The Mathematics of On-Leveling

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1 Abstract

The mathematical foundation of on-leveling premium is explicitly stated. This is combined with an appropriate set of assumptions to derive the formulae for on-leveling premium by rate book (described within) and for using the Parallelogram Method. It is demonstrated in an appendix that this foundation subsumes all works in the bibliography. It is observed that rate book on-leveling has fewer assumptions than the Parallelogram Method and thus, if database granularity permits, the Parallelogram Method should be abandoned.

2 Introduction

On-leveling premium is an important part of any ratemaking exercise. Typical references (e.g. [2] p. 132, 142; [6] p. 73, 80) give three different methods: extension of exposures, the Parallelogram Method, and aggregating policies based on applicable rate level (i.e. rate book on-leveling). Of these, the first is the ideal. Extension of exposures is, by definition, the correct way to on-level. The challenge of the method is operational. It requires a well designed database and maintenance of rating engines. Rate book on-leveling requires an assumption involving the impact of rate changes and database queries of moderate complexity. Finally, the Parallelogram Method, while requiring virtually nothing more beyond periodic reports of earned premium and exposure, requires many assumptions in order to be tractable.

The shortcomings of actuarial literature with respect to on-leveling are fourfold. First, there is a lack of explicitly stated assumptions. Some works (e.g. [2] p. 133; [3] p. 76; [6], p. 73, 75) verbally state assumptions, but do not translate those assumptions into equations. Consequently, there is no demonstration that the verbally stated assumptions are sufficient to derive a formula with which to on-level one's premium. This concern is not merely theoretical. Such papers remark that there are times that the assumptions of the Parallelogram Method do not hold. If this is true, which it undoubltably is at times, then the mathematical formulations of the assumptions are needed so that they may be adjusted to the situation and a new formula derived with which to on-leveled the premium.

Second, explicit formulae are often omitted. For instance, [2], [3], and [6] each illustrate the Parallelogram Method (p. 133 - 141, p. 103 - 108, and p. 74 - 79 respectively) but never explicitly state a formula. The best job is done by Ross ([5]) who has a general integral formula which is then applied to various examples.

Third, there is an unacknowledged use of model functions. While Ross ([5]) explicitly states that the derivative of his written exposure function is constant, it is not observed that derivative of any written quantity function is 0 almost everywhere (this is demonstrated later in this paper). Hence, when proving claims about a function with a non-zero derivative, it is not true that these claims are being made for some class of well behaved written quantity functions, rather they are being made for no written quantity functions at all. Consquently, the work of Miller and Davis ([4]) and Bill ([1]), which use the work of Ross, have this same problem.

Finally, there is a misplaced emphasis on exposure writing and growth. The works ([1], [4], [5]) which sought to provide a theoretical basis for on-leveling devoted much of their text to assumptions regarding the writing and subsequent earning of exposures. However, when on-leveling, one is not concerned with exposure. One is concerned with premium. Assumptions about exposure are only useful insofar as they allow one to calculate various quantities of premium.

The practical applications of this paper are two-fold. First, it provides an explicit record in the actuarial literature for the mathematical foundation of on-leveling. Hence, one may combine assumptions germane to one's situation with this foundation to obtain appropriate on-leveling formulae. Second, it demonstrates that rate book on-leveling has fewer assumptions than the Parallelogram Method. Hence, if one is able to determine in which period each amount of premium is written, one may jetison the Parallelogram Method and obtain a more precise estimate of on-leveled earned premium using rate book on-leveling. While a great deal of mathematics and formulae follow, it is not necessary to examine it in order to make use of this second application. In practice, while application of the extension of exposures method may prove difficult due to differeing structures of the rate order of calculation at different times, it is wholly conceivable that, due to the existence of databases containing transactional data, rate book on-leveling may be used and the subsequent exposition be interesting only to those with mathematical proclivities.

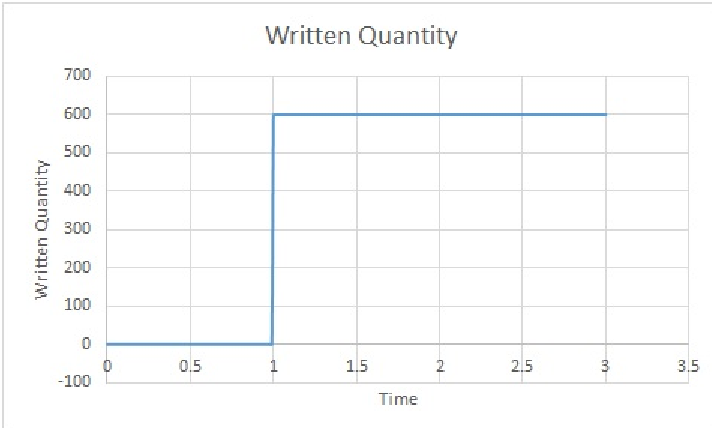
3 Aggregation Of Atomic Transactions

To begin, observe that in insurance, while one is often concerned with policies, policies are not typically the most granular piece of data possessed by the insurance company. Consider the following example.

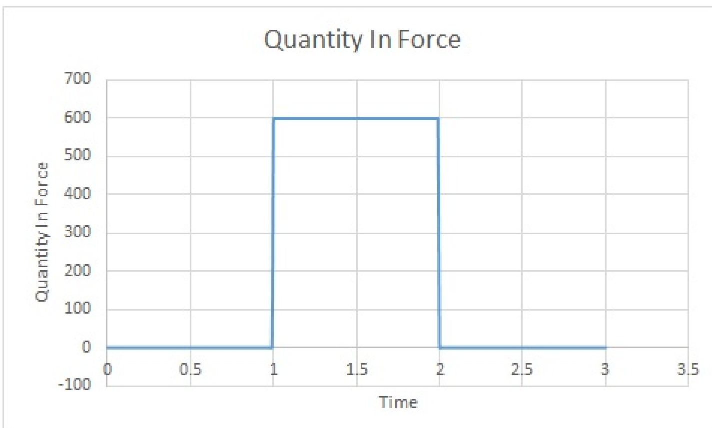
Date	Transaction
01/01/2016	Purchases annual auto policy for \$600
04/01/2016	Adds second identical auto for \$450
07/01/2016	Adds towinig to first auto for \$25
10/01/2016	Cancels second auto's coverage

Three additional changes occured to this policy besides initial writing: a car was added, an endorsement, and a cancelation. The most granular data that a company possesses encompasses transactions such as initial writing of the policy, endorsements to the policy, and cancelation of the policy. The most granular data a company possesses shall, in this paper, be known as **atomic transactions**. In this paper, such transactions will be the basis of considerations of on-leveling premium.

