GENERALIZED EARNED PREMIUM RATE ADJUSTMENT FACTORS

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ABSTRACT

The loss ratio method of ratemaking requires the Actuary to adjust earned
premiums to reflect all rate changes that have been implemented. Many
Actuaries use the parallelogram method of finding the portion of the
exposures earned from a given rate change. The assumption is made that
exposures are being written at a constant level.

Computers are becoming an indispensable tool for the Actuary and the
parallelogram method is cumbersome for computer applications. The purpose
of this paper is to derive a simple general formula for finding the earned
portion of a rate increase for any given policy term, rate effective date,
and evaluation period (period during which the premiums are earned). In the
latter portion of the paper a formula is given based on the assumption that
exposures are increasing at a given growth rate. Finally, a comparison is
made of results produced by the constant exposure model versus the constant
growth rate model.

GENERALIZED EARNED PREMIUM RATE ADJUSTMENT FACTORS

The loss ratio method of ratemaking requires the actuary to adjust earned premiums to reflect all rate changes that have been implemented. Several papers in the Proceedings have addressed the calculation of earned premium at current rates. If there has been just one rate change, the formula for this adjustment is:

Equation (1.1) $\qquad F = \dfrac{1+r}{1+Pr}$

where F is the rate adjustment factor, r is the rate change and P is the portion of the earned exposure that was subject to the new rate. The purpose of this paper is to derive a simple general formula for calculating P for any given policy term, rate effective date, and evaluation period. The evaluation period is defined as the period stated in years during which the premium was earned. For example, if one wanted to convert the first quarter of 1988 earned premiums to current rates, the evaluation period would be .25 regardless of whether the policy term was semiannual, annual, quarterly, etc.

TRADITIONAL METHODS OF FINDING P

Roy Kallop's[1] paper on Workers' Compensation Ratemaking gives a description of the parallelogram method of finding P. This method assumes that exposures are being written at a constant rate. This assumption will be made throughout the remainder of the paper unless otherwise noted.

Calculations are relatively easy, but the Parallelogram method does not lend itself to computer applications. Another method of finding P is to develop precalculated tables based on effective date, policy term, and evaluation

205

period. These tables can become rather large and cumbersome for computer
applications if several policy terms and/or evaluation periods are involved.
Also, they are usually calibrated to rate changes effective at the beginning
of the month and those effective later in the month are not calculated
correctly.

FORMULA METHOD FOR FINDING P

Set the beginning of the evaluation period at 0. Let T equal the policy term,
E equal the evaluation period, and D equal the effective date of a rate
revision relative to the beginning of the evaluation period using years as the
unit of measurement. P can be calculated by the following formula:

Equation (1.2) $$P = 1 - \frac{A^2 - B^2 - C^2}{2ET}$$

Where:

$A = D + T$

$B = MAX (A-E, 0)$

$C = MAX (D, 0)$

Although one can easily memorize this formula, the time saved versus the
parallelogram method is probably minimal if there is only one rate change
involved. The advantage of the formula method is that it can be easily
programmed on a personal computer or a mainframe and multiple rate change
factors for several years can be computed quickly and automatically.

If a complete computerized ratemaking system is being developed, the formula
is much easier to program than alternative methods. Also, the formula is
generalized so that it can be used for any policy term, rate change date, and
evaluation period.

206

EXAMPLE

Assume that one wishes to convert 1988 year-to-date premiums as of August 31, 1988, to current rates. One rate revision was implemented on November 15, 1987, and all policies are on a quarterly policy term. Then:

$D = -1.5/12 = -.125$ $T = .25$ $E = 8/12 = .66667$

$A = D + T = -.125 + .25 = .125$

$B = Max (.125 - .66667,0) = 0$

$C = Max (D,0) = 0$

$$P = 1 - \frac{(.125)^2 - (0)^2 - (0)^2}{(2)(.66667)(.25)}$$
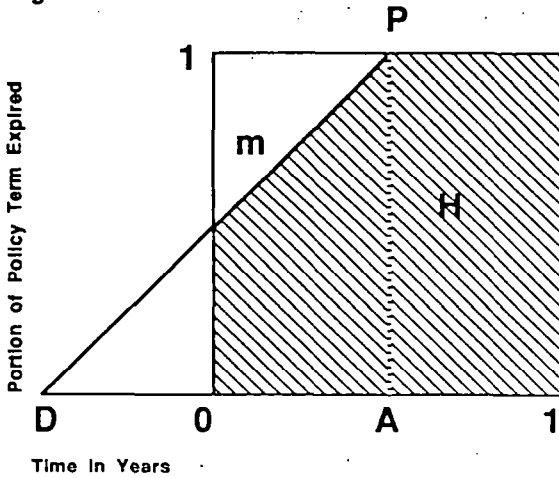
$P = .953125$

