A Unifying Approach to Pricing Insurance and Financial Risk

Andreas Kull

A Unifying Approach to Pricing Insurance and Financial Risk

September 10, 2002

Andreas Kull*

Abstract

The actuarial and the financial approach to the pricing of risk remain different despite the increasingly direct interconnection of financial and insurance markets. The difference can be summarized as pricing based on classical risk theory (insurance) vs. non-arbitrage pricing (finance). However, comparable pricing principles are of importance when it comes to transferring insurance risk to financial markets and vice versa as it is done e.g. by alternative risk-transfer instruments or derivative products. Incompatibilities blur business opportunities or may open up the possibility to arbitrage.

For these situations, the paper aims to bridge the gap between insurance and finance by extending the non-arbitrage pricing principle to insurance. The main obstacle that has to be tackled is related to the incompleteness of the insurance market. It implies that equivalent martingale probabilities are not uniquely defined. By the information theoretical maximum entropy principle a sensible way to choose a particular equivalent martingale measure is found. This approach parallels the successful application of the maximum entropy principle in finance.

The paper pays special attention to the role that investment opportunities beyond risk-free investments play for insurance operations. Equivalent martingale probabilities for the combination of insurance operation risk-free investment and a risky investment are determined. They turn out to be connected to the Esscher measure. This recovers a generalized form of a well-known actuarial premium calculation principle.

The sketched approach is further investigated for typical reinsurance structures like stop-loss and excess-of-loss reinsurance. Arbitrage-free reinsurance premiums are calculated. A numerical example stresses the influence that characteristics of risky investment opportunities have on arbitrage-free premiums.

* Converium Ltd, General Guisan-Quai 26, CH-8022 Zurich, Switzerland (e-mail: Andreas.Kull@converium.com)

1 Introduction

The interconnection of financial and insurance markets has become more direct during the past two decades. This is due to the repositioning of insurance companies as integral financial service providers, increasing exposures that tend to exceed the capacity of the insurance market and finance-related insurance products (e.g. catastrophe-bonds). Presumably, the convergence of insurance and finance has attained its most advanced level in alternative risk transfer contracts, where insurance and financial risks are covered jointly.

Financial and insurance markets and the pricing of respective products differ in many respects. On the market side, the main difference can be summarized as warehousing of risk (insurance) versus intermediation of risk (finance). Insurance pricing on the other hand is in general based on classical risk theory while finance relies on non-arbitrage pricing. Establishing comparable pricing principles is, however, of importance when it comes to transferring insurance risk to financial markets and vice versa. Incompatibilities blur business opportunities or open up the possibility to arbitrage. This is a main motivation for this paper.

Non-arbitrage pricing relies on liquid and efficient markets. Clearly, most of the insurance market is neither liquid nor efficient. Nevertheless, there are situations where in addition to pricing based on classical risk theory the corresponding non-arbitrage value of an insurance contract is of interest, e.g. when part of the risk is transferred to the financial markets or insurance risk is traded. With a growing interconnection of financial and insurance markets this situation becomes more frequent. A pioneering area in this respect is the insurance of credit risk, where warehousing and intermediation of risk overlap in a very natural way.

Pricing of insurance contracts is commonly based on real probabilities P, i.e. probabilities reflecting the actual likelihood of loss events. In the simplest one period case, the premium for an insurance contract covering losses X according to the equivalence principle is

$$\text{premium} = \frac{1}{1+r_l} E^P[X] + S[X] \tag{1.1}$$

where $E^P[X]$ is the loss expectation calculated under the real probability measure P, discounted with a risk-discount rate r_l chosen according to actuarial judgement. $S[X]$ is the safety loading or risk premium. A particular choice of r_l and $S[X]$ should reflect the overall risk related to the contract, the risk-free interest rate, the cost of capital, the expected investment return, market conditions etc. For complex contracts this is usually not an easy task.

Following the seminal work of Black, Scholes and Merton in the early 70's, in finance, the non-arbitrage pricing principle lead to a shift away from the real probability measure P to equivalent martingale measures Q. Valuation of a future (stochastic) cash flow F under the equivalent martingale measure Q takes the form

$$\text{price} = \frac{1}{1+r_{rf}} E^Q[F] \tag{1.2}$$

where $\mathrm{E}^Q[\cdot]$ denotes the expectation operator under the measure Q. Pricing in this context thus invokes again the equivalence principle, however, with respect to the equivalent martingale measure Q. The discounting is performed with the directly observable risk-free interest rate r_{rf} and there is no modification due to a loading. An appealing feature of (1.2) is that once the equivalent martingale measure Q is known, the valuation of the cash flow F is performed without recourse to subjective criteria. Economically, non-arbitrage pricing derives its justification from the existence of a hedge portfolio that creates an overall risk-less position. The existence of a hedge portfolio in turn relies on liquid and efficient markets. Arbitrage-free pricing was pioneered by Black and Scholes (1973). Cox and Ross (1976) and Harrison and Kreps (1979) established the sound theoretical basis in terms of risk-neutral valuation and equivalent martingale measures.

The main difference between classical risk theory and the non-arbitrage approach to pricing is that the non-arbitrage approach substitutes real probabilities and expert knowledge for 'preference-free' probabilities that comply with the non-arbitrage assumption. The fundamental task in both cases remains to find the corresponding probabilities.