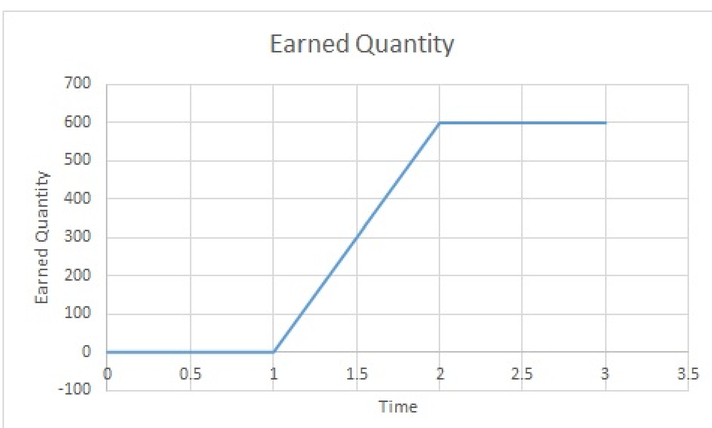
In order to proceed, one must define the set of indicator functions. For all $E \subseteq \mathbb{R}$ define $I_E : \mathbb{R} \to \mathbb{R}$ such that $I_E(x)$ is 1 if $x \in E$ and 0 otherwise. In this way I_E indicates if x is in the set E. For each transaction, one is concerned with the time of the transaction, some amount of some quantity (e.g. exposure or premium), and some length of time (e.g. the term length). Consider an atomic transaction at time $t = t_0$ for amount a and term length τ. Define the amount of written quantity for this transaction from the beginning of the company, at time $t = 0$, through time t, denoted as w, by $w(t) = aI_{([t_0,\infty))}(t)$. A graph of w for the first atomic transaction in the initial example (where time $t = 1$ corresponds to 01/01/2016) is below.



Define the quantity in force for this transaction at time t, denoted by f, by $f(t) = w(t) - w(t - \tau)$. A graph of f is below.



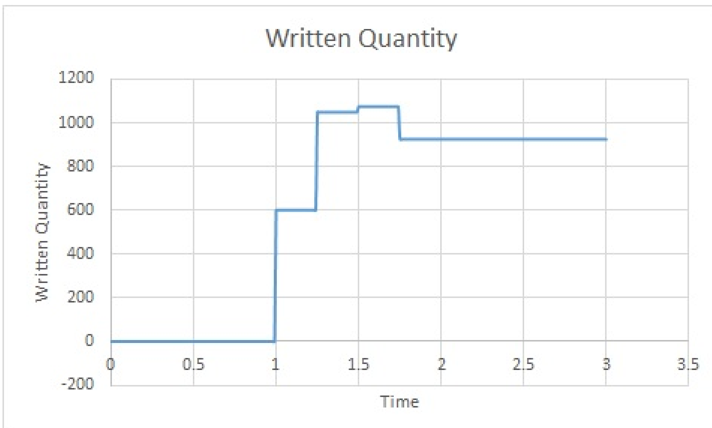
Finally, define the amount of earned quantity for this transaction through time t, denoted as e, by $e(t) = \frac{a}{\tau}(\max(t - t_0, 0) - \max(t - (t_0 + \tau), 0))$. A graph of e is below.

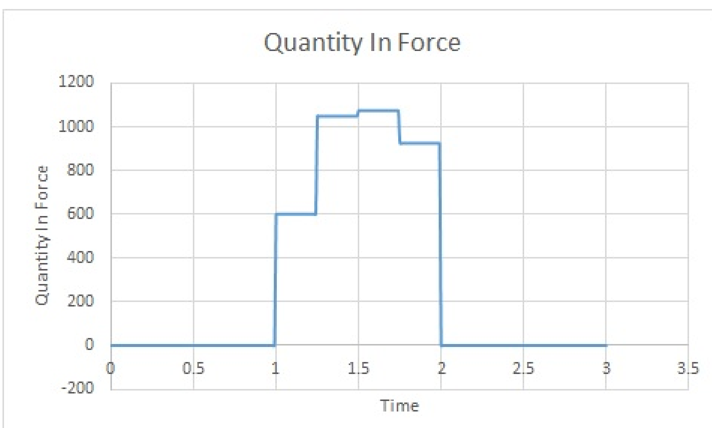


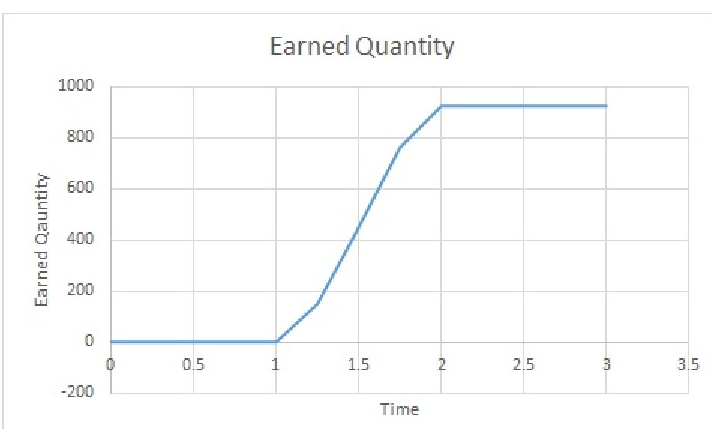
Observe that in the initial example there are four atomic transactions and thus four writtem quantity, quantity in force, and earned quantity functions. If one subscripts functions corresponding to the ith transaction with i, then the functions are given in the following table.

Written Quantity	Quantity In Force	Earned Quantity
$w_1(t) = 600 \cdot I_{[1,\infty)}(t)$	$f_1(t) = 600 \cdot I_{[1,2]}(t)$	$e_1(t) = 600(\max(t-1) - \max(t-2))$
$w_2(t) = 450 \cdot I_{[1.25,\infty)}(t)$	$f_2(t) = 450 \cdot I_{[1.25,2]}(t)$	$e_2(t) = 450(\max(t-1.25) - \max(t-2))$
$w_3(t) = 25 \cdot I_{[1.5,\infty)}(t)$	$f_3(t) = 25 \cdot I_{[1.5,2]}(t)$	$e_3(t) = 25(\max(t-1.5) - \max(t-2))$
$w_4(t) = -150 \cdot I_{[1.75,\infty)}(t)$	$f_4(t) = -150 \cdot I_{[1.75,2]}(t)$	$e_4(t) = -150(\max(t-1.75) - \max(t-2))$

Thus, the total written quantity for the policy is $w_1 + w_2 + w_3 + w_4$, the total quantity in force is $f_1 + f_2 + f_3 + f_4$, and the total earned quantity is $e_1 + e_2 + e_3 + e_4$. The graphs of these three functions are shown below.







