options out-

[7]Even for the pricing of options on equities, for which Black–Scholes is widely used, traders recognize its imperfections. Fischer Black even wrote a paper entitled "How to Use the Holes in Black–Scholes," reprinted in [3]!

side the Black–Scholes framework. This section will illustrate how the synthesis of these ideas can lead to new product concepts outside the current scope of anything widely offered in either the financial or insurance market today.

Options on Reinsurance Premiums

Consider the following. A reinsurance contract can be thought of as an asset, namely the right to recover the monetary value of qualifying insurance claims from a reinsurer.

The price of a reinsurance contract is normally negotiated in the two or three months prior to the inception or anniversary of the contract. Sometimes there is significant uncertainty about the final price until the completion of the negotiations between the ceding company and reinsurers. Under certain circumstances, it might be valuable to a ceding company to fix the cost of its reinsurance coverage at an earlier date, or at least establish an upper bound. Using the option pricing paradigm, it is possible to establish a way to price such a cap.

Since the reinsurance premium, $prem_t$, for coverage incepting at time $t > 0$ (where time 0 would be today) is not known with certainty today, it is a random variable. The pure premium of a call option on $prem_t$ can therefore be calculated using Formula 1.3! Let us use an example to illustrate this.

Suppose the rate on line (i.e., the premium divided by the limit) of a catastrophe reinsurance contract currently in force is 20%. It is six months into the year and there has been a total loss to the layer. There was also a total loss three years ago.

In light of this experience, the premium for renewal will probably be increased, reflecting an upward reassessment by reinsurers of the exposure to loss. The ceding company will also probably be willing to pay a somewhat increased rate to begin to "pay back" reinsurers. However, the new rate will not be established until closer to the renewal date. In the meantime, for the next several months the premium the cedant faces for renewal is unknown and uncertain.

Suppose the market rate on line for renewal, viewed from the point six months prior to renewal, has a mean of 30% and is lognormally distributed with parameters (−1.20,.125). This implies that a rate increase of some size is nearly certain. It also implies about a 10% chance of a price of 35% or greater and about a 1% chance of a renewal price over 40%.

Formula 1.3 can be used to determine the pure premium of a call option to buy the reinsurance at renewal at a 30% rate on line (or any other price). If $r = 5\%$ and $t = .5$ (= 6 months), Formula 1.3 implies an option pure premium of (.975)(1.5%) = 1.46% rate on line, or 4.9% of the strike price of 30% rate on line.

If the ceding company were to buy this call option, it would be certain that the total cost of renewal would be no more than 31.46% rate on line (30% + 1.46%), and it might be less, since if the reinsurance market quotes less than 30%, the cedant would let the option expire unexercised.

Is this reinsurance premium call option a financial derivative or a reinsurance premium? The answer is, it could be either. In the way it was described above, it has the form of a derivatives market instrument. But the concept can also easily be incorporated into a reinsurance contract. Let us assume the renewal date is January 1. The option to buy the 12 months coverage incepting next January 1 can be embedded in a reinsurance contract with a premium payment warranty. If a certain required premium payment is not received before inception, the contract does not come into force.

In periods of significant reinsurance pricing uncertainty, purchasing a premium option will reduce that uncertainty and facilitate a ceding company's reinsurance planning and budgeting process. The specialist reinsurance market for this type of coverage historically has been largely found in London.

Rate Guarantees

The option paradigm can also be used to think properly about multi-year rate guarantees in the primary insurance market. Insureds sometimes seek to negotiate a fixed rate for several years or a limit on future rate increases. In these cases the insured is seeking, in effect, to secure a call option, or series of options, on future rate levels.

Suppose the insured wants a three-year rate guarantee for coverage that would normally be subject to an annual rate review. The current rate (which is guaranteed) is denoted by R_0. The market rates for coverage renewing one year and two years from now, respectively, are random variables R_1 and R_2. If the distributions of R_1 and R_2 can be estimated, it is possible to price the call options the insured is seeking. Then the insured can be charged for the options. Alternatively, the insurer may decide not to charge for the options, and merely use the options pricing exercise to determine the effective rate decrease the three-year guarantee represents.