Gerber (1973) has introduced martingale methods to risk theory. Since then, a number of papers have been investigating martingales in risk theory. Mainly, these papers deal with assessing ruin probabilities. For a review of insurance related use of martingals see e.g. Schmidli (1996). The martingale approach to premium calculation, which is considered here, has been pioneered by Delbaen and Haezendonck (1989). In this paper it was shown that common premium principles can be recovered by martingale methods. Another important paper in this context is by Sondermann (1991), who considers arbitrage-free pricing for reinsurance.

The emphasis of this paper lies on the practical application of martingales to the pricing of insurance contracts whose performances depend on financial markets. The main problem arising is that, due to the incompleteness of the insurance market, equivalent martingale probabilities are not uniquely defined. Another difficulty arises since often the return distribution of insurance contracts is fundamentally different from that of asset returns ('heavy tails'). Thus it is not straightforward to apply standard financial techniques as e.g. the Black-Scholes-Merton framework. We show how the information theoretical maximum entropy principle can be applied to choose in a sensible way a particular equivalent martingale measure in this situation.

The paper pays special attention to the role that investment opportunities beyond risk-free investments play for insurance operations. Equivalent martingale probabilities for the combination of insurance operation risk-free investment and a risky investment are determined. They turn out to be connected to the Esscher measure. This recovers a generalized form of a well-known actuarial premium calculation principle.

The maximum entropy principle has been successfully applied in other fields and to similar problems in finance (e.g. Stutzer, 1996). The methods used here thus are not new. Also, other approaches to unify the actuarial and financial pricing based on different methodology (e.g. deflators, Jarvis et al. 2001) exist.

The paper is organized as follows: Section 2 gives a very brief review of basic concepts used in this paper. In Section 3 a non-arbitrage condition for insurance contracts is formulated. Section 4 tackles the problem of determining equivalent martingale probabilities in an incomplete market. Equivalent martingale probabilities are determined by making use of the maximum entropy principle. Section 5 generalizes the theory to include investment opportunities. Section 6 discusses issues related to unique valuation and implied discounting rates. As an example, simple reinsurance structures are considered in Section 7. This section also contains results of numerical simulations that illustrate some of the main results. Section 8 extends the one-period case of Section 7 to a multi-period framework. The last section presents conclusions.

2 Brief Review of Basic Concepts used in this Paper

This paper aims at bridging the gap between financial and insurance pricing by introducing some of the concepts of modern finance and information theory into insurance. For our purpose, the concepts of non-arbitrage pricing and equivalent martingale probabilities are of importance. They are covered by standard textbooks like e.g. Hull (2000) or Copeland and Weston (1992). In short, non-arbitrage pricing relies on the insight[1] that, in the absence of arbitrage opportunities, the price (or premium) of some contingent claim should match the price for a position perfectly hedging its risk. As has first been shown by Cox and Ross (1976) and Harrison and Kreps (1979), this insight can be reformulated mathematically in terms of 'equivalent martingale probabilities', i.e. in terms of a probability measure Q satisfying for a given random process X

$$E^{Q}[X_T] = X_t \tag{2.1}$$

[1] It was this insight (and its mathematical formulation) that gained Merton and Scholes the Nobel price in 1997.

Equation (2.1) has a straightforward interpretation: The best forecast of future values X_T at $T > t$ is the observed value X_t.

When pricing contingent claims in the non-arbitrage framework, equivalent martingale probabilities substitute the 'real' probabilities P e.g. derived from historical data. Dealing with equivalent martingale probabilities one should keep in mind that they emerge as a consequence of the non-arbitrage assumption and that they're 'artificial' insofar as they do not need to correspond to any real probabilities or beliefs. Second, it is important to note that only for complete and efficient markets there exists a unique equivalent martingale measure. In incomplete markets, however, equivalent martingale measures are not uniquely defined. This reflects the fact that contingent claims can be hedged only partially.

In our context of incomplete insurance markets, basic concepts of information theory come into play when a particular equivalent martingale measure Q is chosen from the infinite possibilities. Information theoretical concepts prove useful by providing a measure of the information a particular equivalent martingale measure Q embodies. In the absence of information other than the non-arbitrage assumption, the rationale is to choose the distribution that embodies least additional information. In a discrete setting, this is achieved by maximizing the entropy[2]

$$S = -\sum_i q_i \cdot \ln[q_i] \tag{2.2}$$

where the q_i's are the probabilities associated with a particular equivalent martingale measure Q Defining probability distributions by maximizing (2.2) has a longstanding history in information theory and statistical physics. The method is known as the *Information Theoretical Maximum Entropy Principle*. A more comprehensive review of information theoretical concepts lies beyond the scope of this paper. However, there are accessible textbooks covering these topics (see e.g. Cover and Thomas (1991)).

3 Formulating the Non-Arbitrage Constraint

Consider the simplest form of insurance: An insurer accepts the liability to pay for the compound loss

$$X = \sum_{j=1}^{N} L_j \tag{3.1}$$

occurring over a period $[0,T]$ where L_j is the (random) jth claim amount during the time period and N is the random number of claims. Here we will assume that the claim amounts L_j are independently and identically distributed and $L_j \equiv L$ refers to the claim distribution without deductible or limit. In exchange for this liability, the insurer receives a premium b. For simplicity it is assumed that there are no costs or investment returns and payments are made at time T only (for a generalization see Section 4). Then, the premium b can be interpreted as a risky asset generating one-period returns

$$R = \frac{b - X}{b} \tag{3.2}$$

The definition of the insurance related return R directly depends on b. To stress this point, b will be referred to as the reference premium in the following.