DERIVATION OF THE FORMULA

Jim Ross, in "Generalized Premium Formulas,"[2] and Miller and Davis, in "A Refined Model for Premium Adjustment,"[3] did an excellent job of developing formulas to find P. They showed several formulas, but did not derive a general formula to calculate P in a straightforward way. By expanding on their work, the formula described above was derived.

PARALLELOGRAM METHOD

Figure 1 gives an illustration of the traditional parallelogram approach
assuming we are working with annual premium (E=1):

Figure 1

D = effective date of rate change

A = time when rate change is fully earned

As in the Miller and Davis paper, the x-axis is time in years and the y-axis
is the portion of policy term expired. For the example shown in Figure (1),
line m indicates the portion of the policy term expired for a policy effective
at time D. The origin corresponds to the beginning of the evaluation period
with the point (1,0) corresponding to the end of the year. D is the effective
date of the rate revision and A is the point in time when the rate revision
becomes fully earned for a policy effective at time D. Line m is drawn
between D and P.

Figure (1) is an illustration of the case where D is less than 0 and 0<A<E.
To find the portion of the rate increase that is earned, we must find the area

208

of the shaded portion of the rectangle in Figure 1 (area H) and compare it to

the total area of the rectangle. Using traditional methods, the shaded area

is subdivided into triangles and rectangles and the area is calculated

manually.

GENERALIZED FORMULA FOR THE AREA

The shaded area can be calculated by integrating the line m from 0 to A and

integrating l from A to 1. If the equation for line m is f(x), then the area

is:

Equation (2.1) $H = \int_{0}^{A} f(x) \, dx + \int_{A}^{1} 1 \, dx$

Let T be the policy term. Note that A = D + T since a policy effective at

time D becomes fully earned at time D + T. The slope of line m would be the

change in Y of 1 divided by the change in X of A - D, or D + T - D, which

yields $\frac{1}{T}$.

Using the general equation of a line given the slope and one point, the

equation for line m becomes:

Equation (2.2) $\quad y = \frac{1}{T} \cdot x - \frac{D}{T} = \frac{x-D}{T} = f(x)$

and Equation (2.1) becomes:

Equation (2.3) $\quad H = \int_{0}^{A} \frac{(x-D)}{T} \, dx + \int_{A}^{1} 1 \, dx$

Where A = D+T

The total area of the rectangle is 1 and the percentage of the rate increase

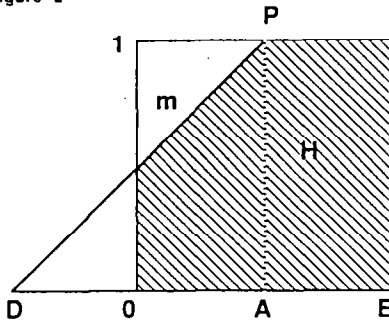earned is equation (2.3) divided by 1.

To arrive at the general case, one must consider all possible values for D and

209

A. Also, one must allow evaluation periods other than one year.

To consider the latter, define E as the policy evaluation period, the length
of time in years of the earned premiums to be converted to current rates.
This could be one year of earned premiums (E=1) or one month of earned
premiums (E=1/12). The only change to our diagram in Figure 1 is that instead
of the rectangle ending at 1, it would end at E as shown below:
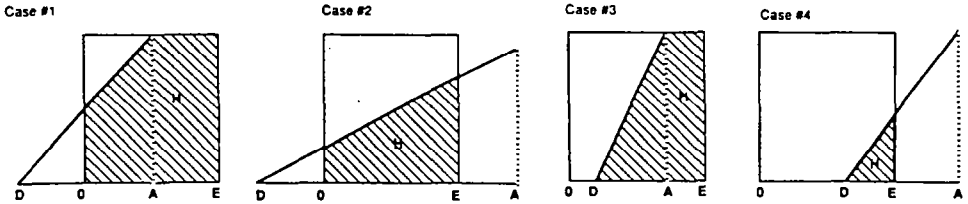
Figure 2

The total area of the rectangle is E which is calculated by multiplying the
length E by the height 1. The percent of the rate increase that is earned is
the area H divided by the area E.