More generally, suppose that n atomic transactions have occured since the beginning of the company and denote the kth specific term length, written quantity, quantity in force, and earned quantity functions by τ_k, w_k, f_k, and e_k respectively. Denote the company's written quantity, quantity in force, and earned quantity functions as w, f, and e respectively and define them as follows

$$w = \sum_{k=1}^{n} w_k, \ f = \sum_{k=1}^{n} f_k, \text{ and } e = \sum_{k=1}^{n} e_k.$$

Since both w and f are linear combinations of indicator functions and each of those functions has a zero time derivative almost everywhere (with respect to Lebesgue measure), then w and f have a zero derivative almost everywhere. This observation becomes important when deriving the Parallelogram Method.

In order to define the policy period quantity functions, fix a period $[t_1, t_2]$. Suppose that there are n atomic transactions pertaining to policies originating in the period $[t_1, t_2]$ and let $\hat{w}_k$, $\hat{f}_k$, and $\hat{e}_k$ be the respective written quantity, quantity in force, and earned quantity functions pertaining to the kth atomic transaction. Similar to how the calendar term functions were defined, denote the company's $[t_1, t_2]$ policy term written quantity, quantity in force, and earned quantity functions as $\hat{w}$, $\hat{f}$, and $\hat{e}$ respectively and define them as follows

$$\hat{w} = \sum_{k=1}^{n} \hat{w}_k, \ \hat{f} = \sum_{k=1}^{n} \hat{f}_k, \text{ and } \hat{e} = \sum_{k=1}^{n} \hat{e}_k.$$

4 On-Leveling

Define $t_0 = 0$ and suppose that the company has implemented n rate changes at times $t_1, t_2, ..., t_n$ where $t_k < t_{k+1}$ for $0 \leq k \leq n-1$. Let $t_{n+1} = \infty$ and $\hat{w}_k$, $\hat{f}_k$, and $\hat{e}_k$ be the policy period written quantity, quantity in force, and earned quantity functions respectively for the period $[t_k, t_{k+1})$ for $0 \leq k \leq n$. For $0 \leq k \leq n$ suppose that there are n_k atomic transactions pertaining to policies originating in the period $[t_k, t_{k+1})$ and for $1 \leq j \leq n_k$ let $\hat{w}_{k,j}$, $\hat{f}_{k,j}$, and $\hat{e}_{k,j}$ be the respective written quantity, quantity in force, and earned quantity functions. Since every atomic transaction is associated to a policy originating in exactly one of these periods, observe that

$$w = \sum_{k=0}^{n} \sum_{j=1}^{n_k} \hat{w}_{k,j} = \sum_{k=0}^{n} \hat{w}_k,$$

$$f = \sum_{k=0}^{n} \sum_{j=1}^{n_k} \hat{f}_{k,j} = \sum_{k=0}^{n} \hat{f}_k,$$

and

$$e = \sum_{k=0}^{n} \sum_{j=1}^{n_k} \hat{e}_{k,j} = \sum_{k=0}^{n} \hat{e}_k.$$

The above equations simply state that any calendar period amount of interest is the sum of all policy period amounts of interest.

Fix a period $[s_1, s_2)$ and consider the task of on-leveling the earned premium in it. Let w_p and e_p be the written premium and earned premium functions and let $\tilde{w}_p$ and $\tilde{e}_p$ be the written premium and earned premium as though the current rates had been in effect since time $t = 0$. Let $\tilde{w}_{p,k}$ and $\tilde{e}_{p,k}$ be the policy period written premium and earned premium for the period $[t_k, t_{k+1})$ for $0 \leq k \leq n$ as though the current rates had been in effect during each of those terms. Hence, while $e_p(s_2) - e_p(s_1)$ is what is recorded in the company's data, the desired, on-leveled quantity is $\tilde{e}_p(s_2) - \tilde{e}_p(s_1)$. Equivalently, one desires the quantity

$$\frac{\tilde{e}_p(s_2) - \tilde{e}_p(s_1)}{e_p(s_2) - e_p(s_1)}.$$

This quantity is often referred to as the on-level factor. If one can re-rate all of its old policies using current rating rules then this number may be computed exactly. If one is unable to do this, then one must make additional assumptions. The first two assumptions, while not possessing names outside of this paper, shall henceforth be referred to as the Classical On-Leveling Assumptions within this paper.

5 The Classical On-Leveling Assumptions

Knowledge of the relationship between $\hat{e}_{p,k}$ and $\tilde{e}_{p,k}$, and thus, the relationship between $\hat{w}_{p,k}$ and $\tilde{w}_{p,k}$, is essential to calculating on-leveled premium. However, it is difficult in practice to know that relationship. This is due to the fact that $\tilde{w}_{p,k}$ may be the result of numerous changes involving reclassifications, rate capping, and other potentially complicated changes. If one is to on-level one's premium while avoiding the use of extension of exposures, one must make some assumption about the relationship between $\hat{w}_{p,k}$ and $\tilde{w}_{p,k}$ with tractable consequences. To proceed, let w_e be the written exposure function, recognize that the mathematical manifestation of the rate is $\frac{dw_p}{dw_e}$, and that implementing a rate change is making a change to $\frac{dw_p}{dw_e}$. For example, suppose that premium is written at a constant rate of \$1000 / exposure. Mathematically, that is expressed as $\frac{dw_p}{dw_e} = 1000$. If the company's first rate change takes place at time $t = 1$ and is a 3% rate change applied to all policies, this would mean that premium is written at a constant rate of \$1030 / exposure after $t = 1$. Mathematically, this would mean that $\frac{dw_p}{dw_e} = 1030$. Since all premium functions are merely sums of atomic transaction functions, we must express the assumption in terms of the atomic transaction written premium functions. To this end, for $0 \leq k \leq n$ and $1 \leq j \leq n_k$, let $\hat{w}_{p,k,j}$ and $\hat{w}_{e,k,j}$ be the written premium and written exposure functions corresponding to the jth atomic transaction associated to a policy originating in the kth policy period. Similarly, let $\tilde{w}_{p,k,j}$ be the written premium function for the jth atomic transaction associated to a policy originating in the kth policy period assuming the current rating structure was in effect. The first classical on-leveling assumption may now be stated in the following way. For $0 \leq k \leq n$ there is some c_k such that

$$\frac{d\tilde{w}_{p,k,j}}{d\hat{w}_{e,k,j}} = c_k \frac{d\hat{w}_{p,k,j}}{d\hat{w}_{e,k,j}} \tag{1}$$

or, more concisely using the chain rule,

$$\frac{d\tilde{w}_{p,k,j}}{d\hat{w}_{p,k,j}} = \frac{d\tilde{w}_{p,k,j}}{d\hat{w}_{e,k,j}} \cdot \frac{d\hat{w}_{e,k,j}}{d\hat{w}_{p,k,j}} = c_k \frac{d\hat{w}_{p,k,j}}{d\hat{w}_{e,k,j}} \cdot \frac{1}{\frac{d\hat{w}_{p,k,j}}{d\hat{w}_{e,k,j}}} = c_k$$