The non-arbitrage theorem can be expressed in different ways. An intuitive formulation in a discrete setting refers to different states i of the world, each of which is characterized by a set of payoffs. These payoffs originate from assets, e.g. in our case from the premium b or a risk-free bond. To formulate the non-arbitrage condition, realizations $l_{j,i}$ and n_i of the single claim amount L and the claim number N are considered. They translate by (3.1) and (3.2) into realizations x_i and r_i of the compound loss X and return R respectively. In addition, a risk-free

[2] Behind this lies the equivalence of the expression for information content and the expression for entropy as

bond with return r_{rf} is considered. Following standard procedures (see e.g. Neftci (2000)) in finance, $(1+r_i)$ and $(1+r_{rf})$ are identified with the insurance and bond related payoffs in the ith state. Grouping the payoffs in a matrix $D \in \Re^{2,K}$ where K is the total number of states leads to

$$D = \begin{bmatrix} (1+r_{rf}) & (1+r_{rf}) & \cdots & (1+r_{rf}) \\ (1+r_1) & (1+r_2) & \cdots & (1+r_K) \end{bmatrix} \tag{3.3}$$

The non-arbitrage theorem now states that if there is no arbitrage, positive constants ψ_i exist such that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1+r_{rf}) & (1+r_{rf}) & \cdots & (1+r_{rf}) \\ (1+r_1) & (1+r_2) & \cdots & (1+r_K) \end{bmatrix} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_K \end{bmatrix} \tag{3.4}$$

holds. With $q_i = \psi_i \cdot (1+r_{rf})$ relation (3.4) becomes

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \dfrac{(1+r_1)}{(1+r_{rf})} & \dfrac{(1+r_2)}{(1+r_{rf})} & \cdots & \dfrac{(1+r_K)}{(1+r_{rf})} \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_K \end{bmatrix} \tag{3.5}$$

from which it is evident that the quantities q_i can be interpreted as probabilities (as the inspection of the first component demonstrates the positive q_i's sum up to one). The q_i's are interpreted as 'risk-adjusted' or 'risk-neutral' probabilities. From (3.5) it follows that under the probability measure Q we have

---

discovered by Shannon (1948) who identified the generation of information with the reduction of entropy.

$$E^Q\left[\frac{1+R}{1+r_{rf}}\right]=\frac{1}{1+r_{rf}}E^Q\left[\frac{X-b}{b}+1\right]=1$$

(3.6)

i.e. under the measure Q, $(1+R)/(1+r_{rf})$ is a martingale and the q_i's are equivalent martingale probabilities.

Given the probabilities q_i, the arbitrage-free pricing of an arbitrary insurance contract referring to the losses L and the claim number N is straightforward. It takes the form

$$\text{non}-\text{arbitrage premium}=\frac{1}{1+r_{rf}}E^Q[f(L,N)]$$

(3.7)

where $f(L,N)$ stands for the loss amount to be paid according to the insurance contract. Relation (3.7) is formally identical to (1.2). According to it, the non-arbitrage premium is the expectation value calculated under the equivalent martingale measure Q, discounted by the risk-free interest rate r_{rf}. No risk premium is added since the risk adjustment is internalized by the change of the probability measure. Pricing any insurance contract whose loss can be written as $f(L,N)$ thus reduces to determine the equivalent martingale probabilities q_i.

In complete and efficient markets, the strict economic justification of (3.7) relies on the fact that the non-arbitrage premium coincides with the (unique) price of a hedging portfolio that yields an overall risk-less position. In incomplete markets (e.g. insurance markets) considered here, however, the situation is more complex. It can be shown that no unique and perfect hedging portfolio exists. This is reflected by the fact that in general the equivalent martingale probabilities q_i are not uniquely defined. The question that arises is how the martingale probabilities should be defined, or, in other words, what particular equivalent martingale measure Q should be chosen out of the infinitely many possibilities. We will address this question in the next section.

4 Equivalent Martingale Measure

The insurance market is neither liquid, effective, nor it is complete. An immediate consequence is that pricing contingent claims by their replication cost is not possible which implies that no unique equivalent martingale measure exists. Other practical difficulties arise since often the loss distribution of L is fundamentally different from that of asset returns. In particular, the loss distribution L may often show heavy tails, i.e. relatively high probabilities of large losses[3]. Thus it is not straightforward to apply standard financial techniques like the Black-Scholes-Merton framework, which implicitly relies on complete markets and log-normal distributed random variables. How can, nevertheless, as much as possible of the appealing properties of non-arbitrage pricing be recovered in this situation?

The equivalent martingale probabilities q_i have to fulfill according to the non-arbitrage assumption the relation (3.5) representing a linear system of two equations and K unknowns q_i. The system is of the form

$$S = \frac{1}{1+r_0} D * q \tag{4.1}$$

For given $S = [1,1]^T$ and D (specified by K realizations r_i), q is not determined uniquely. Indeed, there exist an infinite number of solutions for q_i which reflects market incompleteness.