If the options cannot be priced because the distributions of R_1 and R_2 cannot be estimated with sufficient confidence, perhaps it would be unwise for the insurer to agree to the rate guarantee!

At the time this paper was being prepared, multi-year contracts were beginning to appear in the reinsurance market as well. Obviously the same thought process applies to both insurance and reinsurance.

6. CONCLUSION

This paper has sought to demonstrate the value of the options market paradigm in thinking about and developing new insurance solutions. As the relationship between Formulas 1.1 and 1.3 makes clear, the underlying mathematics of insurance and the broader financial markets is the same. Apart from potential regulatory constraints, there is no logical reason why we should not see a convergence of insurance and other financial services in the coming years. This is especially likely at the wholesale level (e.g., reinsurance), where the relative importance of distribution systems and customer interface recedes and the importance of pure risk characteristics increases.

REFERENCES

[1] Black, Fischer and Myron Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81, May–June 1973, p. 637.

[2] Gerber, Hans and Elias Shiu, "Martingale Approach to Pricing Perpetual American Options," *ASTIN Bulletin* 24, 2, November 1994, p. 195.

[3] Konishi, Atsuo and Ravi Dattatreya, *The Handbook of Derivative Instruments*, Chicago, Probus Publishing Co., 1991.

[4] Hogg, Robert V. and Stuart A. Klugman, *Loss Distributions*, New York, John Wiley & Sons, 1984.

[5] Redhead, Keith, *Introduction to Financial Futures and Options*, Cambridge, England, Woodhead-Faulkener Limited, 1990, pp. 98–102, 161–162.

[6] Peters, Edgar, *Fractal Market Analysis*, New York, John Wiley & Sons, 1994.

[7] Hull, John, *Options, Futures, and Other Derivatives*, Third (International) Edition, London, Prentice Hall International, Inc., 1997.

[8] Kemp, M. H. D., "Actuaries and Derivatives," *British Actuarial Journal* 3, Part I, 1997, p. 51.

APPENDIX A

DERIVATION OF THE BLACK–SCHOLES OPTION PRICING FORMULA FROM A LOGNORMAL ASSET PRICE ASSUMPTION

Let

P_0 = the current market price of the security underlying the option,
t = time (in years) to option expiry,
r = the risk-free interest rate used for continuous compounding (i.e., the force of interest),
x = a random variable for the future market price of the security underlying the option, at time t (expiry).

Assume x is lognormally distributed with parameters $\ln P_0 + rt - 0.5\sigma^2 t$ and $\sigma\sqrt{t}$, and mean $E(x) = P_t = \exp(\ln P_0 + rt)$. This implies $P_t = P_0 \cdot e^{rt}$.

X_t = the actual future market price of the security underlying the option, at expiry.
$c_t(S)$ = the current pure premium (i.e., ignoring transaction costs and risk) for an option to buy the underlying security at a price of S at time t. This is known as a "call option with a strike price of S." Because of its feature of exercise at only one date, it is known as a European option.

The call option $c_t(S)$ will have no intrinsic value at expiry if the market price, X_t, of the security is below the strike price, S. In that case, it is cheaper to buy the security directly at price X_t than to exercise to option to buy at expiry price S. No rational investor would pay a non-zero premium for such an option; hence its nil value.

$c_t(S)$ will have intrinsic value of $X_t - S$ at expiry if the market price X_t exceeds the strike price S. An investor would be indifferent to buying the security directly at price X_t and buying the call option $c_t(S)$ at a price of $X_t - S$ for immediate exercise at price S.

The pure premium of $c_t(S)$ is the probability weighted mean of all possible intrinsic values at expiry, discounted to reflect present value.[8]

If the correct interest rate for discounting is the risk-free rate, the pure premium is expressed as:

$$c_t(S) = e^{-rt} \cdot \int_S^\infty (x - S) \cdot f(x)\,dx \tag{A.1}$$

$$= e^{-rt} \cdot \left(\int_S^\infty x \cdot f(x)\,dx - S \int_S^\infty f(x)\,dx \right) \tag{A.2}$$

$$= e^{-rt} \cdot \left(\int_0^\infty x \cdot f(x)\,dx - \int_0^S x \cdot f(x)\,dx - S\left(1 - \int_0^S f(x)\,dx\right)\right). \tag{A.3}$$

In general, the first moment distribution

$$\frac{\int_0^A x \cdot f(x)\,dx}{E(x)}$$

of a lognormal variate x with parameters (μ, σ) is also lognormal with parameters $(\mu + \sigma^2, \sigma)$.

