GENERALIZED PREMIUM FORMULAE

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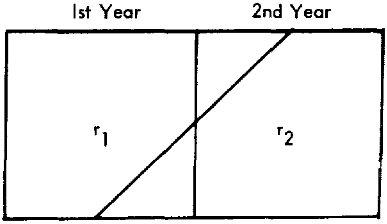
Socrates: But from which line shall we get it? Try and tell us exactly; and if you would rather not reckon it out, just show what line it is.
Boy: Well, on my word, Socrates, I for one do not know.

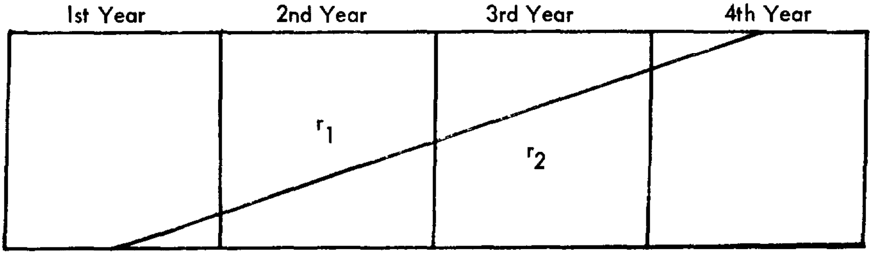
Plato

Fundamental to most ratemaking procedures is the adjustment of historical data to reflect current or anticipated conditions. In ratemaking methods which use premium data it is necessary to adjust the actual premiums to the current rate level. One technique for estimating this adjustment is the parallelogram method, also referred to as the Pro-rata method. This involves drawing a diagram and assigning weights to the different rate levels in proportion to areas on the diagram. In the case where there is an annual policy term the diagram is drawn as follows:



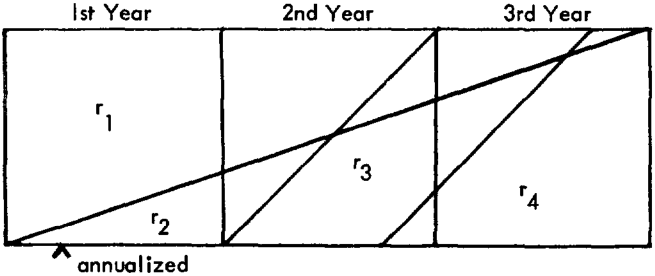
The interpretation of this diagram is that the rate level changed in the middle of the first year from r_1 to r_2. The exposures written at r_1 expire uniformly along the diagonal line and then are renewed at the new rate level r_2. In each year the exposure earned at the new rate level is proportional to the area under the diagonal line. In the first year this is equal to one-eighth of the total area and in the second year is equal to seven-eighths of the total area. Therefore, the average rate level in the first year is given by $7/8r_1 + 1/8r_2$ and the second year is $1/8r_1 + 7/8r_2$. Having determined the average rate level in each year, the factor to adjust to the current rate level is the current rate level divided by the average rate level in that year.

The parallelogram method is also used for policy terms other than annual. In the case of a three year policy term the method is identical except that the slope of the diagonal line becomes one-third rather than one. The diagram becomes:



Again this diagram assumes that a rate change was made in the middle of the first year. The proportion of exposure earned at the new rate is respectively 1/24, 1/3, 2/3, and 23/24 in the first, second, third and fourth years.

The methods described above have been used for many years. This paper had its genesis in a fairly simple problem arising from an application of these methods. Rate level adjustment factors were being calculated from earned premiums and a case was encountered where there had been several rate changes and in addition the policy term had been changed from three years to one year. Using the customary parallelogram approach the diagram looked like this:



In this case the exposure was initially being written on a three year term; the term was changed to annual one-fourth of the way into the first year. Rate changes were made at the beginning of the first and second years and at three fourths of the second year. The problem, of course, is what to do with the crossing lines. Before a solution was found several interesting relationships were discovered and a theoretical framework was developed which may be useful in solving other problems.