The underdetermined nature of (4.1) is formally known as the 'Inverse Problem'. Several methods exist to deal with such situations. One of the most elegant and well-founded ways is provided by the maximum entropy principle of information theory, which we will follow here. In finance, the maximum entropy principle has been successfully applied in a similar context to problems related to option pricing. Rubinstein (1994) used the maximum entropy principle to

[3] A characteristic property of heavy tailed distributions is the non-existence of higher moments $E\left[X^k\right]$ for some $k > k_0$. As an example, consider the Pareto distribution whose second and higher moments do not exist if $\alpha < 2$.

infer martingale measures from observed option prices (Rubinstein 1994). Buchen and Kelly (1996), Stutzer (1996) and Gulko (1999) worked out generalizations and applications of this idea referring to option pricing. In the following, we use to a large extent the same well known information theoretic methods as Stutzer (1996).

4.1 Determining Equivalent Martingale Probabilities

The only information available about the equivalent martingale probabilities q_i is the non-arbitrage constraint as specified by (3.5) and the assumption that all realizations r_i are equally alike. Any other additional information (e.g. the volatility of the equivalent martingale probabilities q_i) would be imposed on the martingale probabilities q_i without justification since relation (3.5) is the only constraint defining properties of the equivalent martingale probabilities. A consistent criterion to make a choice along this line is the additional information a particular distribution is adding besides the constraint (3.5). One should try to minimize it. Minimizing additional information is equivalent to maximize the entropy

$$S = -\sum_i q_i \cdot \ln[q_i] \tag{4.2}$$

associated with a particular distribution $\{q_i\}$ (Jaynes, 1957a,b). This information theoretical *Maximum Entropy Principle* guarantees the chosen distribution to incorporate no information other than specified by the constraint (3.5), which is equivalent to choose the distribution that is 'most unbiased' or is 'embodying least structure'.

The problem of maximizing the entropy associated with the distribution q_i is a constraint maximizing problem:

$$\max S = -\sum_i q_i \cdot \ln[q_i] \tag{4.3}$$

under the constraints

I: $$\sum_i q_i = 1$$ (normalization)

(4.4)

II: $$q_i \geq 0, \quad \forall i$$ (positivity)

(4.5)

III: $$E^Q\left[\frac{1+R}{1+r_{rf}}\right] = \frac{1}{1+r_{rf}}\sum_i q_i(1+r_i) \equiv 1$$ (non-arbitrage condition)

(4.6)

It is solved by using Lagrangian multiplier techniques, i.e. maximizing the expression

$$L = -\sum_i q_i \cdot \ln[q_i] + \gamma\left(\frac{1}{1+r_{rf}}\sum_i q_i(1+r_i)\right) + \beta\left(\sum_i q_i\right) \tag{4.7}$$

where, γ and β represent the Lagrangian multipliers associated with the constraints (4.4) and (4.6). Maximizing (4.7) leads to

$$q_i = \frac{\mathrm{Exp}\left[\hat{\gamma}\dfrac{1+r_i}{1+r_{rf}}\right]}{\sum_j \mathrm{Exp}\left[\hat{\gamma}\dfrac{1+r_j}{1+r_{rf}}\right]} \tag{4.8}$$

where $\hat{\beta}^{-1} = \sum \mathrm{Exp}[\hat{\gamma}(1+r_i)/(1+r_{rf})]$ (constraint (4.4)). The parameter $\hat{\gamma}$ is to be determined numerically by the non-arbitrage constraint (4.6). Since the distribution (4.8) is an exponential, constraint (4.5) is automatically fulfilled.

To recapitulate: The distribution (4.8) is the maximum entropy probability distribution (or 'most unbiased' probability distribution). It has been deduced from the maximum entropy principle and the non-arbitrage constraint. By construction, the probabilities (4.8) are equivalent martingale probabilities. Their distribution is consistent with the non-arbitrage theorem and is otherwise assumption-free.

4.2 A Information Theoretical Justification of the Esscher Premium Principle

As a side remark note that (4.8) is equivalent to a special case of the Esscher Transform

$$F(x,\gamma) = \Pr[X < x, \gamma] = \frac{\int_{-\infty}^{x} \mathrm{Exp}[\gamma \cdot y] \cdot dF(y,t)}{\int_{-\infty}^{\infty} \mathrm{Exp}[\gamma \cdot y] \cdot dF(y,t)} \tag{4.9}$$

where the parameter γ can be chosen to be consistent with a non-arbitrage condition similar to (4.6). As discussed by Gerber and Shiu (1994), resulting equivalent martingale probabilities can be used for option pricing. For discrete probabilities q_i as discussed here, the Esscher Transform becomes

$$F(r,\gamma) = \Pr[R < r, \gamma] = \frac{\sum_{r_i < r} \mathrm{Exp}\left[\gamma \frac{1+r_i}{1+r_{rf}}\right]}{\sum_{r_i} \mathrm{Exp}\left[\gamma \frac{1+r_i}{1+r_{rf}}\right]} \tag{4.10}$$

With $\gamma = \hat{\gamma}$, (4.10) is consistent with the non-arbitrage constraint and (4.8) is recovered. This shows the special case of the Esscher transform to be equivalent to the maximum entropy probability distribution. In this sense, the Esscher transformation (4.10), which lies at the heart of the well-known Esscher premium principle[4], has an information theoretical justification.