Note that, by definition, $c_n = 1$. Since $\tilde{w}_{p,k,j}(0) = \hat{w}_{p,k,j}(0) = 0$, the first classical on-leveling assumption implies that $\tilde{w}_{p,k,j} = c_k \hat{w}_{p,k,j}$. Observe that this equation holds whether one considers these functions as depending on time or written exposure. Hence, for $0 \leq k \leq n$, $\tilde{e}_k = c_k \hat{e}_k$,

$$\tilde{e}_p(s_2) - \tilde{e}_p(s_1) = \sum_{k=0}^{n} c_k(\hat{e}_{p,k}(s_2) - \hat{e}_{p,k}(s_1)),$$

and the on-level factor is

$$\frac{1}{e_p(s_2) - e_p(s_1)} \sum_{k=0}^{n} c_k(\hat{e}_{p,k}(s_2) - \hat{e}_{p,k}(s_1)) = \sum_{k=0}^{n} c_k \frac{\hat{e}_{p,k}(s_2) - \hat{e}_{p,k}(s_1)}{e_p(s_2) - e_p(s_1)}.$$

The on-level factor is a weighted sum of the average of the policy period % increases (i.e. c_k) weighted by the amount of the $[s_1, s_2)$ calendar period earned premium which came from policy period $[t_k, t_{k+1})$. Thus, if for all calendar period earned premium, one is able to determine the policy period with which that premium is associated, one can calculate the above on-level factor. If one's transaction processing database is sufficiently granular, then a simple database query should be able to accomplish this. This observation is made in [6] (p. 80). This formula and the next is all that is needed to on-level by rate book.

A natural question to arise is how to compute the c_ks. To provide a simple answer to that, another definition must be given. For $0 \leq i \leq n$, let ${}_i\tilde{w}_{p,k,j}$ be the written premium function for the jth atomic transaction associated to a policy originating in the kth policy period assuming the rating structure for the time $[t_i, t_{i+1})$ was in effect. Observe that a consequence of this definition is that ${}_n\tilde{w}_{p,k,j} = \tilde{w}_{p,k,j}$ and ${}_k\tilde{w}_{p,k,j} = \hat{w}_{p,k,j}$. The second classical on-leveling assumption may now be made to help calculate the c_ks. For $0 \leq k \leq n-1$

$$\frac{d\ {}_{k+1}\tilde{w}_{p,k,j}}{d\hat{w}_{e,k,j}} = \frac{c_k}{c_{k+1}} \frac{d\hat{w}_{p,k,j}}{d\hat{w}_{e,k,j}}. \tag{2}$$

That is, ${}_{k+1}\tilde{w}_{p,k,j} = (c_k/c_{k+1})\hat{w}_{p,k,j}$. This is assumption is helpful in the following way. Often when implementing a rate change, a rate impact is calculated on the present book of business. That is, the rates that are going to be in effect in the future are applied to the current book of business and compared to current inforce premium. The above assumption is that that rate impact is equal to c_k/c_{k+1}. For example, suppose that there have been two rate changes, one at time $t = 1$ and another at time $t = 2$, and another one planned for time $t = 3$. Suppose that the rate impact at time $t = 1$ was found to be 3% and at time $t = 2$ is 5%. Then assumption (2) and the fact that $c_3 = 1$ by definition yields

$$c_2 = \frac{c_2}{c_3} c_3 = 1.05 \cdot 1 = 1.05 \text{ and } c_1 = \frac{c_1}{c_2} c_2 = 1.03 \cdot 1.05.$$

This assumption is merely a formalization of the usual reasoning used to calculate the cumulative rate level index ([6], p. 76).

6 Parallelogram Method

If one is unable to on-level one's premium using only the classical on-leveling assumptions, then one must make additional simplifying assumptions. One such set of assumptions forms the basis of The Parallelogram Method. Werner and Modlin in [6] on pages 73 and page 75 respectively state these assumptions in words as

1. "that premium is written evenly throughout the time period" and
2. "the distribution of policies written is uniform over time".

Using our previous notation and the assumption that the number of policies written is proportionate to the number of exposures written, the above may be stated mathematically as

$$\text{for } 0 \leq k \leq n \text{ there is some } r_k \text{ such that } \frac{d\hat{w}_{p,k}}{dt} = r_k \text{ on } [t_k, t_{k+1}) \text{ and} \tag{3}$$

$$\text{there is some } r \text{ such that for } 0 \leq k \leq n\ \frac{d\tilde{w}_{p,k}}{d\hat{w}_{e,k}} = r \text{ on } [t_k, t_{k+1}) \text{ and} \tag{4}$$

$$\text{there is some } c \text{ such that for } 0 \leq k \leq n\ \frac{d\hat{w}_{e,k}}{dt} = c \text{ on } [t_k, t_{k+1}). \tag{5}$$

Assumption (4) states that the on-leveled rate per exposure for each of the time periods is the same. This is the mathematical expression of the idea of a "steady mix of business". While not explicity mentioned in [6], this additional assumption is necessary for the Parallelogram Method to work. An example illustrating the insufficiency of only assumptions (3) and (5) is given later. From assumptions (4) and (5) it follows for $0 \leq k \leq n$ that

$$\frac{d\tilde{w}_{p,k}}{dt} = \frac{d\tilde{w}_{p,k}}{d\hat{w}_{e,k}}\frac{d\hat{w}_{e,k}}{dt} = rc$$

on $[t_k, t_{k+1})$ and 0 elsewhere. Thus, $\frac{d\tilde{w}_p}{dt} = \sum_{k=0}^{n} \frac{d\tilde{w}_{p,k}}{dt} = rc$. Hence, the phrase "that premium is written evenly throughout the time period", if applied to on-leveled premium as well as recorded premium, is sufficient to obtain the results of the parallolgram method and no assumption on the writing of exposures need be made.

It shall be now be established that the assumptions of the Parallelogram Method can be used to derive the classical on-leveling assumptions. First, note that

$$\frac{d\tilde{w}_{p,k}}{d\hat{w}_{e,k}} = r = \frac{rc}{r_k} r_k \frac{1}{c} = \frac{rc}{r_k}\frac{d\hat{w}_{p,k}}{dt}\frac{dt}{d\hat{w}_{e,k}} = \frac{rc}{r_k}\frac{d\hat{w}_{p,k}}{d\hat{w}_{e,k}}.$$

This establishes (1). Second, since $c_k = \frac{rc}{r_k}$ as shown above, and

$$\frac{d\hat{w}_{p,k}}{d\hat{w}_{e,k}} = \frac{d\hat{w}_{p,k}}{dt}\frac{dt}{d\hat{w}_{e,k}} = \frac{r_k}{c},$$

then $r_k c_k = rc$ and, by definition, for $0 \leq k \leq n-1$

$$\frac{d\ _{k+1}\tilde{w}_{p,k}}{d\hat{e}_{p,k}} = \frac{r_{k+1}}{c} = \frac{r_{k+1}}{r_k}\frac{r_k}{c} = \frac{c_k}{c_{k+1}}\frac{d\hat{w}_{p,k}}{d\hat{w}_{e,k}}.$$

This establishes (2).