WRITTEN AND EARNED EXPOSURES

Let the function $f(x)$ stand for the rate of exposure writing at time x. Although the writing of exposures is the result of many discrete transactions we will assume that $f(x)$ is at least piecewise continuous. The written exposure between time x_0 and x_1 may be expressed as:

$$WE(x_0, x_1) = \int_{x_0}^{x_1} f(x)dx \tag{1}$$

The case where exposure is being written at a constant rate is equivalent to $f(x) = K$; the written exposure is given by:

$$WE(x_0, x_1) = \int_{x_0}^{x_1} Kdx = K(x_1 - x_0) \tag{2}$$

To calculate the earned exposure it is necessary to take into account the policy term t. The earned amount between x_0 and x_2 may be derived by partitioning the x-axis into segments $\triangle i$; let $x_i E \triangle i$, then the exposure written on the i th partition is approximately equal to $f(x_i) \cdot \triangle i$. Assuming that $x_0 \leq x_1 - t$ (identical results are obtained if $x_0 > x_1 - t$, the proof is very similar), then the earned exposure between x_0 and x_1 is approximated by:

a. 0 if $x_i \leq x_0 - t$

b. $((x_i + t - x_0)/t)\, f(x_i) \cdot \triangle i$ if $x_0 - t < x_i \leq x_0$

c. $f(x_i) \cdot \triangle i$ if $x_0 < x_i \leq x_i - t$

d. $((x_1 - x_i)/t)\, f(x_i) \cdot \triangle i$ if $x_1 - t < x_i \leq x_1$

e. 0 if $x_i > x_1$

Summing and taking the appropriate limits the earned exposure is equal to

$$EE(x_0, x_1) = \int_{x_0 - t}^{x_0} \frac{x + t - x_0}{t} f(x)dx + \int_{x_0}^{x_1 - t} f(x)dx + \int_{x_1 - t}^{x_1} \frac{x_1 - x}{t} f(x)dx \tag{3}$$

Similarly the unearned exposure at any time x_0 is given by

$$U(x_0) = \int_{x_0 - t}^{x_0} \frac{x + t - x_0}{t} f(x)dx \tag{4}$$

Using formulae (1), (3) and (4) it can be shown that

$$EE(x_0, x_1) = U(x_0) + WE(x_0, x_1) - U(x_1) \tag{5}$$

Three special cases are of interest in that they confirm working formulae.

Special Case I: $f(x) = K$

$$\begin{aligned} EE(x_0, x_1) &= \int_{x_0 - t}^{x_0} \frac{x + t - x_0}{t} Kdx + \int_{x_0}^{x_1 - t} Kdx \\ &+ \int_{x_1 - t}^{x_1} \frac{x_1 - x}{t} Kdx \\ &= \tfrac{1}{2} Kt + K(x_1 - t - x_0) + \tfrac{1}{2} Kt \\ &= K(x_1 - x_0) = WE(x_0, x_1) \end{aligned}$$

Thus with a constant rate of writing the earned exposure will equal the written exposure.

Special Case II: $(x_1 - x_0) = t$

$$f(x) = K_1, x_0 - t < x \leq x_0$$

$$f(x) = K_2, x_0 < x \leq x_1$$

$$EE(x_0, x_1) = \tfrac{1}{2} K_1 t + \tfrac{1}{2} K_2 t = \tfrac{1}{2} WE(x_0 - t, x_0) + \tfrac{1}{2} WE(x_0, x_1)$$

This is the earned exposure calculated by the "annual pro-rata" method with annual term.

Special Case III: $(x_1 - x_0) = \tfrac{1}{3} t$

$$f(x) = K_1, x_0 - t < x \leq x_0 - \tfrac{2}{3} t$$

$$f(x) = K_2, x_0 - \tfrac{2}{3} t < x \leq x_0 - \tfrac{1}{3} t$$

$$f(x) = K_3, x_0 - \tfrac{1}{3} t < x \leq x_0$$

$$f(x) = K_4, x_0 < x \leq x_1$$

$$EE(x_0, x_1) = \frac{K_1 t}{18} + \frac{K_2 t}{9} + \frac{K_3 t}{9} + \frac{K_4 t}{18}$$

$$= WE(x_0 - t, x_0 - \tfrac{2}{3}\, t)/6 + WE(x_0 - \tfrac{2}{3}\, t, x_0 - \tfrac{1}{3}\, t)/3$$

$$+ WE(x_0 - \tfrac{1}{3}\, t, x_0)/3 + WE(x_0, x_1)/6$$

This is recognizable as a version of the "annual pro-rata" method but with three year term. Other common formulae which are based on the assumption of constant writings over various periods of time may also be derived. It is of interest to note that the various pro-rata formulae also hold true when $f(x) = a + bx$, a and b constants; however, the earned exposure will no longer equal the written exposure.