With respect to points D and A, there are two trivial and four nontrivial
possibilities not considering cases where D or A exactly equals zero or E.
The two trivial cases are:

1. D and A are both less than zero and the rate increase is fully earned
    and;

2. D and A are both greater than E and none of the rate increase is earned.

ACA12/053089

Illustrated in Figure 3 are the four nontrivial cases.

Figure 3

Case #1   Case #2   Case #3   Case #4

Case 1 is the example we have been using and the area of the rectangle is:

Equation (2.4) $$H = \int_{O}^{A} \frac{(x-D)}{T} \, dx + \int_{A}^{E} 1 \, dx$$

For case 2, A is greater than E. The formula would be the same except the second term in equation 2.4 drops out and the integration of the first term extends from 0 to E. To include both cases 1 and 2 equation 2.4 becomes:

Equation (2.5) $$H = \int_{O}^{Min(A,E)} \frac{(x-D)}{T} \, dx + \int_{Min(A,E)}^{E} 1 \, dx$$

Note that for A > E the second term is the integral from E to E. This equals zero as expected.

For cases 3 and 4, D is greater than 0 and the integration begins at D rather than 0. Therefore, in the general case we begin our integration at 0 or D, whichever is greater and formula 1.5 becomes:

Equation (2.6) $$H = \int_{Max(0,0)}^{Min(A,E)} \frac{(x-D)}{T} \, dx + \int_{Min(A,E)}^{E} 1 \, dx$$

See appendix 1 for the actual evaluation of this integral. The formula for P is found by evaluating this integral and dividing by the area of the total rectangle which is the length E multiplied by the height of one. Using

211

appendix 1 and considering the trivial cases the formula for P is:

Equation (2.7)

$$P = 1 \qquad\qquad\qquad A < 0$$

$$P = 0 \qquad\qquad\qquad D > E$$

$$P = 1 - \frac{A^2 - B^2 - C^2}{2ET} \qquad A > 0 \text{ and } D < E$$
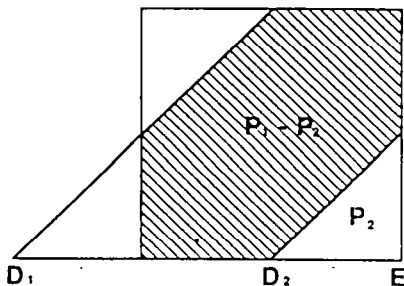
Where:

$$A = D + T$$

$$B = \text{Max } (A-E, 0)$$

$$C = \text{Max } (D,0)$$

MULTIPLE RATE CHANGES

Assume there are two rate changes during the period, effective at D_1 and D_2 as shown below.

Figure 4

Let P_1 be the portion of the total earned exposure from policies written between D_1 and E and let P_2 be the portion of the earned exposure from policies written between D_2 and E. It follows that the portion of the earned exposure from policies written between D_1 and D_2 is $P_1 - P_2$. The general equation for the average rate level during the evaluation period E is:

APPENDIX II
DERIVATION OF RATE ADJUSTMENT FACTORS

$$AR = (1 - P_1) + (P_1 - P_2)R_1 + (P_2 - P_3)R_2 + \cdots + (P_{n-1} - P_n) R_{n-1} + P_n R_n$$

Where AR equals the average rate level with rates R_1, R_2, R_3, $\cdots R_n$ rates indexed to the initial rate.

