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TWO STUDIES IN AUTOMOBILE INSURANCE RATEMAKING

BY

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Section A, Effectiveness of Merit Rating and Class Rating, uses the Canadian experience for private passenger automobiles to show (1) that merit rating is almost as effective as the class plan in separating the better risks from the poorer risks, (2) that both merit rating and class rating leave unanalyzed a considerable amount of variation among risks and (3) that certain available evidence supports the conclusion that annual mileage, which has long been felt to be an important measure of hazard, is a very significant cause of this unanalyzed variation among risks.

Section B, Improved Methods for Determining Classification Rate Relativities, presents a method for obtaining relativities among groups on which a multiple classification system has been imposed. The customary method of calculating class relativities uses the total experience for each class with all subdivisions within the classes added together. With the customary method it is difficult to make a completely accurate adjustment for different distributions by territory or merit rating, because any change in the class relativities disturbs the other sets of relativities and conversely. It is shown that even if such an adjustment were made, the customary method of calculating relativities one set at a time does not reflect the relative credibility of each subgroup and does not produce the best fit to the actual data. Moreover it produces differences between the actual data and the fitted values which are far too large to be caused by chance. In addition, for private passenger automobile insurance in Canada, it is shown that two sets of relativities which are multiplied together cannot produce the best fit to the actual data, and some of the consequences of trying to do so are explained. Some methods are advanced whereby all sets of relativities for classes, merit ratings, territories, and so forth, can be calculated simultaneously, which will overcome all the deficiencies in the customary method. These improved methods use the technique of minimizing a measure (technically known as the Chi-square test) of the differences between the actual data and the fitted values. Some applications to other lines of insurance are mentioned.

Section A: Effectiveness of Merit Rating and Class Rating

Introduction

Private passenger automobile insurance uses a multiple classification system. We classify by age (under or over 25 years) and within each age we classify by occupation (farm or non-farm). We also classify by use and sex. On top of all this we classify by territory. And now we have begun to classify by previous accident and conviction record which is popularly called the "merit rating plan." There is no basic difference between merit rating and class rating if the rates for each merit rating group are based on the subsequent experience of cars previously classified according to their accident and conviction record, just as the rates for each class are based on the subsequent experience of cars previously classified according to the characteristics of the class plan. In actual fact, merit rating is a class rating plan and is part of the multiple classification system. However, in this paper, as a matter of convenience, and not implying a basic distinction, we will follow the common usage in the United States by referring to classification according to previous accident and conviction record as "the merit rating plan," and to classification according to age, sex, use and occupation as "the class plan."

A class plan which uses age, sex, use and occupation does not precisely classify each risk according to its true value. Underwriters have long recognized this, and it is further substantiated by the Canadian merit rating experience which shows that risks which have been accident-free for three or more years have better experience in the following year than the average for their class. Likewise a class plan which uses only the previous accident record would not precisely classify each risk[1]. This is shown by the fact that in the Canadian merit rating experience, the cars which qualified for the best merit rating have different accident frequencies depending on which class they are in.

This means that private passenger automobile risks vary considerably from each other and that the class plan and the merit rating plan are both attempts to separate the better risks from the poorer risks. Neither plan is perfect, but we would like to discuss the question, "How do merit rating and class rating compare with each other in their ability to separate the better risks from the poorer risks?" After discussing their comparative effectiveness, we shall then discuss their absolute effectiveness.

Comparative Effectiveness of Merit Rating and Class Rating

Table 1 at the end of this section shows the Canadian automobile experience[2] arranged to show what it would have looked like if there had been (1) merit rating without class rating and (2) class rating without merit rating. The premiums have been adjusted to what they would have been if all the cars had been written at 1B rates, by use of the approximate relativities:

[1] See also "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records" by Lester Dropkin, CAS XLVI, p. 165.

[2] The Canadian experience includes that of virtually every insurance company operating in Canada and is collated by the Statistical Agency (Canadian Underwriters' Association —Statistical Department) acting under instructions from the Superintendent of Insurance.

Merit Rating Definition	Relativity
A — licensed and accident free three or more years	65
X — licensed and accident free two years	80
Y — licensed and accident free one year	90
B — all others	100
Class Definitions	
1 — pleasure, no male operator under 25	100
2 — pleasure, non-principal male operator under 25	165
3 — business use	165
4 — unmarried owner or principal operator under 25	240
5 — married owner or principal operator under 25	165

The purpose of any classification plan is to reduce the rates for the better risks and to offset this reduction with an appropriate increase in the rates for the more hazardous risks. We will define "effectiveness" of a classification plan in this paper to be the extent to which the plan separates the better risks from the overall average. This definition of effectiveness was applied in making an evaluation of a one-year merit rating plan where the better risks would get only a 1.6% reduction from the average rate if a 15% discount were given for a one-year accident-free record. Because the reduction of 1.6% was so small, the plan was considered to be ineffective.[3]

Since both merit rating and class rating in Canada include about the same proportion, 80%, of the cars in the lowest rated class, a measure of the comparative effectiveness of the two is the percentage reduction of the lowest rated class from the over-all average.