5 Including Costs and Investment Return

Costs and investments play a crucial role for the profitability of insurance operations. While costs enter premium calculations in a straightforward way, this is less obvious for investments. Pricing of contracts and assessing reserves depends directly on the anticipated investment return. Usually, investment opportunities are taken into account by a risk-adjusted discounting factor in (1.1) whose determination is left for actuarial judgment. From the market perspective, the difficulty lies in finding a market conforming discount factor for the combined risk of insurance and investment operation or, in other terms, the market price of the overall risk. A way to address this problem is to consider arbitrage-free pricing of insurance operations that reflects and incorporates investment possibilities.

The maximum entropy approach provides an elegant way to account for investment returns and cost simply by modifying or adding constraints. Fixed costs c modify the return related to insurance (risky asset b) according to

$$R = \frac{b-c-X}{b} \tag{5.1}$$

Further, assume that the insurer is investing the amount b during the period $[0,T]$ in which no payments are made in a risky asset with (stochastic) return Y. The overall return of the insurance operation then becomes

[4] The Esscher premium principle defines (as e.g. the Variance premium principle) a particular way of calculating a safety loading or risk premium. Interestingly enough, the Esscher premium principle relies on 'distorted' probabilities in much the same way as non-arbitrage pricing relies on equivalent martingale probabilities. Indeed, in our case here, the martingale probabilities can be seen as originating as a special case from the Esscher premium principle. Besides this, there exist other economically relevant links (e.g. to the framework of Pareto-optimal risk exchange, see Bühlmann (1980)). I thank Peter Blum for stressing this point.
The connection of the Esscher premium principle to information theory, non-arbitrage pricing, and Pareto-optimal risk exchange certainly warrants a more detailed study.

$$R^{tot} = \frac{b - c - X}{b} + Y$$

(5.2)

Note that the overall return R^{tot} of the insurance operation will reflect diversification effects that are present due to the non (or only partially) correlated nature of the returns R and Y.

Again, equivalent martingale probabilities are calculated as outlined in Section 2 and 3. The only differences are that the constraint (4.6) is modified to reflect the cost c and that a new non-arbitrage constraint

III': $$E^Q\left[\frac{1+Y}{1+r_{rf}}\right] = \frac{1}{1+r_{rf}} \sum_i q_i (1+y_i) = 1$$

(5.3)

for the investment return is added. Because of linearity, $(1+R^{tot})/(1+r_{rf})$ is automatically a martingale, i.e. no modification for R^{tot} is needed. Maximizing the entropy in analogy to (4.7) leads to

$$q_i = \frac{\mathrm{Exp}\left[(1+r_{rf})^{-1}(\gamma_1(1+r_i)+\gamma_2(1+y_i))\right]}{\sum_i \mathrm{Exp}\left[(1+r_{rf})^{-1}(\gamma_1(1+r_i)+\gamma_2(1+y_i))\right]}$$

(5.4)

where the parameters γ_1 and γ_2 are determined by the constraints (4.6) and (5.3), respectively. The non-arbitrage value of an insurance contract is defined according to (3.7) again. Note that the discounting is done with respect to the risk-free interest rate r_{rf}, i.e. the dependence on the investment return is internalized.

Evaluating (3.7) with the equivalent martingale measure (5.4) will account for investment opportunities. Specifically, it will yield the non-arbitrage value of the combination of insurance and investment operation and, by this, the market price of the overall risk. Thus, the non-arbitrage pricing principle (3.7) together with the equivalent martingale measure (5.4) provides a unified valuation of assets and liabilities.

The equivalent martingale measure (5.4) will also reflect diversification effects present due to the non (or only partially) correlated nature of the returns R and Y. To see this, consider that (5.4) depends on the correlation of R and Y. How the correlation affects the non-arbitrage premium, however, is not easy to guess since it depends on the parameters γ_1 and γ_2 that depend on the correlation themselves. What can be inferred from (5.4) is that for negative, fixed $\hat{\gamma}_1$ and $\hat{\gamma}_2$ a positive correlation between R and Y results in relatively more weight being put on low returns. Vice versa, no correlation or negative correlation will decrease the relative weight on low returns. This feature of (5.4) can be interpreted in terms of a portfolio effect. We will come back to these issues in Section 7.

To end this Section, note that relation (5.4) can be interpreted as a generalization of the Esscher transformation (4.10). The transformed measure Q in (5.3) depends not on a single measure P but on two measures P_1 and P_2 related to insurance and investment returns. Relation (5.4) can easily be extended to cover more complex situation with additional investment (or insurance) opportunities (provided that the number of states of the world is bigger than the number of incorporated risk factors plus two).

6 Unique Valuation and Implied Discounting

In comparison to real probabilities P, the equivalent martingale probabilities Q will put more weight on low returns. This connects in a natural way to the insurance related concept of the safety loading. In Delbaen and Haezendonck (1989) it was shown that common premium and loading principles can be recovered by suitably choosing martingale probabilities. The case of a compound Poisson process was considered. The situation here is different in two respects: First, in addition to the insurance return, the stochastic investment return is considered. Second, we will be interested in the discounting factor that is implied by the overall non-arbitrage value of the contract and a particular loading. The motivation for this stems from the fact that according to standard finance, the non-arbitrage value coincidences with the (unique) market price. While this interpretation relies on liquidity, effectiveness and completeness of the market that don't fully apply in the current case, the implied discounting rate may nevertheless shed some light on how the discounting factor in (1.1) has to be chosen.