$$AR = 1 - P_1 + P_1R_1 - P_2R_1 + P_2R_2 - P_3R_2 + P_3R_3 - P_4R_3 + \cdots + P_{n-1} R_{n-1} - P_nR_{n-1} + P_nR_n$$

$$AR = 1 + P_1 [R_1 - 1] + P_2 [R_2 - R_1] + P_3 [R_3 - R_2] + P_4 [R_4 - R_3] + \cdots + P_{n-1} [R_{n-1} - R_{n-2}] + P_n [R_n - R_{n-1}]$$

$$R_i = R_{i-1} (1 + r_i)$$

Where r_i is the ith rate change

$$AR = 1 + P_1 [1 + r_i - 1] + P_2 [(1 + r_2) R_1 - R_1] + P_3 [(1 + r_3) R_2 - R_2] + P_4 [(1 + r_4) R_3 - R_3] + \cdots + P_{n-1} [(1 + r_{n-1}) R_{n-2} - R_{n-2}] + P_n [(1 + r_n) R_{n-1} - R_{n-1}]$$

$$AR = 1 + P_1r_1 + P_2R_1r_2 + P_3R_2r_3 + P_4R_3r_4 + \cdots + P_{n-1} R_{n-2} r_{n-1} + P_nR_{n-1}r_n$$

Using Equation (3.2)

$$AR = 1 + P_1r_1 + P_2(1 + r_1) r_2 + P_3 (1 + r_1)(1 + r_2) r_3 + \cdots + P_n(1 + r_1)(1 + r_2) \cdots (1 + r_{n-1}) r_n$$

The rate level adjustment factor equals the current rate R_n divided by the average rate level or:

$$= \frac{R_n}{AR} = \frac{(1 + r_1)(1 + r_2) \cdots (1 + r_n)}{1 + P_1r_1 + P_2 (1 + r_1) r_2 + \cdots + P_n (1 + r_1)(1 + r_2) \cdots (1 + r_{n-1}) r_n}$$

Equation (3.1)

$$(1-P_1) R_0+(P_1-P_2) R_1+(P_2-P_3) R_2 + \ldots + (P_{n-1}-P_n) R_{n-1}+P_nR_n$$

Where R_0 is the initial rate with R_1, R_2, R_3, ... R_n subsequent rates.

Please note that any P_i would be 1 if the rate change were fully earned before the evaluation period and would be 0 if the rate change were effective after the end of the evaluation period.

If one indexes all rates to the initial rate R_0, then $R_0 = 1$. By definition $R_i = (R_{i-1}) (1+r_i)$ where r_i is the ith rate change implemented at time D_i. It follows that

Equation (3.2)

$$R_i = (1+r_1)(1+r_2)\ldots\ldots(1+r_i)$$

Using equations 3.1 and 3.2, the following formula for the rate level adjustment factor can be derived (derivation shown in appendix 2):

Equation (3.3)

$$F = \frac{(1+r_1)(1+r_2)\ldots(1+r_n)}{1+P_1r_1+P_2(1+r_1)r_2+\ldots+P_n(1+r_1)(1+r_2)\ldots(1+r_{n-1})r_n}$$

This formula is probably somewhat easier to program in a computer than equation 3.1. Note that if there is only one rate change during the period equation 3.3 yields:

$$\frac{1+r_1}{1+P_1r_1}$$

which is the familiar formula shown in equation (1.1).

213

EXPOSURES NOT WRITTEN AT A CONSTANT RATE

Up to this point we have assumed that exposures are being written at a
constant rate throughout the period. In practice, the exposures are not
written at a constant rate and the formulas above do not produce exactly the
correct results. In most cases the error is negligible. Jim Ross in
"Generalized Premium Formulas"[2] developed an integral for calculating P
based on the assumption that exposures are growing at a constant compound
growth rate. However, each individual rate change had to be integrated to
calculate P for that rate change. By evaluating his integrals using
generalized limits as we did in the constant exposure formula, a truly
generalized formula can be developed for this model.

Assume that exposures are increasing at a constant rate and let V equal one
plus the growth rate (V≠1). The formula for P, excluding the trivial cases,
is:

    Equation (4.1)

$$P = \frac{\ln V [T(1-V^C)+BV^{B+E-T}-(A-C)V^{D-C}]+V^{D-C}+V^E(1-V^{B-T})-1}{(V^E-1)(1-V^{-T})}$$

Where:

A = D + T

B = Max (A-E,0)

C = Max (D,0)

EXAMPLE OF CONSTANT GROWTH RATE MODEL

Assume that the evaluation period is calendar year 1988 with annual policies and that a rate change was implemented on July 1, 1987. Also, assume that exposures are increasing at the rate of 60 percent per year.