Rating Plan	Reduction of lowest rated class from average	Relative Reduction	Proportion of cars in lowest rated class
Merit rating alone	10.5%	77	80.9%
Class rating alone	13.7%	100	80.1%
Merit and class rating combined	21.4%	156	66.4%

This means that the merit rating plan is 77% as effective as the class plan. The Canadian merit rating plan could be improved by extending it from three years to five (which was done during the latter part of 1959) and by including convictions. Something also could be gained if the merit rating plan gave extra weight to a loss exceeding, say $1000, since it was noted that there is a positive correlation between the loss ratio and the average size of loss. Likewise the Canadian class plan, which is similar to the plans used in the United States, could also be improved. But the point remains that merit rating is almost as effective as the class plan in separating the better risks from the poorer risks and a substantial improvement is realized when they are used in combination.

[3] See Muir, J. M., "Principles and Practices in Connection With Classification Rating Systems for Liability Insurance As Applied to Private Passenger Automobiles", CAS XLIV, pp. 32 and 33.

Our previous paper showed the experience for each class subdivided by merit rating.[4] This was a natural format because the class plan was here first and merit rating was being imposed on top of the already existing class plan. Table 2 at the end of this section shows how the experience would have been presented if merit rating had been here first and the class plan was being imposed on top of the already existing merit rating plan. Losses are used this time instead of number of claims because there is a much greater difference in average claim costs among the classes than among the merit ratings.

The relative loss ratio for Class 1 within each merit rating is slightly lower than the corresponding ratio shown in our previous paper for merit rating A within each class, indicating a greater effectiveness for class rating. The class plan is most effective in the worst merit rating, B, just as merit rating was shown to be most effective in the worst class, 4.

Absolute Effectiveness of Merit Rating and Class Rating

Thus far, this paper has shown, based on the Canadian experience, that merit rating is almost as effective as class rating in separating the better risks from the poorer risks. But it has not shown in absolute terms just how effective either rating plan is.

In order to determine the absolute effectiveness of a rating plan, an analytical expression of the distribution of risks according to their "inherent hazard" is needed. Mr. Dropkin's paper on the negative binomial distribution[5] provides a valuable tool for this purpose. His paper shows that inherent hazards of individual risks are much more widely distributed than was commonly supposed. The class plan reduces this wide distribution very little. This is illustrated by the fact that merit rating will give the best risks a reduction of 10.5% from the average when there is no class plan and will still give the best risks within Class 1 a reduction of 8.9%[6] from the average Class 1 rate. This means that Class 1 has almost as much variation within it as there is among all classes combined.

This demonstrates what has often been recognized, that while merit rating and class rating are effective tools in a relative sense, in an absolute sense both merit rating and class rating are quite ineffective in separating the better risks from the poorer risks. There remains a considerable amount of unanalyzed variation among risks.

Cause of the Unanalyzed Variation Among Automobile Risks

It is one thing to show there is variation among risks and another thing to find the cause of variation.

In our previous paper we listed three possible reasons why the empirical credibilities discussed there for 1, 2 and 3 years of merit rating were not in the expected ratio of 1 : 2 : 3. They were:

[4] Bailey, Robert A. and Simon, LeRoy J., "An Actuarial Note on the Credibility of Experience of a single Private Passenger Car", CAS XLVI; Table 1, p. 162.

[5] *Op. Cit.*

[6] Bailey, Robert A. and Simon, LeRoy J., *Op. Cit.*, Table 4, p. 163.

(1) new risks entering a class,
(2) an individual risk's chance of having an accident varying from year to year, and
(3) a markedly skew distribution of risks.

With the help of the negative binomial distribution, we can check the third alternative. Using the formula derived by Mr. Bailey[7] for the credibility

$$Z = \frac{n}{n+a}$$

where n = number of accident-free years and
a is a parameter in the distribution of risks,

we find that the relative credibilities for 1, 2 and 3 years should be in the ratio of

$$1 : 2\left(\frac{1+a}{2+a}\right) : 3\left(\frac{1+a}{3+a}\right)$$

By setting the one year credibility for Class 1 cars of .055, shown in Table 4 of our previous paper,[8] equal to $\frac{1}{1+a}$, we obtain $a = 17.2$. Therefore the relative credibilities for 1, 2 and 3 years should be in the ratio of 1:1.90:2.70 which are close to 1:2:3 as we had expected. But the actual relative credibilities also shown in Table 4 of our previous paper are in the ratio of 1:1.38:1.62. Therefore while the distribution of risks is definitely skew, it is not skew enough to account for such large discrepancies, and we may cross out the third alternative listed above.

We know that new risks entering the class account for some of the discrepancy, but we do not feel that new risks can account for such large discrepancies. Therefore we feel that the evidence strongly supports the conclusion that the individual risk's chance of having an accident does vary significantly from year to year.

Thus far we have shown that merit rating and class rating are of about equal effectiveness and that a substantial improvement is realized when they are used in combination. However, both of them leave unanalyzed a considerable amount of variation among risks. In our investigation of the characteristics of this unanalyzed variation we have eliminated certain factors from consideration and now feel we have reached the point where we may state that the still unanalyzed cause (or causes) of variation among individual risks:

(1) has a wide dispersion,
(2) varies significantly from year to year for an individual risk, and
(3) is measured only to a limited extent by the class plan and the merit rating plan.

Annual mileage, which has long been felt to be an important measure of hazard,

[7] Bailey, Robert A. Discussion, "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Record", CAS XLVII, p. 152.