Another illustrative example is provided in the case where the rate of exposure writing is changing at a uniform rate. In this case $f(x) = Ke^{cx}$, where c is the rate of change. From (1) the written exposure is given by

$$WE(x_0, x_1) = \frac{K}{c}\,(e^{cx_1} - e^{cx_0})$$

From either equations (3) or (5) it can be shown that

$$EE(x_0, x_1) = WE(x_0, x_1)\,(1 - e^{-ct})/ct$$

Note that as $C \to O$, $EE(x_0, x_1) \to WE(x_0, x_1)$

For a number of typical values the ratio of earned exposures to written exposure when there is a constant rate of change in writings is as follows:

	Ratios of Earned to Written Annual Rate of Change in Writings[2]			
Term	−20%	−10%	+10%	+20%
6 mos	1.0470	1.0242	.9765	.9558
12 mos	1.0970	1.0492	.9538	.9141
36 mos	1.3310	1.1576	.8697	.7702

The concept of importance for what follows is that of earned contribution to the interval (x_0, x_1) from the writings over the interval (y_0, y_1). Roughly this is the portion of $WE(y_0, y_1)$ which is earned between x_0 and x_1. More precisely, first define the function $g(x)$ as follows:

$$g(x) = \begin{cases} O, x \leq y_0 \\ f(x), y_0 < x \leq y_1 \\ O, x > y_1 \end{cases}$$

The earned contribution to the interval (x_0, x_1) from the writings over (y_0, y_1) is equal to

$$EC(y_0, y_1; x_0, x_1) = \int_{x_0 - t}^{x_0} \frac{x + t - x_0}{t} g(x)dx + \int_{x_0}^{x_1 - t} g(x)dx + \int_{x_1 - t}^{x_1} \frac{x_1 - x}{t} g(x)dx \tag{6}$$

In general it can be shown that the following are true:

$$EC(-\infty, x_0; x_0, x_1) + EC(x_0, x_1; x_0, x_1) = EE(x_0, x_1)$$

and when $x_0 < x_1 < x_2$

$$EC(y_0, y_1; x_0, x_1) + EC(y_0, y_1; x_1, x_2) = EC(y_0, y_1; x_0, x_2)$$

[2] The annual rate of change is given by $a = e^c - 1$

and when $y_0 < y_1 < y_2$

$$EC(y_0, y_1; x_0, x_1) + EC(y_1, y_2; x_0, x_1) = EC(y_0, y_2; x_0, x_1)$$

Also of interest are the following:

Let $f(x) = K, \mathfrak{z} > 0$

$$EC(-\infty, a; a, a + \mathfrak{z}) = \tfrac{1}{2} Kt \text{ when } \mathfrak{z} \geq t$$

$$EC(a, a + \mathfrak{z}; a, a + \mathfrak{z}) = \tfrac{1}{2} K \mathfrak{z}^2/t$$

That a change in term will have an immediate effect on exposures written is obvious; the same policies are being written but more or less exposure is being booked depending upon whether the term was lengthened or shortened. However, a change in term should not affect exposures earned; this fact allows us to determine the change in $f(x)$ due to a change in t.

If the change from t_0 to t_1 is made at time x_0, then if $f(x) = f_0(x)$ when $x < x_0$ the new function $f(x) = f_1(x)$ for $x \geq x_0$ may be determined by using the following equation with $\mathfrak{z} > 0$

$$EC(-\infty, x_0; x_0, x_0 + \mathfrak{z}) + EC(x_0, x_0 + \mathfrak{z}; x_0, x_0 + \mathfrak{z}) = EE(x_0, x_0 + \mathfrak{z}) \tag{7}$$