In the present case, x is lognormal $(\ln P_0 + rt - 0.5\sigma^2 t, \sigma\sqrt{t})$ and its first moment distribution has parameters $(\ln P_0 + rt + 0.5\sigma^2 t, \sigma\sqrt{t})$. Accordingly, the second term within the main

[8]The justification for use of the risk-free rate is described in footnote 2 in the body of the paper.

brackets of Formula A.3 can be restated as follows:

$$\int_0^S x \cdot f(x)\,dx = \mathrm{E}(x) \cdot N\left(\frac{\ln S - (\ln P_0 + rt + 0.5\sigma^2 t)}{\sigma\sqrt{t}}\right)$$
$$= P_t \cdot N\left(\frac{\ln S - (\ln P_0 + rt + 0.5\sigma^2 t)}{\sigma\sqrt{t}}\right),$$

where N is the cumulative distribution function of the standard normal distribution.

Evaluation of the other terms of Formula A.3 is straightforward, and this formula can now be rewritten as:

$$c_t(S) = P_t e^{-rt} \cdot \left(1 - N\left(\frac{\ln S - (\ln P_0 + rt + 0.5\sigma^2 t)}{\sigma\sqrt{t}}\right)\right)$$
$$- Se^{-rt} \cdot \left(1 - N\left(\frac{\ln S - (\ln P_0 + rt - 0.5\sigma^2 t)}{\sigma\sqrt{t}}\right)\right)$$
$$= P_0\left(1 - N\left(\frac{\ln S - \ln P_0 - (r + 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)\right)$$
$$- Se^{-rt} \cdot \left(1 - N\left(\frac{\ln S - \ln P_0 - (r - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)\right)$$
$$= P_0\left(1 - N\left(\frac{\ln(S/P_0) - (r + 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)\right)$$
$$- Se^{-rt} \cdot \left(1 - N\left(\frac{\ln(S/P_0) - (r - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)\right); \quad \text{(A.4)}$$

and, since $1 - N(z) = N(-z)$,

$$c_t(S) = P_0 \cdot N\left(\frac{\ln(P_0/S) + (r + 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)$$
$$- Se^{-rt} \cdot N\left(\frac{\ln(P_0/S) + (r - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right). \quad \text{(A.5)}$$

Let

$$d_1 = \frac{\ln(P_0/S) + (r + 0.5\sigma^2)t}{\sigma\sqrt{t}},$$

and

$$d_2 = \frac{\ln(P_0/S) + (r - 0.5\sigma^2)t}{\sigma\sqrt{t}}.$$

Then Formula A.5 can be restated as

$$c_t(S) = P_0 \cdot N(d_1) - Se^{-rt} \cdot N(d_2). \qquad \text{(A.6)}$$

This is the Black–Scholes option pricing formula.

APPENDIX B

VALUATION OF CALL, PUT, AND CYLINDER SPREADS

Call Spreads

The value of a *call spread* $c_t(T_1,T_2)$ with $T_2 > T_1$ and time t to expiry is given by

$$\begin{aligned} c_t(T_1,T_2) &= c_t(T_1) - c_t(T_2) \\ &= e^{-rt}\left[\int_{T_1}^{\infty}(x-T_1)\cdot f(x)dx - \int_{T_2}^{\infty}(x-T_2)\cdot f(x)dx\right] \\ &= e^{-rt}\left[\int_{T_1}^{T_2}(x-T_1)\cdot f(x)dx + \int_{T_2}^{\infty}(x-T_1)\cdot f(x)dx \right. \\ &\qquad \left. - \int_{T_2}^{\infty}(x-T_2)\cdot f(x)dx\right] \\ &= e^{-rt}\left[\int_{T_1}^{T_2}(x-T_1)\cdot f(x)dx + \int_{T_2}^{\infty}(T_2-T_1)\cdot f(x)dx\right]. \end{aligned} \tag{B.1}$$

Note the similarity to the formulas used to work with excess layers in insurance applications.