[8] *Op. cit.*

fits all these requirements better than any other single cause. The distribution of risks according to mileage is widely dispersed.[9] Mileage varies significantly from year to year. Farmers, for example, have less mileage than average[10] and business use risks have more mileage than average. The discount for two or more cars in one family is a reflection of mileage. Accident frequencies (and even conviction frequencies) are a crude indication of mileage. Mileage is certainly not the whole story because there is conclusive evidence that newly licensed drivers and youthful drivers have a higher accident rate per mile than other drivers and that other things such as drinking and irresponsibility play a part, but the evidence supports the conclusion that mileage is a very significant cause of variation among individual risks.

TABLE 1

Canada excluding Saskatchewan
Policy Years 1957 & 1958 as of June 30, 1959
Private Passenger Automobile Liability - Non Farmers

Merit Rating	Earned Car Years	Earned Prem. at Present 1B Rates	Losses Incurred	Loss Ratio	Relative Loss Ratio
		Classes 1, 2, 3, 4 & 5 Combined			
A	3,356,480	192,881,000	87,094,000	.452	.895
X	175,553	10,518,000	6,233,000	.593	1.174
Y	219,597	13,118,000	8,461,000	.645	1.277
B	398,445	24,152,000	19,633,000	.813	1.610
Total	4,150,075	240,669,000	121,421,000	.505	1.000
A + X	3,532,033	203,399,000	93,327,000	.459	.909
A + X + Y	3,751,630	216,517,000	101,788,000	.470	.931
		Merit Ratings A, X, Y & B Combined			
Class					
1	3,325,714	194,106,000	84,607,000	.436	.863
2	168,998	9,385,000	6,505,000	.693	1.372
3	321,327	20,627,000	13,684,000	.663	1.313
4	252,397	12,390,000	14,199,000	1.146	2.269
5	81,639	4,161,000	2,426,000	.583	1.154
Total	4,150,075	240,669,000	121,421,000	.505	1.000
1A	2,757,520	159,108,000	63,191,000	.397	.786

[9] See DeSilva, Harry R. *Why We Have Automobile Accidents.* John Wiley & Sons, New York, 1942, p. 12.

[10] *Ibid.*, p. 13.

TABLE 2

Canada excluding Saskatchewan
Policy Years 1957 & 1958 as of June 30, 1959
Private Passenger Automobile Liability - Non Farmers

Class	Earned Car Years	Earned Prem. at Present 1B Rates	Losses Incurred	Loss Ratio	Relative Loss Ratio
		Merit Rating A - licensed and accident-free 3 or more years			
1	2,757,520	159,108,000	63,191,000	.397	.878
2	130,535	7,175,000	4,598,000	.641	1.418
3	247,424	15,663,000	9,589,000	.612	1.354
4	156,871	7,694,000	7,964,000	1.035	2.290
5	64,130	3,241,000	1,752,000	.541	1.197
Total	3,356,480	192,881,000	87,094,000	.452	1.000
		Merit Rating X - licensed and accident-free 2 years			
1	130,706	7,910,000	4,055,000	.513	.865
2	7,233	431,000	380,000	.882	1.487
3	15,868	1,080,000	701,000	.649	1.094
4	17,707	888,000	983,000	1.107	1.867
5	4,039	209,000	114,000	.545	.919
Total	175,553	10,518,000	6,233,000	.593	1.000
		Merit Rating Y - licensed and accident-free 1 year			
1	163,544	9,862,000	5,552,000	.563	.873
2	9,726	572,000	439,000	.767	1.189
3	20,369	1,382,000	1,011,000	.732	1.135
4	21,089	1,052,000	1,281,000	1.218	1.888
5	4,869	250,000	178,000	.712	1.104
Total	219,597	13,118,000	8,461,000	.645	1.000
		Merit Rating B - all other			
1	273,944	17,226,000	11,809,000	.686	.844
2	21,504	1,207,000	1,088,000	.901	1.108
3	37,666	2,502,000	2,383,000	.952	1.171
4	56,730	2,756,000	3,971,000	1.441	1.772
5	8,601	461,000	382,000	.829	1.020
Total	398,445	24,152,000	19,633,000	.813	1.000

SECTION B: IMPROVED METHODS FOR DETERMINING CLASSIFICATION RATE RELATIVITIES

Multiple classification systems are quite prevalent in the insurance industry. For example, in fire insurance we classify the simple dwelling risks by town grading as well as by construction, resulting in a 10 x 2 system (typically). Other lines similarly involve multiple classification systems, but automobile is probably the best example. We have used a class plan and a territorial plan in automobile and now we have introduced the merit rating plan. It has been customary to determine a countrywide set of class relativities. Under the merit rating plan it will be necessary to determine relativities here, too. Assuming these relativities are to continue to be applied in series as multipliers on a "base" pure premium, the problem then arises as to how to determine the best set of relativities. The customary procedure* is to sum over all variables except the one we are interested in and then compute our relativities. For example, to get the class relativities, get the total mass of experience broken down only by class. Then the ratio of the experience for each class (usually adjusted in some manner for differences in the distribution by territory and merit rating) to the overall average experience will give the individual class relativity. The same steps would be followed for the merit rating classes and for territories. The subdivisions within each class are added together because individually they are usually not fully credible. Combining them is a means of obtaining a credible volume of experience. This process of combining subgroups results in a loss of some information because any combination yields less information than the aggregate information yielded by the individual subgroups. A method for obtaining relativities which is able to avoid combining the subgroups and is able to use each subgroup individually would produce a better set of relativities.