For the purposes of computation, observe that the above also implies $c_k = c_{k+1}\left(\frac{r_{k+1}}{c} \div \frac{r_k}{c}\right)$ for $0 \leq k \leq n-1$. This means simply that c_k is proportional to the rate impact of the $(k+1)$st rate change. This, along with the fact that $c_n = 1$, allows one to compute all c_ks.

It should be mentioned that assumptions (3) and (5) present two theoretical difficulties. As mentioned in the section on atomic transactions, each of those functions has a zero derivative almost everywhere, and thus the sum of those functions has a zero derivative almost everywhere. Hence, if r_k or c is anything other 0, then the functions mentioned in the above assumptions are truly model written quantity functions, not actual written quantity functions. This leads to the second theoretical difficulty. If the functions in the assumptions above are model quantity functions and not tied to actual observed quantities or atomic transactions, then an additional assumption must be made in order to derive an earned quantity from a written quantity. This leads to the fourth Parallelogram Method assumption.

There is some τ such that all atomic transactions have a term length of τ. (6)

Functionally, this means that no policies will be canceled and no endorsements will be added or canceled. Using Assumption (6) and the notation from the first section, an equation linking written quantity and earned quantity functions may be derived by observing

$$f(t) = \sum_{k=1}^{n} f_k(t) = \sum_{k=1}^{n} w_k(t) - w_k(t-\tau_k) = \sum_{k=1}^{n} w_k(t) - w_k(t-\tau) = w(t) - w(t-\tau)$$

and

$$e(t) = \sum_{k=1}^{n} e_k(t) = \sum_{k=1}^{n} \frac{a_k}{\tau_k}(\max(t-t_k, 0) - \max(t-(t_k+\tau_k), 0)) = \sum_{k=1}^{n} \int_0^t \frac{1}{\tau_k}(w_k(s) - w_k(s-\tau_k))ds$$
$$= \int_0^t \frac{1}{\tau}\sum_{k=1}^{n} w_k(s) - w_k(s-\tau)ds = \int_0^t \frac{1}{\tau}(w(s) - w(s-\tau))ds = \int_0^t \frac{1}{\tau}f(s)ds.$$

Based on the above equations, if w is a model written quantity function and τ is the term length for all transactions, then define f by $f(t) = w(t) - w(t-\tau)$ and e by $e(t) = \int_0^t \frac{1}{\tau}(w(t) - w(t-\tau))dt$.

Using assumption (3) and the fact that $\hat{w}_{p,k}(t_k) = 0$, formulae for policy term earned premium may be developed. In particular,

$$\hat{w}_{p,k}(t) = \begin{cases} 0 & t \le t_k \\ r_k(t-t_k) & t_k \le t \le t_{k+1}. \\ r_k(t_{k+1}-t_k) & t_{k+1} \le t \end{cases}$$

The premium in force and earned premium functions for the cases $t_{k+1} - t_k \le \tau$ and $\tau \le t_{k+1} - t_k$ are slightly different but may be compactly stated as below. Their development is given in Appendix A. The general form of the premium in force is

$$\hat{f}_{p,k}(t) = \begin{cases} 0 & t \le t_k \\ r_k(t-t_k) & t_k \le t \le \min(t_k+\tau, t_{k+1}) \\ r_k(\min(t_{k+1},t) - \max(t_k, t-\tau)) & \min(t_k+\tau, t_{k+1}) \le t \le \max(t_k+\tau, t_{k+1}), \\ r_k(t_{k+1}+\tau-t) & \max(t_k+\tau, t_{k+1}) \le t \le t_{k+1}+\tau \\ 0 & t_{k+1}+\tau \le t \end{cases}$$

and the $[t_k, t_{k+1}]$ policy term earned premium through time t is

$$\hat{e}_{p,k}(t) = \begin{cases} 0 & t \le t_k \\ .5\frac{r_k}{\tau}(t-t_k)^2 & t_k \le t \le \min(t_k+\tau, t_{k+1}) \\ .5\frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau)^2 & \min(t_k+\tau, t_{k+1}) \le t \le \max(t_k+\tau, t_{k+1}) \\ +\frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau) & \\ \cdot(t-(\min(t_{k+1}-t_k,\tau)+t_k)) & \\ \frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau)^2 & \max(t_k+\tau, t_{k+1}) \le t \le t_{k+1}+\tau \\ +\frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau) & \\ \cdot(\max(t_{k+1}-t_k,\tau)-\min(t_{k+1}-t_k,\tau)) & \\ -.5\frac{r_k}{\tau}(t_{k+1}+\tau-t)^2 & \\ \frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau)^2 & t_{k+1}+\tau \le t. \\ +\frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau) & \\ \cdot(\max(t_{k+1}-t_k,\tau)-\min(t_{k+1}-t_k,\tau)) & \end{cases}$$

7 Parallelogram Examples

7.1 Example 1

For an example, consider the "simple Example" given in [6] starting on p.72. In this example the policy term is annual (i.e. that is $\tau = 1$).

Rate Level Group	Effective Date	Overall Average Rate Change
1	Initial	$--$
2	07/01/2010	5%
3	01/01/2011	10%
4	04/01/2012	-1%

The task is to on-level calendar year 2011 earned premium. Suppose that time $t = 0$ corresponds to 01/01/2010. Let $\hat{w}_{p,0}$, $\hat{w}_{p,1}$, $\hat{w}_{p,2}$, and $\hat{w}_{p,3}$ be the policy term written premium functions corresponding to the terms $[0, .5]$, $[.5, 1]$, $[1, 2.25]$, and $[2.25, \infty)$ respectively and let $\hat{e}_{p,0}$, $\hat{e}_{p,1}$, $\hat{e}_{p,2}$, and $\hat{e}_{p,3}$ be the corresponding policy term earned premium functions. Based on the table of rate changes, we have that

$$\begin{aligned} c_0 &= 1.05 \cdot 1.1 \cdot .99 \\ c_1 &= 1.1 \cdot .99 \\ c_2 &= .99 \\ c_3 &= 1. \end{aligned}$$