The implied discounting factor is obtained by solving the equation

$$\frac{1}{1+r_i}\mathrm{E}^P[f(L)] + S[f(L)] = \frac{1}{1+r_{rf}}\mathrm{E}^Q[f(L)] \tag{6.1}$$

for $(1+r_i)$ where $f(L)$ stands for the loss occurred by an insurance structure. It reconnects the actuarial and financial view. By definition, it is consistent with the chosen loading principle and the market value of the overall risk of insurance and investment operation as implied by non-arbitrage pricing.

Relation (6.1) defines the implied discounting factor $(1+r_i)$ in a unique way. One has to keep in mind however that the right hand side of equation (6.1) depends on the martingale measure Q, which is, due to the incompleteness of the insurance market, not uniquely defined. As well, Q is linked by construction to the reference premium b. Thus, the implied discounting factor $(1+r_i)$ is relative to the reference premium b as defined 'ground-up' by (1.1)

$$B = \frac{1}{1+r_i}E^P[X] + S[X]$$

Other definitions for b can be considered. And indeed, in a real life situation likely a reference premium would be considered that is referring to $E^P[f'(L)]$ and $S(f'(L))$, or simply to a premium of an already placed contract.

7 An Example: Simple Reinsurance Structures

An illustrative example of non-arbitrage pricing with methods discussed here is reinsurance. Sonderman (1991) has pioneered pricing of reinsurance contracts with martingales. Premiums are calculated under the assumption of an arbitrage-free reinsurance market and in the context of the continuous Lundberg model. Considering discounted cash flows based on stochastic interest rates links the insurance and finance side. However, there is no explicit non-arbitrage condition for investments in financial assets. By these assumptions, the model of Sondermann (1991) differs in two respects from the approach followed here. First, we developed a unified view of insurance and investment operations. There is no discounting of cash flows that creates the somehow artificial bridge between insurance and finance. Investments in financial and insurance-related risky assets are considered on an equal footage. Second, a discrete setting is investigated which reflects the reality of basic reinsurance policies that are renewable once a year. The quarterly traded catastrophe insurance futures traded at the Chicago Board of Trade give a comparable real-life example.

Two simple reinsurance structures will be considered here, one of which is related to compound claim amounts X (stop-loss insurance) and the other one to single claim amounts L_i (excess of loss insurance). Both refer to the same underlying asset b with stochastic one-period returns defined by (3.2).

7.1 Stop-Loss Reinsurance

In the case of stop-loss reinsurance, an insurer cedes the risk of being exposed to compound losses X surpassing some threshold d to a reinsurer. For simplicity, it is again assumed that no payments are made in the time period $[0,T]$. What is the non-arbitrage value of reinsurance in this situation and its relation to the premium b? The incurred loss, which at time T is ceded to the reinsurer, is

$$f(L,N) = f(X) = \max(X - d,0) \tag{7.1}$$

where d is the deductible. Note that (7.1) is basically the payoff of a call option. On the other hand, the insurer incurs the loss

$$\min(X, d) \tag{7.2}$$

Taking the expectation value of (7.1) with respect to the equivalent martingale measure Q and discounting with the risk-free rate leads to the non-arbitrage reinsurance premium

$$b_{reins} = \frac{\mathrm{E}^{Q}[\max(X - d, 0)]}{1 + r_{rf}} \tag{7.3}$$

Because of the relation

$$\max(X - d, 0) = X - \min(X, d) = X - (d - \max(d - X, 0)) \tag{7.4}$$

the reinsurance premium b_{reins} can also be expressed as

$$b_{reins} = \frac{(b - d)}{1 + r_{rf}} + \frac{\mathrm{E}^{Q}[\max(d - X, 0)]}{1 + r_{rf}} \tag{7.5}$$

The first term in (7.5) is the discounted value of a cash amount $(b - d)$ while the second term corresponds to the value of a 'put option on insured losses' with strike d. Thus, the value of reinsurance for the insurer equals to holding (or lending) the cash amount $(b - d)$ at the risk-free rate r_{rf} and selling a put option with strike d. In this context it is important to keep in mind that by the equivalent martingale measure Q expression (7.3) (or (7.5)) for the reinsurance premium b_{reins} implicitly takes into account the investment return Y which means that the equivalent martingale measure Q depends on both, the insurance return related to X and the investment return Y.

Relation (7.5) shows how the insurance premium b and reinsurance premium b_{reins} are interconnected when no arbitrage is present. Other relations can be deduced. An illustrative reminiscence to the well known 'call–put parity' in finance following from (7.4) is

$$b + \mathrm{E}^{\mathrm{Q}}[\max(d - X, 0)] = d + \mathrm{E}^{\mathrm{Q}}[\max(X - d, 0)] \tag{7.6}$$

With respect to the premium b and the deductible d, equation (6.6) interconnects the value of put and call options with equal strike (i.e. deductible) on compound losses X. Similar relations for more complicated reinsurance structures can be deduced.