For purposes of illustration we'll solve the following problem: What is the best set of class relativities and merit rating relativities to use in Canada? The data is presented in Table B in a loss ratio form (all at Class 1B rates) and in Table D as relative loss ratios. We will assume that the territorial factor is properly reflected in this data because we are dealing with loss ratios. A better way would be to use pure premiums and to work out territorial relativities at the same time as class and merit rating relativities. However, such data is not available to the authors, but the procedure would be similar in either case. To determine what is an acceptable set of relativities we must establish the criteria which a set should meet:

Criterion 1. It should reproduce the experience for each class and merit rating class and also the overall experience; i.e., be *balanced* for each class and in total.

Criterion 2. It should reflect the relative *credibility* of the various groups involved.

Criterion 3. It should provide a minimal amount of *departure* from the raw data for the maximum number of people.

Criterion 4. It should produce a rate for each sub-group of risks which is close enough to the experience so that the differences could reasonably be caused by *chance*.

*For example, see "Current Rate Making Procedures for Automobile Liability Insurance", Stern, Phillip K., CAS XLIII, p. 127ff.

A set which meets these four criteria will be judged to be a "best" set of relativities. If more than one set satisfactorily meets the four criteria, the choice among sets may be made on a non-mathematical basis such as (a) simplicity of application, (b) similarity to existing sets, (c) ease of explanation to non-technical personnel or (d) the actuary's personal preference.

Let us define x_i as the class relativity for the i^{th} class ($i = 1,2,3,4,5$) and y_j as the class relativity for the j^{th} merit rating class ($j = 1,2,3,4$ representing A, X, Y and B respectively). Let r_{ij} be the actual relative loss ratio for persons classified as class i and merit rating class j; $r_{.j}$ is the relative loss ratio of the j^{th} merit rating class where all i classes are combined; $r_{i.}$ is the relative loss ratio of the i^{th} class where all j merit rating classes are combined; and finally $r_{..}$ is the relative loss ratio for all classes and merit rating classes combined and thus equals 1.00. Let us also define n_{ij} as the number of earned car years of exposure. The n_{ij} are shown in Table A.

Relativities calculated by the customary method, which we will call "Method 1", are as follows:

and
$$\left.\begin{aligned} x_i &= r_{i.} \\ y_j &= r_{.j} \end{aligned}\right\} \qquad (1)$$

and are shown in Table C.

The estimated relative loss ratio is then $x_i y_j$, and, if multiplied by the overall loss ratio, will produce the estimated loss ratio for the i, j class. Or, if $x_i y_j$ is multiplied by the overall pure premium, it would produce the estimated pure premium for the i, j class. The estimated relative loss ratios, $x_i y_j$ obtained by Method 1 are shown in Table D. When compared with the actual relative loss ratios, r_{ij}, also shown in Table D, it is evident that there are some undesirably large differences. Moreover, all $x_i y_1$ are too low and all $x_i y_4$ are too high.

To test the balance (Criterion 1 above) we calculate

$$\Sigma n_{ij} x_i y_j / \Sigma n_{ij} r_{ij} \qquad (2)$$

summing over each i, each j and total.

A set of relativities is balanced if equation (2) equals 1.000. The balance as determined by equation (2) is shown in Table E. Method 1 is out of balance in total and far out of balance for the individual classes. If the off-balance in the total is corrected, the classes will still be far out of balance. The reason why the classes are so far out of balance is that in our calculation of x_i and y_j, no adjustment was made for differences in the distribution by class or merit rating class. This illustrates what happens if a merit rating plan is imposed on an already existing class plan without any adjustment in the class relativities. If we had made some tentative adjustment, the off-balance by class and merit rating class would have been reduced. To make a completely accurate adjustment in the class relativities is difficult, however, because any adjustment in the class relativities disturbs the relativities for the merit rating classes and conversely, thus requiring an adjustment process which zig-zags back and forth. However, even if such an adjustment were made so that Criterion 1 would be satisfied, Method 1 would still not satisfy Criteria 2, 3 and 4, as will be shown later.

Again speaking in general, in order to reflect the relative credibility of the various groups involved (Criterion 2), the indicated proportional departure of each group

$$\frac{\text{actual loss ratio} - \text{expected loss ratio}}{\text{expected loss ratio}}$$

should be given a weight proportional to the square root of the expected number of losses for the group. This is based on the fact that the indication of each group should be given a weight inversely proportional to the standard deviation of the indication. The standard deviation of the indication is inversely proportional to the square root of the expected number of losses for the group. An equivalent credibility procedure would be to give the square of the indication a weight proportional to the expected number of losses.

Criterion 2 (Credibility) is not met by the customary relativities (Method 1) because when all the data is added together for, say, class i, to obtain $r_{i.}$ each subgroup r_{ij} is given a weight approximately proportional to the expected number of claims instead of the square root of the expected number of claims. This is one of the reasons why Method 1 does not satisfy Criteria 3 and 4. Moreover, if each entry in a row of r_{ij} is of low credibility, the resulting r_i will not be too trustworthy. Nevertheless, the resulting $r_{i.}$ will be treated as 100% credible by Method 1 in the determination of x_i. Methods 2, 3 and 4 developed below will remove these defects. Each r_{ij} will contribute to the final set in proportion to its relative credibility in relationship to all other r_{ij} in the table and not just in relationship to the other members of its row or column, and conversely each x_i and y_j will be influenced by all the r_{ij} and not just by one row or column of r_{ij}.