Since we wish to on-level the 2011 calendar year, we calculate

$$\begin{aligned} \hat{e}_{p,0}(2) - \hat{e}_{p,0}(1) &= \frac{1}{1}(r_0 \cdot .5^2 + r_0 \cdot .5(1-.5)) - \frac{1}{1}(.5r_0 \cdot .5^2 + r_0 \cdot .5(1-.5)) = .5r_0 - .375r_0 = .125r_0 \\ \hat{e}_{p,1}(2) - \hat{e}_{p,1}(1) &= \frac{1}{1}(r_1 \cdot .5^2 + r_1 \cdot .5(1-.5)) - \frac{1}{1}.5r_1(1-.5)^2 = .5r_1 - .125r_1 = .375r_1 = .375 \cdot 1.05r_0 \\ \hat{e}_{p,2}(2) - \hat{e}_{p,2}(1) &= \frac{1}{1}.5r_2 \cdot 1^2 - \frac{1}{1} \cdot 0 = .5r_2 = .5 \cdot 1.05 \cdot 1.1r_0 \\ \hat{e}_{p,3}(2) - \hat{e}_{p,3}(1) &= 0 - 0 = 0. \end{aligned}$$

Let $\tilde{e}_p(t)$ be the earned premium function assuming that all premium has been written at current rates. Then

$$\tilde{e}_p(2) - \tilde{e}_p(1) = \frac{1}{1}(.5r_3 \cdot 1 + r_3 \cdot 1(2-1)) - \frac{1}{1} \cdot .5r_3 \cdot 1^2 = r_3 = 1.05 \cdot 1.1 \cdot .99r_0.$$

Hence, the on-level factor for calendar year 2011 is

$$\frac{1.05 \cdot 1.1 \cdot .99r_0}{.125r_0 + .375 \cdot 1.05r_0 + .5 \cdot 1.05 \cdot 1.1r_0} \approx 1.0431.$$

7.2 Example 2

The second example is like the first example except that the policy term is 6 months. (i.e. that is $\tau = .5$). Using the same notation as before, we calculate

$$\begin{aligned} \hat{e}_{p,0}(2) - \hat{e}_{p,0}(1) &= \frac{1}{.5}(r_0 \cdot .5^2 + r_0 \cdot .5 \cdot 0) - \frac{1}{.5}(r_0 \cdot .5^2 + r_0 \cdot .5 \cdot 0) = 0 \\ \hat{e}_{p,1}(2) - \hat{e}_{p,1}(1) &= \frac{1}{.5}(r_1 \cdot .5^2 + r_1 \cdot .5 \cdot 0) - \frac{1}{.5}(.5r_1 \cdot .5^2) = .25r_1 = .25 \cdot 1.05r_0 \\ \hat{e}_{p,2}(2) - \hat{e}_{p,2}(1) &= \frac{1}{.5}(.5r_2 \cdot .5^2 + r_2 \cdot .5 \cdot (2 - (.5+1))) - \frac{1}{.5}0 = .75r_2 = .75 \cdot 1.05 \cdot 1.1r_0 \\ \hat{e}_{p,3}(2) - \hat{e}_{p,3}(1) &= \frac{1}{.5}0 - \frac{1}{.5}0 = 0. \end{aligned}$$

Let $\tilde{e}_p(t)$ be the earned premium function assuming that all premium has been written at current rates. Then

$$\tilde{e}_p(2) - \tilde{e}_p(1) = \frac{1}{.5}(.5r_3 \cdot .5^2 + r_3 \cdot .5(2-.5)) - \frac{1}{.5}(.5r_3 \cdot .5^2 + r_3 \cdot .5(1-.5)) = r_3 = 1.05 \cdot 1.1 \cdot .99r_0.$$

Hence, the on-level factor for calendar year 2011 is

$$\frac{1.05 \cdot 1.1 \cdot .99 r_0}{.25 \cdot 1.05 r_0 + .75 \cdot 1.05 \cdot 1.1 r_0} \approx 1.0130.$$

7.3 Example 3

In this example, we demonstrate the necessity of assumption (4) in the Parallelogram Method. To this end, suppose that policies are annual (e.g. $\tau = 1$), there was a single rate change on 01/01/2011. Suppose that time $t = 0$ corresponds to 01/01/2010. Let $\hat{w}_{p,0}$ and $\hat{w}_{p,1}$ be the policy term written premium functions corresponding to the terms $[0, 1]$ and $[1, 2]$ respectively and let $\hat{e}_{p,0}$ and $\hat{e}_{p,1}$ be the corresponding policy term earned premium functions. Assume further that

$$\begin{aligned}
\frac{d\hat{w}_{e,1}}{dt} &= \frac{d\hat{w}_{e,2}}{dt} = 1000 \\
\frac{d\hat{w}_{p,1}}{d\hat{w}_{e,1}} &= 100 \\
\frac{d\tilde{w}_{p,1}}{d\hat{w}_{e,1}} &= 105 \\
\frac{d\tilde{w}_{p,2}}{d\hat{w}_{e,2}} &= \frac{d\hat{w}_{p,2}}{d\hat{w}_{e,1}} = 110.
\end{aligned}$$

That is, 1000 exposures are being written each year. In 2010, the written premium per exposure is \$100 and is \$105 at current rate level. In 2011, the written premium per exposure is \$110. Hence,

$$\begin{aligned}
\hat{e}_{p,1}(2) - \hat{e}_{p,1}(1) &= 100 \cdot 1000 \cdot 1^2 + 100 \cdot 1000 \cdot 1 \cdot (1-1) - .5 \cdot 100 \cdot 1000 \cdot (1+1-2) - .5 \cdot 100 \cdot 1000 \cdot 1^2 \\
&= 50000 \\
\hat{e}_{p,2}(2) - \hat{e}_{p,2}(1) &= .5 \cdot 110 \cdot 1000 \cdot (2-1)^2 - 0 = 55000.
\end{aligned}$$

Let $\tilde{e}_p(t)$ be the earned premium function assuming that all premium has been written at current rates. Then

$$\begin{aligned}
\tilde{e}_p(2) - \tilde{e}_p(1) &= (\tilde{e}_{p,1}(2) - \tilde{e}_{p,1}(1)) + (\tilde{e}_{p,2}(2) - \tilde{e}_{p,2}(1)) \\
&= (105 \cdot 1000 \cdot 1^2 + 105 \cdot 1000 \cdot 1 \cdot (1-1) - .5 \cdot 105 \cdot 1000 \cdot (1+1-2) - .5 \cdot 105 \cdot 1000 \cdot 1^2) \\
&+ 55000 = 52500 + 55000 = 107,500.
\end{aligned}$$

Hence, the on-level factor for calendar year 2011 is

$$\frac{107500}{105000} \approx 1.0238 \neq 1.0244 \approx= \frac{1.05}{.5 + .5 \cdot 1.05}.$$

Observe that the factor does not match the form typically prescribed by the Parallelogram Method.