$D = -.5$

$V = 1.6$

$E = 1$

$T = 1$

Therefore:

$A = .5$

$B = 0$

$C = 0$

Then P equals:

$$P = \frac{\ln(1.6)[-(.5)(1.6)^{-.5}]+(1.6)^{-.5}+(1.6)(1-(1.6)^{-1})-1}{(1.6-1)(1-(1.6)^{-1})}$$

$P = .91$

The following table illustrates the magnitude of the error when exposures are growing at a constant rate and the traditional parallelogram method is used (which assumes that exposures are being written at a constant rate). Assume we are converting calendar year 1988 earned premiums to current rates.

ACA12/053089

PORTION OF RATE INCREASE EARNED

Effective Date	Fixed Exposure	Growth Rate		
		20%	40%	60%
Annual Policy Term				
4/01/87	.969	.973	.976	.979
7/01/87	.875	.890	.901	.910
10/01/87	.719	.744	.764	.781
1/01/88	.500	.530	.556	.578
4/01/88	.281	.307	.330	.351
7/01/88	.125	.141	.155	.168
10/01/88	.031	.036	.041	.045
Semiannual Policy Term				
10/01/87	.938	.944	.949	.953
1/01/88	.750	.769	.784	.797
4/01/88	.500	.525	.545	.563
7/01/88	.250	.269	.286	.301
10/01/88	.063	.069	.076	.081

CONCLUSION

Computers are becoming an indespensible tool for the actuarial profession. The formulas derived in this paper can greatly simplify the task of programming the computer to convert earned premiums to current rates. These formulas are particularly helpful when one is dealing with multiple policy terms, and evaluation periods, or evaluation periods of different lengths.

216

FOOTNOTES

[1]Kallop, R., "A Current Look at Workers' Compensation Ratemaking," PCAS LX11 (1975)

[2]Ross, J. P., "Generalized Premium Formulas," PCAS LX11 (1975)

[3]Miller, D. L. and Davis, G. E., "A Refined Model for Premium Adjustment," PCAS LX111, (1976)

ACA12/053089

INTEGRATION OF RATE ADJUSTMENT FORMULA

From Equation (2.6):

$$H = \int_{Max(D,0)}^{Min(A,E)} \frac{x-D}{T} \, dx + \int_{Min(A,E)}^{E} 1 \, dx$$

$$H = \left[\frac{x^2}{2T} - \frac{Dx}{T} \right]_{Max(D,0)}^{Min(A,E)} + \left[x \right]_{Min(A,E)}^{E}$$

$$H = \frac{Min(A,E)^2}{2T} - \frac{D \cdot Min(A,E)}{T} - \frac{Max(D,0)^2}{2T} + \frac{D \cdot Max(D,0)}{T} + E - Min(A,E)$$

Note that $D \cdot Max(D,0) = Max(D,0)^2$

$$H = E + \frac{Min(A,E)^2 - 2D \cdot Min(A,E) - Max(D,0)^2 + 2\,Max(D,0)^2}{2T} - \frac{2T \cdot Min(A,E)}{2T}$$

$$H = E + \frac{Min(A,E)^2 - 2D \cdot Min(A,E) - 2T \cdot Min(A,E) + Max\,(D,0)^2}{2T}$$

Let $C = Max(D,0)$

$$H = E + \frac{Min(A,E)^2 - 2\,Min(A,E)(D+T) + C^2}{2T}$$

Since $A = D+T$ and completing the square of the first two terms in the numerator we have:

$$H = E + \frac{(Min(A,E) - A)^2 - A^2 + C^2}{2T}$$

Note that $Min(A,E) - A = Min(0,E-A)$ and for all E and A, $Min(0,E-A) = (-1)(Max(A-E,0))$

Therefore:

$$H = E + \frac{Max(A-E,0)^2 - A^2 + C^2}{2T}$$

Let $B = Max(A-E,0)$ and factoring out a negative one we have:

$$H = E - \frac{A^2 - B^2 - C^2}{2T}$$

The formula for P is found by dividing H by the area of the rectangle which is the length E multiplied by the height 1. Therefore, dividing the above equation by E yields

$$P = \frac{H}{E} = 1 - \frac{A^2 - B^2 - C^2}{2ET}$$