There is no assurance that Criteria 3 and 4 are met by the customary relativities (Method 1). In the paragraphs that follow we will show clearly that this set of relativities results in an average departure that is far from minimal and further, that the individual departures are too large to be caused by chance.

As a test of a set of relativities for compliance with Criterion 3, let us calculate how much error the average policyholder will have in his estimated relativity by calculating,

$$\sum_{i,j} n_{ij} \,|\, r_{ij} - x_i y_j \,|\, / \sum_{i,j} n_{ij} r_{ij} \tag{3}$$

The result of this calculation is shown in Table E.

Equation (3) endeavors to measure how much "inequity" the set has. The farther a policyholder's rate is from the indications of the raw data, the more "inequity" is involved. Anyone who has dealt directly with insureds at the time of a rate increase, knows that you can be much more positive when the rate for his class is very close to the indications of experience. The more persons involved in a given sized inequity, the more important it is.

To test a set of relativities for compliance with Criterion 4 (differences between the raw data and the estimated relativities should be small enough to be caused by chance), the Chi-square test is appropriate. It is shown in the Appendix that in terms of relative loss ratios, exposures and relativities,

$$\chi^2 = K \sum_{i,j} \frac{n_{ij}(r_{ij} - x_i y_j)^2}{x_i y_j} \tag{4}$$

where K is a constant dependent on the data and for the Canadian data, K equals approximately 1/200. The values of χ^2 are shown in Table E.

It should be noticed that the χ^2 formula (4) is equivalent to giving the square of the indication a weight proportional to the expected number of claims.

$$\chi^2 = K \sum_{i,j} \frac{n_{ij}(r_{ij} - x_i y_j)^2}{x_i y_j} = K \sum_{i,j} n_{ij} x_i y_j \left(\frac{r_{ij} - x_i y_j}{x_i y_j} \right)^2$$

This means that a set of x_i, y_j, which is specifically designed to produce a minimum χ^2 will automatically reflect the relative credibility of each group involved (Criterion 2). This is accomplished without a credibility weighting process involving tabular credibilities. Moreover, since a set of x_i, y_j, which produces a minimum χ^2 will very likely also satisfy Criterion 3 (minimal average amount of departure) and will come very close to satisfying Criterion 1 (balance), it seems evident that the best set of relativities will be those which are designed specifically to produce a minimum χ^2. These relativities can be obtained by setting the partial derivatives of χ^2 equal to zero.

$$\frac{\partial \chi^2}{\partial x_i} = K \sum_j n_{ij} y_j - K \sum_j \frac{n_{ij} r^2_{ij}}{x_i^2 y_j} = 0 \qquad (5)$$

Solving for x_i, we obtain

$$x_i = \left[\sum_j \frac{n_{ij} r_{ij}^2}{y_j} \Big/ \sum_j n_{ij} y_j \right]^{\frac{1}{2}} \qquad (6)$$

and similarly,

$$y_j = \left[\sum_i \frac{n_{ij} r_{ij}^2}{x_i} \Big/ \sum_i n_{ij} x_i \right]^{\frac{1}{2}} \qquad (7)$$

This gives us nine equations in nine unknowns. Since the equations are not of a simple, rational form, the easiest way to arrive at a numerical solution is by a method of iteration, as follows:

1. Take $r_{i.}$ (the customary method of obtaining x_i) as the first estimates of x_i.
2. Use these values in the right hand side of (7) to obtain the first estimates of y_j.
3. Use the first estimates of y_j in the right hand side of (6) to obtain the second estimates, of x_i.
4. Repeat this process until two consecutive sets of solutions are identical (or substantially so).

Notice that there are an infinite number of solutions for x_i and y_j, all of which, however, produce the same set of $x_i y_j$. This is true because each x_i may be multiplied by a constant if each y_j is divided by the same constant. The results of this method, which we shall call "Method 2" are shown in Table C. The estimated relative loss ratios, $x_i y_j$, are shown in Table D and the tests of Criteria 1, 3 and 4 are shown in Table E.

It is evident that Method 2, which derives all sets of relativities simultaneously, solves the difficult problem of obtaining relativities which are balanced in total and by class. It automatically satisfies Criterion 2 (credibility) and it also reduces substantially the average error and χ^2 (Criteria 3 and 4). But in spite of this improvement, the average error of .0317 still does not compare very favorably with a profit margin of .050 or thereabouts, especially for a company that writes a disproportion of business in one class. Moreover, χ^2, although much less than for Method 1, is still too high to be the result of chance. This means that a set of factors which are multiplied together, $x_i y_j$, cannot satisfactorily represent the actual data for Canadian private passenger automobiles, although it may be satisfactory for other lines or types of data.

Turning to the actual data, shown in Table D, it can be seen that the *percentage* difference between the lowest and the highest merit rating decreases as the rate for the class increases, ranging from 73% for class 1 down to 39% for class 4. With these conditions present in the basic Canadian data, it is little wonder that the multiplicative relativities do not fit satisfactorily.