8 Conclusion

On-leveling premium is merely an arithmetic problem. The impediment to the extension of exposures is technology and data driven. Thus, the pace of technology and availability of data will soon make papers like this obselete. Until then, if one is unable to apply the extension of exposures technique, then one must make some assumptions. The most popular set of assumptions, as indicated by a review of the literature, are those which give rise to the Parallelogram Method. The popularity of this method is unsurprising given that it originated in a time when there was not access to fast databases and that it can be applied using periodic reports of earned premium and exposure. However, this is a different time. Today, even if one does not have all of the data or the rating engines to use the extension of exposures, one can certainly, with a few database queries determine under what rate book every piece of earned premium was written. Hence, one may, with only a modicum of effort, on-level premium by rate book. Moreover, as shown by this paper, the assumptions governing the Parallelogram Method are much stronger than those needed to on-level by rate book. Therefore, it is the opinion of this author that the reader experiment by comparing premium on-leveled by the three different methods and determine whether on-leveling by the Parallelogram Method or by rate book is closer to the correct answer of on-leveling by extension of exposures. The importance of this answer lies in its ability to allow one to price more accurately. The more accurate one's on-leveling procedure, the less one's premium trend calculations must account for one's on-leveling inaccuracies, and the more precise one's projection of premium will be.

9 Appendix - Derivation Of Parallelgram Method

In order to obtain the general expression of the premium in force function for the Parallelogram Method, first consider the case $t_{k+1} - t_k \leq \tau$. In this case,

$$\hat{f}_{p,k}(t) = \begin{array}{ll} 0 & t \leq t_k \\ r_k(t - t_k) & t_k \leq t \leq t_{k+1} \\ r_k(t_{k+1} - t_k) & t_{k+1} \leq t \leq t_k + \tau \\ r_k(t_{k+1} + \tau - t) & t_k + \tau \leq t \leq t_{k+1} + \tau \\ 0 & t_{k+1} + \tau \leq t \end{array}$$

Next, consider the case $\tau \leq t_{k+1} - t_k$. In this case,

$$\hat{f}_{p,k}(t) = \begin{array}{ll} 0 & t \leq t_k \\ r_k(t - t_k) & t_k \leq t \leq t_k + \tau \\ r_k\tau & t_k + \tau \leq t \leq t_{k+1} \\ r_k(t_{k+1} + \tau - t) & t_{k+1} \leq t \leq t_{k+1} + \tau \\ 0 & t_{k+1} + \tau \leq t \end{array}$$

Combining the above results yields the general form of the premium in force function as

$$\hat{f}_{p,k}(t) = \begin{array}{ll} 0 & t \leq t_k \\ r_k(t - t_k) & t_k \leq t \leq \min(t_k + \tau, t_{k+1}) \\ r_k(\min(t_{k+1}, t) - \max(t_k, t - \tau)) & \min(t_k + \tau, t_{k+1}) \leq t \leq \max(t_k + \tau, t_{k+1}). \\ r_k(t_{k+1} + \tau - t) & \max(t_k + \tau, t_{k+1}) \leq t \leq t_{k+1} + \tau \\ 0 & t_{k+1} + \tau \leq t \end{array}$$

Similarly, the derivation of the general form of the earned premium function must be split into two cases. In the first case, $t_{k+1} - t_k \leq \tau$ and $\min(t_k + \tau, t_{k+1}) = t_{k+1}$. Hence,

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^t 0 = 0 \text{ for } t \leq t_k,$$

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^{t_k} 0 + \int_{t_k}^t \frac{r_k}{\tau}(t - t_k) = .5\frac{r_k}{\tau}(t - t_k)^2 \text{ for } t_k \leq t \leq \min(t_k + \tau, t_{k+1}),$$

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^{\min(t_k+\tau,t_{k+1})} \frac{1}{\tau}\hat{f}_{p,k}(s)ds + \int_{\min(t_k+\tau,t_{k+1})}^t \frac{r_k}{\tau}(\min(t_{k+1}, s) - \max(t_k, s - \tau))ds$$

$$= .5\frac{r_k}{\tau}(t_{k+1} - t_k)^2 + \int_{t_{k+1}}^t \frac{r_k}{\tau}(t_{k+1} - t_k)ds = .5\frac{r_k}{\tau}(t_{k+1} - t_k)^2 + \frac{r_k}{\tau}(t_{k+1} - t_k)(t - t_{k+1})$$

$$= .5\frac{r_k}{\tau}\min(t_{k+1} - t_k, \tau)^2 + \frac{r_k}{\tau}\min(t_{k+1} - t_k, \tau)(t - (\min(t_{k+1} - t_k, \tau) + t_k))$$

for $\min(t_k + \tau, t_{k+1}) \leq t \leq \max(t_k + \tau, t_{k+1})$,

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^{\max(t_k+\tau,t_{k+1})} \frac{1}{\tau}\hat{f}_{p,k}(s)ds + \int_{\max(t_k+\tau,t_{k+1})}^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds$$

$$= .5\frac{r_k}{\tau}(t_{k+1} - t_k)^2 + \frac{r_k}{\tau}(t_{k+1} - t_k)(t_k + \tau - t_{k+1}) + \int_{t_k+\tau}^t \frac{r_k}{\tau}(t_{k+1} + \tau - s)ds$$

$$= .5\frac{r_k}{\tau}(t_{k+1} - t_k)^2 + \frac{r_k}{\tau}(t_{k+1} - t_k)(t_k + \tau - t_{k+1}) + .5\frac{r_k}{\tau}((t_{k+1} - t_k)^2 - (t_{k+1} + \tau - t)^2)$$

$$= \frac{r_k}{\tau}(t_{k+1} - t_k)^2 + \frac{r_k}{\tau}(t_{k+1} - t_k)(t_k + \tau - t_{k+1}) - .5\frac{r_k}{\tau}(t_{k+1} + \tau - t)^2$$

$$= \frac{r_k}{\tau}\min(t_{k+1} - t_k, \tau)^2 + \frac{r_k}{\tau}\min(t_{k+1} - t_k, \tau)(\max(t_{k+1} - t_k, \tau) - \min(t_{k+1} - t_k, \tau)) - .5\frac{r_k}{\tau}(t_{k+1} + \tau - t)^2$$

for $\max(t_k + \tau, t_{k+1}) \leq t \leq t_{k+1} + \tau$, and

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^{t_{k+1}+\tau} \frac{1}{\tau}\hat{f}_{p,k}(s)ds + \int_{t_{k+1}+\tau}^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds$$

$$= \frac{r_k}{\tau}(t_{k+1} - t_k)^2 + \frac{r_k}{\tau}(t_{k+1} - t_k)(\max(t_{k+1} - t_k, \tau) - \min(t_{k+1} - t_k, \tau)) + \int_{t_{k+1}+\tau}^t 0ds$$

$$= \frac{r_k}{\tau}\min(t_{k+1} - t_k, \tau)^2 + \frac{r_k}{\tau}\min(t_{k+1} - t_k, \tau)(\max(t_{k+1} - t_k, \tau) - \min(t_{k+1} - t_k, \tau)) \text{ for } t_{k+1} \leq t.$$

In the second case, $\tau \leq t_{k+1} - t_k$ and $\min(t_k + \tau, t_{k+1}) = t_k + \tau$. Hence,