The first term on the left side of equation (7) and the term on the right side are calculated using t_0 and $f_0(x)$, while the second term on the left side of the equation contains t_1 and $f_1(x)$. Take the case where $f_0(x) = K_0$ and $\mathfrak{z} < t_1 < t_0$ then we have:

$$\tfrac{1}{2}\, K_0 t_0 + K_0\,(t_0 - \mathfrak{z})\ \left(\tfrac{1}{2}\ (t_0 + \mathfrak{z})/t_0 - 1\right) +$$
$$\int_{x_0}^{x_0 + \mathfrak{z}} \frac{x_0 + \mathfrak{z} - x}{t_1}\ f_1(x)dx = K_0\mathfrak{z}$$

simplifying results in the integral equation

$$\int_0^{\mathfrak{z}} (\mathfrak{z} - y)\ f_1\ (y)\ dy = \tfrac{1}{2}\ K_0\ (t_1/t_0)\ \mathfrak{z}^2$$

which has solution:

$$f_1 = K_0(t_1/t_0)$$

Where f_1 has domain $x_0 < x \leq x_0 + t_1$

The general solution to the problem when $f_0(x) = K_0$ may be obtained by repeated applications of formula (7) with $\mathfrak{z}$ increasing. If N is the largest integer such that $Nt_1 \leq t_0$, it can be shown that $f(x)$ will have the following values:

$$f(x) = K_0(it_1/t_0) \text{ when } x_0 + (i-1)\ t_1 < x \leq x_0 + it,$$
$$i = 1, 2 \cdot \cdot \cdot N$$

and $f(x) = K_0(N+1)(t_1/t_0)$ when $x_0 + Nt_1 < x < x_0 + t_0$

$$f(x) = K_0\, N(t_1/t_0) \text{ when } x_0 + t_0 \leq x < x_0 + (N+1)\ t_1$$

and $f(x) = K_0(N+1)(t_1/t_0)$ when $x_0 + (N+j)\ t_1 \leq x < x_0 + t_0 + jt_1$

$$f(x) = K_0\, N(t_1/t_0) \text{ when } x_0 + t_0 + jt_1 \leq x < x_0 + (N+j+1)t$$
$$j = 1, 2 \cdot \cdot \cdot$$

A simple example may be helpful at this point: assume at time x_0 the term was changed from three to one; the exposure prior to x_0 had been written at a constant rate of K_0. We then have N equal to three and $f(x)$ is as follows:

$$
\begin{aligned}
f(x) &= \tfrac{1}{3}\ K_0 \quad , \quad x_0 < x \leq x_0 + 1 \\
&= \tfrac{2}{3}\ K_0 \quad , \quad x_0 + 1 < x \leq x_0 + 2 \\
&= \quad K_0 \quad , \quad x_0 + 2 < x \leq x_0 + 3 \\
&= \quad K_0 \quad , \quad x > x_0 + 3
\end{aligned}
$$

This confirms what we would expect, the written exposure drops to one-third the prior rate for one year rises to two-thirds the following year and then as all policies are converted to the one year basis the rate of writing returns to the prior rate.

The change from six month policies to annual policies illustrates another phenomenon; assuming again a constant rate K_0 prior to the change in term, we have N equal to zero and the following:

$$\begin{aligned} f(x) &= K_0\ (N+1)\ (1/0.5) = 2\,K_0\,, & x_0 < x \leq x_0 + 0.5 \\ &= K_0\,N\ (1/.05) = 0\,, & x_0 + 0.5 < x \leq x_0 + 1 \\ &= K_0\ (N+1)\ (1/0.5) = 2\,K_0, & x_0 + 1 < x \leq x_0 + 1.5 \\ &= K_0\,N\ (1/.05) = 0\,, & x_0 + 1.5 < x \leq x_0 + 2 \end{aligned}$$

. . . .