A possible method, which we will call "Method 3", is to let the estimated relative loss ratio be $x_i + y_j$, where the relativities are added instead of multiplied. The χ^2 formula becomes

$$\chi^2 = K \sum_{i,j} \frac{n_{ij}(r_{ij} - x_i - y_j)^2}{x_i + y_j} \tag{8}$$

And setting the partial derivatives of χ^2 equal to zero we have:

$$\frac{\partial \chi^2}{\partial x_i} = K \sum_j n_{ij} - K \sum_j \frac{n_{ij} r_{ij}^2}{(x_i + y_j)^2} = 0 \tag{9}$$

For convenience let us write (9) as $f(x_i) = 0$. If we first obtain an estimate of x_i, we can obtain a correction, Δx_i, to be added to x_i by the use of Newton's method; that is,

$$\Delta x_i = -\frac{f(x_i)}{f'(x_i)}$$

where $f'(x_i)$ is the derivative of $f(x_i)$. Using this procedure we obtain

$$\Delta x_i = \frac{\sum_j n_{ij} \left(\frac{r_{ij}}{x_i + y_j} \right)^2 - \sum_j n_{ij}}{2 \sum_j n_{ij} \left(\frac{r_{ij}}{x_i + y_j} \right)^2 \left(\frac{1}{x_i + y_j} \right)} \tag{10}$$

The expression for Δy_j is the same as for Δx_i except that the summations are taken over i instead of j. The x_i and y_j are derived as follows:

1. Select a set of first estimates of x_i and y_j.
2. Use these values in (10) to obtain Δx_i and Δy_j.
3. Add Δx_i to x_i and Δy_j to y_j to obtain the second estimates of x_i and y_j.
4. Repeat this process until all Δx_i and Δy_j are equal to zero.

It should be noted here again that there are an infinite number of solutions for x_i and y_j, all of which, however, produce the same set of x_i+y_j. This is true because a constant may be added to all the x_i if the same constant is subtracted from all the y_j, and this will not change any of the estimated relative loss ratios, x_i+y_j. In fact, if we let the estimated relative loss ratios be (x_i+y_j-1) and alter formula (10) accordingly, we can use the values of x_i and y_j obtained by Method 1 as our first estimates to be used in (10).

It may be well at this point to emphasize how little absolute meaning can be attached to a given set of relativities. Whether they were based on a minimum χ^2 or not, or whether they were multiplicative or additive, a simple transformation can change their individual values and, of course, the happenstance of our choice of initial values in solving either (6) or (10) will produce one solution instead of another. It is quite natural for us to attempt to attach a special meaning to a developed set of relativities; that is, to impart to them some special quality in and of themselves. However, they can only be regarded in relationship to the coordinate system in which they find themselves.

The values of x_i and y_j obtained by Method 3 are shown in Table C, the estimated relative loss ratios, x_i+y_j, are shown in Table D and the tests of Criteria 1, 3 and 4 are shown in Table E.

It is evident that Method 3 not only satisfies Criteria 1 and 2 (balance and credibility) but it also reduces the average error to .0098 which is much better than Methods 1 and 2, and it produces a χ^2 which could very easily be the result of chance. This means that while the actual data cannot be represented satisfactorily by a set of relativities which are multiplied, x_iy_j, the actual data can be satisfactorily represented by a set of relativities which are added, x_i+y_j.

Another method of obtaining relativities which we will call "Method 4" is a compromise between Methods 2 and 3. Let the estimated relative loss ratios be $ax_iy_j-(a-1)$ and then minimize

$$\chi^2=K\sum_{i,j}\frac{n_{ij}[r_{ij}-(ax_iy_j-a+1)]^2}{ax_iy_j-(a-1)} \tag{11}$$

If $a=1$, (11) reduces to (4) which is Method 2. With the proper selection of a, greater than 1, results can be produced which are very similar to Method 3. It seems that the only practical way to obtain the optimum value of a is by judgment. Basing our judgment on the four corner values of r_{11}, r_{14}, r_{41} and r_{44}, we selected $a=3$. For computational purposes, equation (11) was translated to the form of equation (4) by adding 2 to each r_{ij} and dividing the results by 3. The relativities, x_i and y_j, were then obtained by the iterative process described for Method 2 and are shown in Table C. The estimated relative loss ratios, $3x_iy_j-2$, are shown in Table D and the tests of Criteria 1, 3 and 4 are shown in Table E. It can be seen that Method 4 produces results very similar to Method 3, and for the Canadian data that both Methods 3 and 4 satisfy all four criteria listed at the beginning of this section. Moreover, they both are methods of calculating all sets of relativities simultaneously.

We have developed only a two dimensional problem (x by y) here, but the methods can easily be extended to include more dimensions such as farm versus non-farm and territorial relativities. A small computer would be very useful in performing the tedious calculations which would be involved.

Consequences of Using Multiplicative Relativities

When the attempt is made to fit a set of relativities which are multiplied, $x_i y_j$, to a set of data that should be fitted by a set of relativities which are added, $x_i + y_j$, rates are produced for the lowest rated class that are too high in the lowest merit rating and too low in the highest merit rating and rates are produced for the highest rated class that are too low in the lowest merit rating and too high in the highest merit rating. This can be seen in Table D by comparing Method 2 with the actual data or with Method 3.