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^t 0 = 0 \text{ for } t \le t_k,$$

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^{t_k} 0 + \int_{t_k}^t \frac{r_k}{\tau}(t-t_k) = .5\frac{r_k}{\tau}(t-t_k)^2 \text{ for } t_k \le t \le \min(t_k+\tau, t_{k+1}),$$

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^{\min(t_k+\tau,t_{k+1})} \frac{1}{\tau}\hat{f}_{p,k}(s)ds + \int_{\min(t_k+\tau,t_{k+1})}^t \frac{r_k}{\tau}(\min(t_{k+1},s)-\max(t_k,s-\tau))ds$$

$$= .5\frac{r_k}{\tau}\tau^2 + \int_{t_k+\tau}^t \frac{r_k}{\tau}\tau ds = .5\frac{r_k}{\tau}\tau^2 + \frac{r_k}{\tau}\tau(t-(t_k+\tau))$$

$$= .5\frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau)^2 + \frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau)(t-(\min(t_{k+1}-t_k,\tau)+t_k))$$

for $\min(t_k+\tau, t_{k+1}) \le t \le \max(t_k+\tau, t_{k+1})$,

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^{\max(t_k+\tau,t_{k+1})} \frac{1}{\tau}\hat{f}_{p,k}(s)ds + \int_{\max(t_k+\tau,t_{k+1})}^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds$$

$$= .5\frac{r_k}{\tau}\tau^2 + \frac{r_k}{\tau}\tau(t_{k+1}-(t_k+\tau)) + \int_{t_{k+1}}^t \frac{r_k}{\tau}(t_{k+1}+\tau-s)ds$$

$$= .5\frac{r_k}{\tau}\tau^2 + \frac{r_k}{\tau}\tau(t_{k+1}-(t_k+\tau)) + .5\frac{r_k}{\tau}(\tau^2-(t_{k+1}+\tau-t)^2)$$

$$= \frac{r_k}{\tau}\tau^2 + \frac{r_k}{\tau}\tau(t_{k+1}-(t_k+\tau)) - .5\frac{r_k}{\tau}(t_{k+1}+\tau-t)^2$$

$$= \frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau)^2 + \frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau)(\max(t_{k+1}-t_k,\tau)-\min(t_{k+1}-t_k,\tau)) - .5\frac{r_k}{\tau}(t_{k+1}+\tau-t)^2$$

for $\max(t_k+\tau, t_{k+1}) \le t \le t_{k+1}+\tau$, and

$$\int_0^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds = \int_0^{t_{k+1}+\tau} \frac{1}{\tau}\hat{f}_{p,k}(s)ds + \int_{t_{k+1}+\tau}^t \frac{1}{\tau}\hat{f}_{p,k}(s)ds$$

$$= \frac{r_k}{\tau}\tau^2 + \frac{r_k}{\tau}\tau(\max(t_{k+1}-t_k,\tau)-\min(t_{k+1}-t_k,\tau)) + \int_{t_{k+1}+\tau}^t 0ds$$

$$= \frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau)^2 + \frac{r_k}{\tau}\min(t_{k+1}-t_k,\tau)(\max(t_{k+1}-t_k,\tau)-\min(t_{k+1}-t_k,\tau)) \text{ for } t_{k+1} \le t.$$

10 Appendix - Reconciliation To Previous Papers

There are three main papers which seek to give a systematic treatment to on-leveling premium. It should be noted that the expressions given in this paper, under a change of notation, yield the same results as those papers. Also, these papers are considering model written exposure functions instead of those related to actual atomic transactions. The first paper, [5], is that of Ross from 1975. On p. 52 in [5], Ross presents a formula, denoted as $EE(x_0, x_1)$, for the amount of exposures earned between time x_0 and x_1, denotes time and policy term length using the letters x and t respectively, and writes "Let the function $f(x)$ stand for the rate of exposure writing at time x". In order to see the equivalence of formulae, note that the set of points in the plane which describe the region $\{(x,s) : x_0 \le x \le x_1, x-\tau \le s \le x\}$ can also be described by

$$\begin{array}{ll} x_0 \leq s \leq x_0 & x_0 \leq x \leq s+\tau \\ x_0 \leq s \leq x_1-\tau & s \leq x \leq s+\tau \\ x_1-\tau \leq s \leq x_1 & s \leq x \leq x_1. \end{array}$$

Using his notation, the equivalence between his formula the one developed in this paper is as follows:

$$\begin{aligned} e_e(x_1)-e_e(x_0) &= \frac{1}{\tau}\int_{x_0}^{x_1} f_e(x)dx = \frac{1}{\tau}\int_{x_0}^{x_1} w_e(x)-w_e(x-\tau)dx = \frac{1}{\tau}\int_{x_0}^{x_1}\int_{x-\tau}^{x} f(s)dsdx \\ &= \frac{1}{\tau}\left(\int_{x_0-\tau}^{x_0}\int_{x_0}^{s+t} f(s)dxds + \int_{x_0}^{x_1-\tau}\int_{s}^{s+t} f(s)dxds + \int_{x_1-\tau}^{x_1}\int_{s}^{x_1} f(s)dxds\right) \\ &= \int_{x_0-\tau}^{x_0}\frac{s+\tau-x_0}{\tau}f(s)ds + \int_{x_0}^{x_1-\tau} f(s)ds + \int_{x_1-\tau}^{x_1}\frac{x_1-s}{\tau}f(s)ds = EE(x_0,x_1). \end{aligned}$$

The second paper, [4], is that of Miller and Davis from 1976. On p. 121 in [4], Miller and Davis, giving a geometric interpretation to the work of Ross, derived a formula equivalent to his. They used notation similar to his except that they denoted the term length as k. Using their notation, equivalence between their formula and the one developed in this paper is as follows:

$$\begin{aligned} e_e(x_1)-e_e(x_0) &= \frac{1}{\tau}\int_{x_0}^{x_1} f_e(x)dx = \frac{1}{\tau}\int_{x_0}^{x_1} w_e(x)-w_e(x-\tau)dx = \frac{1}{\tau}\int_{x_0}^{x_1}\int_{x-\tau}^{x} f(s)dsdx \\ &= \int_{x_0}^{x_1}\int_{x}^{x-\tau} -\frac{1}{\tau}f(s)dsdx = \int_{x_0}^{x_1}\int_{0}^{1} f(x-\tau s)dsdx = EE(x_0,x_1). \end{aligned}$$

The third paper, [1], is that of Bill from 1989. Bill's work is an application of the formulae of Ross (p. 207). Hence, the work of Bill can also be derived from the formulae presented in this paper.

Finally, since the formulae implicitly referenced in [2], [3], and [6] are stated explicitly and developed from a more general set of equations, the claim set forth in the abstract, that this paper subsumes all works in the bibliography, is established.

References

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