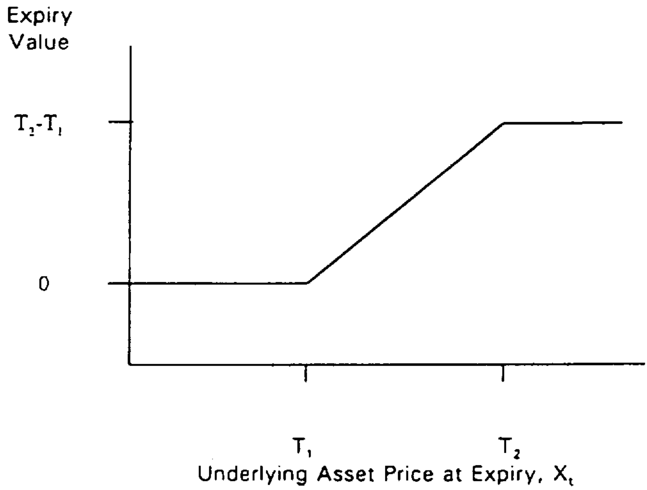
If the actual price of the underlying asset at expiry of the option is X_t, the value of the long call spread position at expiry is given by

$$\begin{array}{ll} T_2 - T_1, & X_t \geq T_2; \\ X_t - T_1, & T_2 > X_t > T_1; \\ 0, & T_1 \geq X_t. \end{array}$$

This is shown graphically in Figure B-1.

FIGURE B-1

EXPIRY VALUE PROFILE: CALL OPTION SPREAD $c_t(T_1,T_2)$



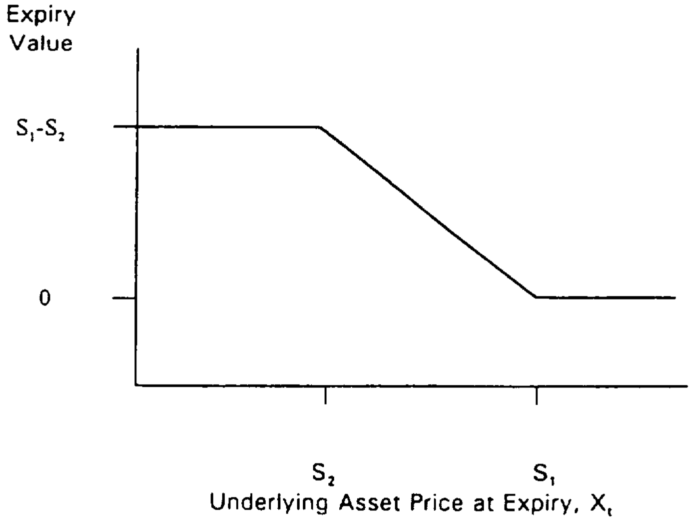
Put Spreads

The value of a *put spread* $p_t(S_1,S_2)$ with $S_1 > S_2$ and time t to expiry is given by

$$\begin{aligned} p_t(S_1,S_2) &= p_t(S_1) - p_t(S_2) \\ &= e^{-rt}\left[\int_0^{S_1}(S_1 - x)\cdot f(x)dx - \int_0^{S_2}(S_2 - x)\cdot f(x)dx\right] \\ &= e^{-rt}\left[\int_0^{S_2}(S_1 - x)\cdot f(x)dx + \int_{S_2}^{S_1}(S_1 - x)\cdot f(x)dx \right. \\ &\qquad \left. - \int_0^{S_2}(S_2 - x)\cdot f(x)dx\right] \\ &= e^{-rt}\left[\int_{S_2}^{S_1}(S_t - x)\cdot f(x)dx + \int_0^{S_2}(S_1 - S_2)\cdot f(x)dx\right]. \end{aligned} \tag{B.2}$$

FIGURE B-2

EXPIRY VALUE PROFILE: PUT OPTION SPREAD $p_t(S_1, S_2)$



The value of the long put spread position at expiry is given by

$$\begin{array}{ll} 0, & X_t \geq S_1; \\ S_1 - X_t, & S_1 > X_t > S_2; \\ S_1 - S_2, & S_2 \geq X_t. \end{array}$$

This is shown graphically in Figure B-2.