It is evident that the same difficulty occurs in private passenger automobile insurance in the United States when a countrywide set of class relativities is multiplied by a set of territory relativities. Several attempts have been made to correct this difficulty. Two sets of countrywide class relativities are used, one for large cities and one for all other territories.* The relativities for large cities have a smaller spread between the lowest and the highest rated classes. This is quite likely not caused by a difference in classification experience between high-rated territories and low-rated territories but it is the result of trying to use two sets of relativities which are multiplied when it is quite likely that the two sets of relativities should be added instead of multiplied. Another example of an attempt to correct this situation is the fact that in New York City, which is about the highest rated territory in the United States and where, in addition, the experience has enough volume to be credible, a special set of class relativities is used which has much less spread between the lowest and highest rated classes than the sets of relativities used elsewhere.

The reason this difficulty has not become more noticeable in other territories is that very few territories have sufficient volume to be credible for each class. But it is very likely that multiplying countrywide class relativities by territory relativities has produced and is producing rates which are too high for Class 1 in very low-rated territories and too low for Class 1 in very high-rated territories. This situation will become worse if three sets of relativities, for territories, classes and merit rating classes, are all multiplied together, $x_i y_j z_k$. The introduction of merit rating makes it all the more important to use a method of obtaining relativities which will satisfy the four criteria listed at the beginning of this section.

The methods developed in this paper, designated Methods 2, 3 and 4, have possibilities of wide application in many lines of insurance. For example, the non-reviewed workmen's compensation classes could be treated on a nationwide basis with relativities established by class and state. General Liability classes, which often involve a limited amount of exposure, could similarly be treated on a nationwide basis with relativities by class and territory. A & H involves many relativities. In automobile insurance itself the excess limits tables could be tested to determine whether the limits changes are, in fact, multiplicative with the basic rates or are more properly included as some other function. One can also visualize Homeowners rate making on a pure premium basis per homeowner with relativities for protection grading, construction and policy size

*See Stern, *Op. Cit.*, p. 154 and Livingston, G. R., & Carlson, T. O., discussion of "Principles and Practices in Connection with Classification Rating Systems for Liability Insurance as Applied to Private Passenger Automobiles". CAS XLV, p. 230.

(this latter item is a quantitative characteristic of the experience and would introduce an interesting facet into the problem). A multitude of similar practical problems could also be solved through this technique.

TABLE A

Array of Number of Earned Car Years of Exposure**
n_{ij} with 000 omitted

Merit Rating Class

		A	X	Y	B	
	i\j	1	2	3	4	Total
	1	2758	131	164	274	3327
	5	64	4	5	9	82
Class*	3	247	16	20	38	321
	2	131	7	10	22	170
	4	157	18	21	57	253
	Total	3357	176	220	400	4153

TABLE B

Array of Loss Ratios at 1B Rates**

Merit Rating Class

		A	X	Y	B	
	i\j	1	2	3	4	Total
	1	.397	.513	.563	.686	.436
	5	.541	.545	.712	.829	.583
Class*	3	.612	.649	.732	.952	.663
	2	.641	.882	.767	.901	.693
	4	1.035	1.107	1.218	1.441	1.146
	Total	.452	.593	.645	.813	.505

*These classifications have been rearranged so that the "Total" column in Table B is in ascending order.

**Source: Table 2 at end of Section A.

TABLE C

Relativities

		Method 1 (Customary*)	Method 2 (Min χ^2 on x_iy_j)	Method 3 (Min χ^2 on x_i+y_j)	Method 4 (Min χ^2 on $3x_iy_j-2$)
Class	x_1	.863	.881	.869	.958
	x_5	1.154	1.161	1.145	1.049
	x_3	1.313	1.309	1.291	1.099
	x_2	1.372	1.367	1.352	1.118
	x_4	2.269	2.125	2.172	1.384
Merit Rating Class	y_1	.895	.906	—.083	.971
	y_2	1.174	1.113	.135	1.040
	y_3	1.277	1.215	.237	1.076
	y_4	1.610	1.462	.512	1.167

*Source: Total column and Total row in Table B divided by .505.

TABLE D

Arrays of Relative Loss Ratios

Actual, r_{ij}
(Table B divided by .505)

Merit Rating Class

	i\j	A 1	X 2	Y 3	B 4
Class	1	.786	1.016	1.115	1.358
	5	1.071	1.079	1.410	1.642
	3	1.212	1.285	1.450	1.885
	2	1.269	1.747	1.519	1.784
	4	2.050	2.192	2.412	2.853

Method 1, x_iy_j
(Customary)

i\j	1	2	3	4
1	.772	1.013	1.102	1.389
5	1.033	1.355	1.474	1.858
3	1.175	1.541	1.677	2.114
2	1.228	1.611	1.752	2.209
4	2.031	2.664	2.898	3.653

Method 2, x_iy_j
(Minimum χ^2 on x_iy_j)

i\j	1	2	3	4
1	.798	.981	1.070	1.288
5	1.052	1.292	1.411	1.697
3	1.186	1.457	1.590	1.914
2	1.239	1.521	1.661	1.999
4	1.925	2.365	2.582	3.107