7.2 Excess-of-Loss Reinsurance

Up to now, only a compound claim amount X has been considered. In the most common cases of insurance and reinsurance, however, the amount to be paid does depend on single claim amounts. As an example consider excess-of-loss reinsurance where the amount for the ith claim to be paid by the reinsurer is

$$\max(L_i - d, 0) \tag{7.7}$$

The total amount paid in the period $[0,T]$ is obtained by summing over i

$$f(L,N) = \sum_{i=1}^{N} \max(L_i - d, 0) \tag{7.8}$$

Here, N is the (random) number of claims occurring in the period $[0,T]$. Interpreting (7.8) as a payoff function $f(L,N)$ of the underlying asset b, the non-arbitrage premium is calculated by taking the expectation value of (7.9) with respect to the equivalent martingale measure Q

$$b_{reins} = \frac{\mathrm{E}^{\mathrm{Q}}\left[\sum_{i=1}^{N} \max(L_i - d, 0)\right]}{1 + r_{rf}} \tag{7.9}$$

The expectation value $E^Q[\cdot]$ in expression (7.9) can be written (see e.g. Daykin, Pentikäienen and Pensonen (1994)) as

$$b_{reins} = \frac{E^Q[N] \cdot E^Q[\max(L_i - d, 0)]}{1 + r_{rf}} \tag{7.10}$$

and, in analogy to Section 7.1, an equivalent expression

$$b_{reins} = \frac{\left(b - E^Q[N] \cdot d\right)}{1 + r_{rf}} + \frac{E^Q[N] \cdot E^Q[\max(d - L, 0)]}{1 + r_{rf}} \tag{7.11}$$

can be obtained. In the same manner, relation (7.6) now becomes

$$E^Q[\max(L - d, 0)] = \left(\frac{b}{E^Q[N]} - d\right) + E^Q[\max(d - L, 0)] \tag{7.12}$$

The interpretation is similar to the one in the case of compound losses. With respect to the premium b, the deductible d and the expected value of number of claims evaluated under the equivalent martingale measure Q, equation (7.12) interconnects the value of put and call options with equal strike (i.e. deductible) on individual losses L_i.

7.3 Numerical Examples

For illustrative purposes, this section presents numerical results for the reinsurance examples of Section 7.1 and 7.2. Arbitrage-free reinsurance premiums are calculated as a function of the deductible d. Particular attention is paid to the role of investment possibilities.

We assume that the investment return Y over the period T is a normally distributed with mean μ_I and standard deviation σ_I. The dependence of non-arbitrage premiums on investment opportunities is investigated by comparing non-arbitrage premiums for different parameter values μ_I and σ_I. The numerical examples are based on loss amounts generated from Poisson distributed claim numbers N and lognormal distributed claim amounts L. The corresponding parameter values are given in Table 1, together with the characterization of the investment return Y. Investment returns Y for different parameter values μ_I and σ_I have been obtained by scaling and shifting. The numerical results are based on 50 scenarios (i.e. states of the world) that correspond to about 200 single losses[5]. These numbers are sufficient to sample a large part of the compound loss distribution X.

The compound loss distribution is shown in Figure 1 together with an example of equivalent martingale probabilities. The equivalent martingale probabilities are shown for parameter values μ_I =6% and σ_I = 5%. Figure 2 and Figure 3 present non-arbitrage premiums for stop-loss reinsurance as a function of the deductible d and various combinations of μ_I and σ_I. Corresponding non-arbitrage premiums for excess-of-loss reinsurance are shown in Figure 4 and Figure 5, again as a function of the deductible d. This time however the deductible d refers to single claim amounts. The Figures show generic features: By construction of the equivalent martingale probabilities, all non-arbitrage premiums coincide for d =0. This reflects the fact that as discussed in Section 6 non-arbitrage premiums are defined relative to a given premium b. It should be noted that the particular choice d =0 is arbitrary. In principle, corresponding results for any reference premium b' can be derived. For $d \neq 0$ the non-arbitrage premiums reflect the investment characteristics: For a fixed volatility σ_I the higher the average investment return μ_I is, the higher is the non-arbitrage premium, i.e. the arbitrage-free value of the contract. For a fixed average return μ_I and varying volatility σ_I, a reversed situation is encountered: The higher the volatility σ_I, the lower is the non-arbitrage premium. These relations are the same

[5] The investment return time-series and the loss scenarios correspond to the typical data that is needed in real life examples. Historical data can be used in a straightforward way.

independently of the nature of the reinsurance contract (i.e. excess-of-loss or stop-loss) considered here.

Figure 2 to Figure 5 in addition show the expected loss $E^P[f(X)]$ as a function of the deductible d and sample insurance premiums calculated from the variance premium principle

$$\text{premium} = \frac{1}{1+r_{rf}} E^P[f(X)] + S[f(X)] = \frac{1}{1+r_{rf}} E^P[f(X)] + \alpha \cdot Var[f(X)]$$

with α =0.015. By equating the insurance premium with the non-arbitrage premium, the implied discounting factor $(1+r_i)$ can be deduced. Implied discount factors corresponding to Figure 5 are plotted in Figure 6. The arbitrage-free premiums (and the related implied discounting factor $(1+r_i)$) have to be understood as originating from the combined insurance and investment operations. This explains the somewhat counterintuitive fact the higher investment returns correspond to lower implied discounting rates. In the present one-period case, these implied discount factors have little meaning besides reconnecting the actuarial and financial premium calculation.

The example considered here is one in which the insurance premium is in general higher than the non-arbitrage premium. Obviously, depending on the loading $S[f(X)]$ and other parameters, the situation could be reversed. In such a situation, the question of how investment related value could be transferred back to the insured by lowering the premium becomes important. A discussion of these issues lies beyond the scope of this paper. However we note that a consistent way of valuing the passing of investment generated value to the primary insurer should be based on the equivalent martingale measure Q, i.e. on an arbitrage-free valuation.