Put-Call Parity

There is an important relationship between the value of calls and puts known as "put-call parity." Consider two portfolios. The first consists of an asset with a value of P_0 and a related put option worth $p_t(T_1)$. The second consists of a T-bill valued at $T_1 \cdot e^{-rt}$ and a call option on the asset in the first portfolio, valued at $c_t(T_1)$.

These two portfolios have identical expiry value profiles (namely, $\max(T_1, P_t)$), so unless there are obstacles to arbitrage trading, they must have equal market values for any $T_1 \geq 0$:

$$P_0 + p_t(T_1) = T_1 e^{-rt} + c_t(T_1). \tag{B.3}$$

We can use put-call parity to derive the analogous relationship between put and call spreads:

Since

$$T_1 e^{-rt} = P_0 + p_t(T_1) - c_t(T_1)$$

and

$$T_2 e^{-rt} = P_0 + p_t(T_2) - c_t(T_2),$$

then

$$\begin{aligned}(T_2 - T_1)e^{-rt} &= p_t(T_2) - c_t(T_2) - p_t(T_1) + c_t(T_1) \\ &= c_t(T_1, T_2) + p_t(T_2, T_1). \end{aligned} \tag{B.3a}$$

A brief analysis of Formula B.3a shows that it is consistent with using the risk-free rate for discounting European option pure premiums. If we restate Formula B.3a in terms of integrals and treat the interest rate to be used for discounting the right side of the equation as an unknown, i, we obtain:

$$\begin{aligned}(T_2 - T_1)e^{-rt} \\ = e^{-it}\Bigg(&\int_{T_1}^{T_2}(x - T_1)\cdot f(x)dx \\ &+ \int_{T_2}^{\infty}(T_2 - T_1)\cdot f(x)dx + \int_0^{T_1}(T_2 - x)\cdot f(x)dx \\ &+ \int_{T_1}^{T_2}(T_2 - x)\cdot f(x)dx - \int_0^{T_1}(T_1 - x)\cdot f(x)dx\Bigg)\end{aligned}$$

TABLE 3

Expiry Price	Long Call Value	Short Put Value	Cash Value	Short Put+ Cash Value
$X_t \geq T_2$	$T_2 - T_1$	0	$T_2 - T_1$	$T_2 - T_1$
$T_2 > X_t > T_1$	$X_t - T_1$	$-(T_2 - X_t)$	$T_2 - T_1$	$X_t - T_1$
$T_1 \geq X_t$	0	$-(T_2 - T_1)$	$T_2 - T_1$	0

$$= e^{-it}\left(\int_0^{T_2}(x - T_1)\cdot f(x)dx + \int_{T_2}^{\infty}(T_2 - T_1)\cdot f(x)dx + \int_0^{T_2}(T_2 - x)\cdot f(x)dx\right)$$

$$= e^{-it}\left(\int_0^{T_2}(T_2 - T_1)\cdot f(x)dx + \int_{T_2}^{\infty}(T_2 - T_1)\cdot f(x)dx\right)$$

$$= e^{-it}\int_0^{\infty}(T_2 - T_1)\cdot f(x)dx$$

$$= e^{-it}(T_2 - T_1),$$

which implies $i = r$.

Formula B.3a also implies a definition for a call spread in terms of a put spread and T-bills:[9]

$$C_t(T_1,T_2) = (T_2 - T_1)e^{-rt} - p_t(T_2,T_1). \qquad \text{(B.3b)}$$

This means that it is possible to achieve a synthetic call spread position using put spreads and vice versa. In particular, Formula B.3b says that selling a put spread, $p_t(T_2,T_1)$, and holding the present value of $T_2 - T_1$ in T-bills is equivalent to buying a call spread, $c_t(T_1,T_2)$. To see this, Table 3 compares the expiry values of these two positions.

[9]Note that formulas B.3a and B.3b imply a put-call parity relationship for spreads that, unlike the ordinary put-call parity formula, has no reference to P_0.