As can be seen a permanent distortion in the written exposure has resulted from the change in term. Within six months all policies are on an annual basis and none will be renewed for an additional six months. This is generally true whenever the new term does not evenly divide the old term. For example, a change from five year term to three year term, with the customary assumptions, will have the following effect on written exposures:

$$\begin{aligned} f(x) &= K_0\,(3/5)\,, & x_0 < x \leq x_0 + 3 \\ &= K_0\,(6/5)\,, & x_0 + 3 < x \leq x_0 + 5 \\ &= K_0\,(3/5)\,, & x_0 + 5 < x \leq x_0 + 6 \\ &= K_0\,(6/5)\,, & x_0 + 6 < x \leq x_0 + 8 \\ &= K_0\,(3/5)\,, & x_0 + 8 < x \leq x_0 + 9 \\ &= K_0\,(6/5)\,, & x_0 + 9 < x \leq x_0 + 11 \\ &= K_0\,(3/5)\,, & x_0 + 11 < x \leq x_0 + 12 \end{aligned}$$

Here the pattern of one year writing at $K_0\ (3/5)$ followed by two years at $K_0\ (6/5)$ continues indefinitely.

EARNED PREMIUMS AND RATE ADJUSTMENT FACTORS

Earned premiums are the result of both earned exposures and rates; by rate we will mean the charge for some fixed amount of exposure, thus a change in term in itself does not result in a change in rate.

With constant rate r the earned premium is given by

$$EP\ (x_0, x_1; r) = rEE\ (x_0, x_1)$$

When there have been different rates $r_1, r_2 \ldots r_n$ which have been in effect on the intervals $(y_0, y_1), (y_1, y_2) \ldots (y_{n-1}, y_n)$ then the earned premium is given by

$$EP\ (x_0, x_1) = \sum_{i=1}^{n} r_i\, EC\ (y_{i-1}, y_i; x_0\, x_1)$$

Example:

$f(x) = K$

r_1: $f\ x < x_0$

r_2: $f\ x \geq x_0$

$x_1 - x_0 = t = 1$

$$\begin{aligned} EP\ (x_0, x_1) &= r_1\, EC\ (-\infty, x_0; x_0, x_1) + r_2\, EC\ (x_0, x_1; x_0, x_1) \\ &= \tfrac{1}{2}\, r_1\, Kt + \tfrac{1}{2}\, r_2\, Kt = \tfrac{1}{2}\ (r_1 + r_2)\ Kt \end{aligned}$$

Which says that with an annual term a rate change at the beginning of the year will result in one-half of the premium earned at the old rate and one-half at the new rate.

The rate level adjustment factor, which is simply the factor to multiply the actual earned premium by to arrive at what the earned premium would have been if it were all written at a constant rate r_i is given by:

$$AF(x_0, x_1; r_i\text{Q} = r_i\, EE(x_0, x_1)/EP(x_0, x_1)$$

From the example above we have

$$AF(x_0, x_1; r_2) = r_2\, EE(x_0, x_1)/EP(x_0, x_1) = 2\, r_2/(r_1 + r_2)$$

THE ORIGINAL PROBLEM

We now have all the tools necessary to solve the original problem. The problem is to determine the rate level adjustment factors when the following conditions apply:

Rates:

$r_1, x < x_0$

$r_2, x_0 \leq x < x_0 + 1$

$r_3, x_0 + 1 \leq x < x_0 + 7/4$

$r_4, x_0 + 7/4 \leq x$

Terms:

$t = 3, x < x_0 + 1/4$

$t = 1, x \geq x_0 + 1/4$

With the assumption that the exposure was being written at a constant rate K_0 prior to annualization we have $f(x)$ as follows:

$$f(x) = \begin{cases} K_0, x < x_0 + 1/4 \\ 1/3\, K_0, x_0 + 1/4 \leq x < x_0 + 5/4 \\ 2/3\, K_0, x_0 + 5/4 \leq x < x_0 + 9/4 \\ K_0, x \geq x_0 + 9/4 \end{cases}$$

The earned premium at rate level r_4 would have been $r_4 K_0$ in each year. The actual earned premiums are estimated as:

$$\begin{aligned} EP(x_0, x_0 + 1) &= r_1\, EC(-\infty, x_0; x_0, x_0 + 1) \\ &\quad + r_2\, EC(x_0, x_0 + 1/4\,; x_0, x_0 + 1) \\ &\quad + r_2\, EC(x_0 + 1/4\,, x_0 + 1; x_0, x_0 + 1) \\ &= 5/6\, r_1 K_0 \\ &\quad + 7/96\, r_2 K_0 \\ &\quad + 9/96\, r_2 K_0 \end{aligned}$$