Method 3, x_i+y_j
(Minimum χ^2 on x_i+y_j)

i\j	1	2	3	4
1	.786	1.004	1.106	1.381
5	1.062	1.280	1.382	1.657
3	1.208	1.426	1.528	1.803
2	1.269	1.487	1.589	1.864
4	2.089	2.307	2.409	2.684

Method 4, $3x_iy_j-2$
(Minimum χ^2 on $3x_iy_j-2$)

i\j	1	2	3	4
1	.787	.988	1.090	1.354
5	1.057	1.276	1.387	1.675
3	1.198	1.429	1.543	1.846
2	1.255	1.489	1.606	1.915
4	2.029	2.320	2.464	2.845

TABLE E

Tests of Criteria 1, 3 and 4

Criterion 1, Balance

		Method 1	Method 2	Method 3	Method 4
Class	x_1	.9886	1.0007	1.0011	.9979
	x_5	1.0099	1.0014	1.0024	1.0008
	x_3	1.0195	1.0006	.9993	.9982
	x_2	1.0230	1.0027	1.0027	1.0005
	x_4	1.1067	1.0027	.9974	.9994
Merit Rating Class	y_1	.9806	1.0006	1.0015	.9978
	y_2	1.0589	1.0026	1.0083	.9996
	y_3	1.0536	1.0015	1.0020	.9986
	y_4	1.1122	1.0025	.9931	1.0002
Total		1.0103	1.0011	1.0006	.9983
		Criterion 3, Average Error			
Total		.0401	.0317	.0098	.0111
		Criterion 4, Chi-Square			
χ^2		98	34	10	8
Degrees of freedom for χ^2		12	12	12	11
Probability [$\chi^2 >$ observed]		less than .001	about .001	.60	.70

APPENDIX

Harald Cramér in his book, "Mathematical Methods of Statistics", pages 233 and 234, shows that if $\xi_1 \ldots\ldots \xi_n$ are n independent random variables each of which is normal with a mean of 0 and a variance of 1, then

$$\chi^2 = \sum_{i=1}^{n} \xi_i^2$$

is distributed according to the well-known Chi-square distribution.

For the Canadian data,

$$\xi = \frac{\text{actual relative loss ratio-expected relative loss ratio}}{\text{standard deviation of the actual relative loss ratio}}$$

has a mean of 0 and a variance of 1 and is very close to normal when the actual relative loss ratio is based on the average of a large number of car years. The actual relative loss ratio is r_{ij}, the expected relative loss ratio is $x_i y_j$. The variance of the actual relative loss ratio is approximately $\frac{200 x_i y_j}{n_{ij}}$ and is developed as follows:

Letting $C =$ value of claim

$n =$ number of car years

$S_p^2 =$ variance of the pure premium for one car year

$S_c^2 =$ variance of the claim cost

$m_f =$ mean claim frequency per car year

$m_c =$ mean claim cost

$m_p =$ mean pure premium per car year

Then $S_p^2 = \Sigma C^2/n - (\Sigma C/n)^2$ by definition

$$= m_f[(\Sigma C/nm_f)^2 + \Sigma C^2/nm_f - (\Sigma C/nm_f)^2] - (\Sigma C/n)^2$$

$$= m_f(m_c^2 + S_c^2) - m_f^2 m_c^2 \qquad (12)$$

Equation (12) agrees with Mr. R. E. Beard, "Analytical Expressions of the Risks Involved in General Insurance", in Transactions XVth International Congress of Actuaries, 1957, Vol. II, p. 233.

If we let the time interval be less than one year and approach zero as a limit, which is appropriate if S_c^2 is based on a distribution of claims where each claim is listed separately regardless of how close in time they may have occurred, the second term in equation (12) becomes insignificant and we obtain

$$S_p^2 = m_f(m_c^2 + S_c^2) \qquad (13)$$

which agrees with Mr. A. L. Bailey, "Sampling Theory in Casualty Insurance", CAS XXIX, p. 60.

Formula (13) can be written

$$S_p^2 = \frac{m_p^2}{m_f}\left(1 + \frac{S_c^2}{m_c^2}\right) \qquad (14)$$

Equation (14) is the variance of the pure premium for one car year. The variance of the mean pure premium per car year based on a group of n car years, where each mean is divided by the overall mean pure premium, P, is therefore

$$S_r^2 = \frac{m_p^2}{nP^2 m_f}\left(1 + \frac{S_c^2}{m_c^2}\right) \qquad (15)$$

Since $P = M_c M_f$, where M is the overall mean, and m_p is approximately equal to $M_c m_f$, and $x_i y_j = \frac{m_p}{P}$, equation (15) can be written

$$S_{r_{ij}}^2 = \left(\frac{x_i y_j}{n_{ij}}\right)\left(\frac{1}{M_f}\right)\left(1 + \frac{S_c^2}{m_c^2}\right) \qquad (16)$$

It is estimated that for the Canadian data, which is total limits for BI and PD combined, $1 + \frac{S_c^2}{m_c^2}$ equals approximately 20. This is only a rough estimate based on the limited data available to the authors. $M_f = .097$. Therefore for the Canadian data

$$\chi^2 = \frac{1}{200} \sum_{i,j} n_{ij} \frac{(r_{ij} - x_i y_j)^2}{x_i y_j}, \text{ approximately.}$$

Notice that the same constant, K, is produced regardless of whether x_i, y_j, are chosen as multiplicative or as some other form.