TABLE 4

Expiry Price	Long Put Value	Short Call Value	Cash Value	Short Call+ Cash Value
$X_t \geq T_2$	0	$-(T_2 - T_1)$	$T_2 - T_1$	0
$T_2 > X_t > T_1$	$T_2 - X_t$	$-(X_t - T_1)$	$T_2 - T_1$	$T_2 - X_t$
$T_1 \geq X_t$	$T_t - T_1$	0	$T_2 - T_1$	$T_2 - T_1$

Alternatively, since

$$p_t(T_2,T_1) = (T_2 - T_1)e^{-rt} - c_t(T_1,T_2),$$

buying a put spread $p_t(T_2,T_1)$ is equivalent to selling a call spread $c_t(T_1,T_2)$ and holding the present value of $T_2 - T_1$ in T-bills, as shown in Table 4.

Cylinder Spreads

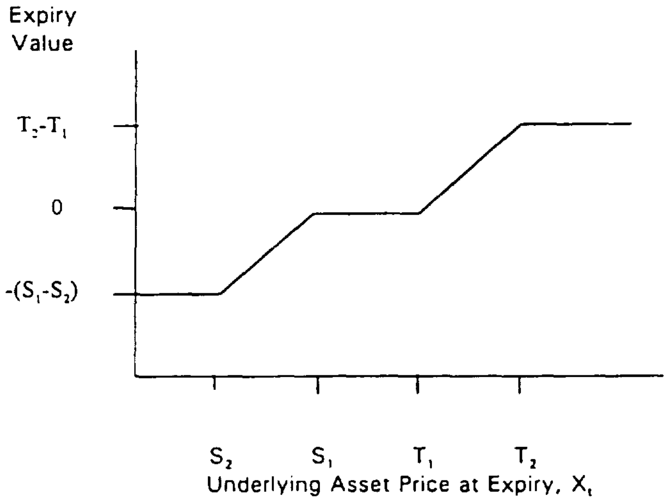
The *bull cylinder spread*, $\text{cyl}_t(S_1,S_2;T_1,T_2)$, created from the call and put spreads defined above, where $T_2 > T_1 > S_1 > S_2$, has the following value:

$$\begin{aligned}\text{cyl}(S_1,S_2;T_1,T_2) &= c_t(T_1,T_2) - p_t(S_1,S_2)\\ &= e^{-rt}\left[\int_{T_1}^{T_2}(x - T_1)\cdot f(x)dx + \int_{T_2}^{\infty}(T_2 - T_1)\cdot f(x)dx\right.\\ &\quad - \int_{S_2}^{S_1}(S_1 - x)\cdot f(x)dx\\ &\quad \left.- \int_{0}^{S_2}(S_1 - S_2)\cdot f(x)dx\right]. \qquad \text{(B.4)}\end{aligned}$$

The value of $\text{cyl}_t(S_1,S_2;T_1,T_2)$ depends on the choices of S_1, S_2, T_1 and T_2. These parameters can be chosen to create a cylinder structure that produces the desired cylinder value at time t to

FIGURE B-3

EXPIRY VALUE PROFILE: BULL CYLINDER OPTION SPREAD
$\mathrm{cyl}_t(S_1, S_2, T_1, T_2)$



expiry. Additional flexibility can be introduced in the cylinder structure by relaxing the requirement that the same number of call and put spreads are used. If Q is defined as the ratio of the number of puts to the number of calls, then the value of $\mathrm{cyl}_t(S_1, S_2; T_1, T_2)$ is given by

$$\mathrm{cyl}(S_1, S_2; T_1, T_2) = e^{-rt}\left[\int_{T_1}^{T_2}(x - T_1)\cdot f(x)\,dx + \int_{T_2}^{\infty}(T_2 - T_1)\cdot f(x)\,dx - Q\cdot\left(\int_{S_2}^{S_1}(S_1 - x)\cdot f(x)\,dx + \int_0^{S_2}(S_1 - S_2)\cdot f(x)\,dx\right)\right].$$

At expiry the value of the bull cylinder spread position is given by

$$\begin{array}{ll} T_2 - T_1, & X_t \geq T_2; \\ X_t - T_1, & T_2 > X_t \geq T_1; \\ 0, & T_1 > X_t > S_1; \\ -Q \cdot (S_1 - X_t), & S_1 \geq X_t > S_2; \\ -Q \cdot (S_1 - S_2), & S_2 \geq X_t. \end{array}$$

This is illustrated for $Q = 1$ in Figure B-3.