$$
\begin{aligned}
EP(x_0+1, x_0+2) &= r_1\, EC(-\infty, x_0; x_0+1, x_0+2) \\
&+ r_2\, EC(x_0, x_0+1/4; x_0+1, x_0+2) \\
&+ r_2\, EC(x_0+1/4, x_0+1; x_0+1, x_0+2) \\
&+ r_3\, EC(x_0+1, x_0+5/4; x_0+1, x_0+2) \\
&+ r_3\, EC(x_0+5/4, x_0+7/4; x_0+1, x_0+2) \\
&+ r_4\, EC(x_0+7/4, x_0+2; x_0+1, x_0+2) \\
&= 48/96\, r_1 K_0 \\
&+ 8/96\, r_2 K_0 \\
&+ 15/96\, r_2 K_0 \\
&+ 7/96\, r_3 K_0 \\
&+ 16/96\, r_3 K_0 \\
&+ 2/96\, r_4 K_0
\end{aligned}
$$

$$
\begin{aligned}
EP(x_0+2, x_0+3) &= r_1\, EC(-\infty, x_0; x_0+2, x_0+3) \\
&+ r_2\, EC(x_0, x_0+1/4; x_0+2, x_0+3) \\
&+ r_2\, EC(x_0+1/4, x_0+1; x_0+2, x_0+3) \\
&+ r_3\, EC(x_0+1, x_0+5/4; x_0+2, x_0+3) \\
&+ r_3\, EC(x_0+5/4, x_0+7/4; x_0+2, x_0+3) \\
&+ r_4\, EC(x_0+7/4, x_0+9/4; x_0+2, x_0+3) \\
&+ r_4\, EC(x_0+9/4, x_0+3; x_0+2, x_0+3) \\
&= 16/96\, r_1 K_0 \\
&+ 8/96\, r_2 K_0 \\
&+ 0/96\, r_2 K_0 \\
&+ 1/96\, r_3 K_0 \\
&+ 16/96\, r_3 K_0 \\
&+ 28/96\, r_4 K_0 \\
&+ 27/96\, r_4 K_0
\end{aligned}
$$

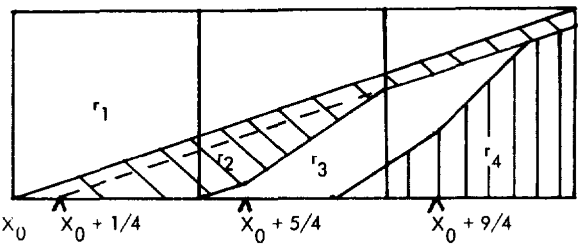
The rate adjustment factors are given by

$$AF(x_0, x_0 + 1; r_4) = r_4 \div [(5/6)\ r_1 + (1/6)\ r_2]$$

$$AF(x_0 + 1, x_0 + 2; r_4)$$
$$= r_4 \div [(1/2)\ r_1 + (23/96\ r_2 + (23/96)\ r_3 + 2/96)\ r_4]$$

$$AF(x_0 + 2, x_0 + 3; r_4)$$
$$= r_4 \div [(1/6)\ r_1 + (8/96)\ r_2 + (17/96)\ r_3 + (55/96)\ r_4]$$

Interestingly, the solution to this problem may be translated into a diagram which would look as follows:



Note that the line separating r_2 and r_3 changes slope at $x_0 + 5/4$ from 1/3 to 2/3; and the line separating r_3 and r_4 changes slope at $x_0 + 9/4$ from 2/3 to 1.

CONCLUSION

When $f(x)$ is other than a very simple formula, many of the equations become quite cumbersome; however, this presents no problem to a computer. More accurate rate level adjustment factors can be determined by making more realistic assumptions regarding the rate of exposure writings.

Throughout this paper it has been assumed that the premiums to be adjusted were calendar year premiums and that the changes in rates or term affected policies as they come due for renewal. In practice, other variations occur; it may be necessary to adjust policy year premiums rather than calendar year premiums. Also the rate or term changes may affect all outstanding policies rather than just renewal policies. These situations require techniques slightly different than those developed in this paper.

Aside from the relatively minor problem with the rate level adjustment factors and possible applications to corporate model building or more general areas where income or costs are deferred, the insight gained in the relationships between term, writings and earnings is of value in itself.