Finally, Figure 7 demonstrates the presence of a portfolio effect. It shows excess-of-loss non-arbitrage premiums for different correlations between insurance and investment return. The different cases have been obtained by considering insurance and investment returns with rank correlation 1, approximately, 0 and –1, respectively. These cases thus represent the extremes. The graphs show a pronounced and expected dependence on the correlation between insurance and investment return.

8 Multi-Period Contracts

The one-period setting considered here is easily generalized to the multi-period case. Basically, the constraints (3.6) and (5.3) have to capture multi-period payoffs. As well, the discounting with respect to the risk-free interest rate r_{rf} should account for multiple periods. In the simplest case where no term structure is present, constraint (3.6) e.g. becomes

$$E^{Q}\left[\sum_{t}\frac{1+R_t}{(1+r_{rf})^t}\right] = 1$$

Considering multiple periods will allow for the valuation of long-term contracts, which often come with complex option-like structures (e.g. commutation features, guarantees). Recently, with the pressure from under-performing financial markets, issues related to the valuation of options became especially important in life insurance.

9 Conclusions

The relevance of arbitrage-free pricing relies on the liquidity and efficiency of the insurance market. Liquidity and efficiency are not features of the insurance market in general and thus the non-arbitrage approach to pricing of insurance contracts may be inadequate. The situation is different when insurance risk is traded or transferred to the financial markets and vice versa. In this case, the implicit assumptions underlying the non-arbitrage approach may be satisfied in that a (at least partial) hedge can be set up. Even for traditional insurance structures the financial risk due to investments can become comparable to the insurance risk itself (e.g. life insurance). In these cases the price for the overall risk should reflect in some way the price financial markets are putting on risk. In this paper we tried to tackle this problem by extending the non-arbitrage principle to the insurance side. Due to incompleteness of the insurance market, no unique equivalent martingale measure exists. The information theoretical maximum entropy principle is applied to make a sensible choice in this situation.

Equivalent martingale probabilities consistent with the combined non-arbitrage conditions for insurance and investment operations for a one-period horizon are calculated. They turn out to be linked to a generalized form of the Esscher measure and show correlation dependence. The latter illustrates the presence of a portfolio effect. These findings are illustrated by a numerical example referring to common reinsurance-like structures.

In summary, a practicable and well-motivated way to infer overall market prices for combined insurance and investment operations based on the non-arbitrage pricing principle has been outlined. While the relevance of this approach may be limited for traditional insurance structures with only limited exposure to financial risk, the market price is clearly of relevance for insurance structures whose performances depend heavily on the performance of financial markets.

Acknowledgements

I thank Peter Blum (Converium), Victor Choi and Sergei Esipov (Centre Solutions) for valuable comments and stimulating discussions. Part of this work has been performed at Centre Solutions. The views expressed here are those of the author and not Converiums' or Centre Solutions'. All remaining errors are the authors' responsibility.

References

Black, F. and M. Scholes (1973), The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, Vol. 81, pp. 637-659.

Buchen, P., and M. Kelly (1996), The maximum entropy distribution of an asset inferred from option prices, Journal of Financial and Quantitative Analysis, Vol. 31, pp. 143 – 159.

Bühlmann, H. (1980), An Economic Premium Principle, *ASTIN Bulletin*, Vol. 11, 1980, pp. 52-60.

Copeland, T. E. and Weston, J. F. (1992), *Financial Theory and Corporate Policy*, Addison Wesley.

Cover, T. M. and Thomas, J. A. (1991), *Elements of Information Theory*, John Wiley.

Daykin, C.D., Pentikainen, T. and Pesonen, M. (1994), *Practical Risk Theory for Actuaries.*, Chapman & Hall: London, New York.

Delbaen, F. and Haezendonck, J. M., (1989), A martingale approach to premium calculation principles in an arbitrage-free market, *Insurance: Mathematics and Economics,* Vol. 8, 269-277.

Gerber, H. U. and Shiu, S. W. (1994), Martingale Approach to Pricing Perpetual American Options, *ASTIN Bulletin*, Vol. 24, 1994, pp. 195-220.

Gerber, H. U. and Shiu, S. W. (1994), Option Pricing by Esscher Transforms, *TSA*, Vol. 16, pp. 99-140.

Golan, A., G. G. Judge, and D. Miller (1997), The Maximum Entropy Approach To Estimation and Inference: An Overview, *Advances in Econometrics*, Vol. 12, pp. 3-24.

Gulko, L. (1997), Dart Board and Asset Prices: Introducing the Entropy Pricing Theory, *Advances in Econometrics*, Vol. 12, pp. 237-276.

Harrison, J. M., and Kreps, D. M., 1979, Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory*, Vol. 20, 381-408.

Hull, J.C. (2000), *Options, Futures and other Derivatives*, Prentice Hall.

Jaynes, E. T. (1957a), Information Theory and Statistical Mechanics, *Physics Review*, Vol. 106, pp. 620-630.

Jaynes, E. T. (1957b), Information Theory and Statistical Mechanics II, *Physics Review*, Vol. 108, pp. 171-190.

Jarvis S., Southall F. and Varnell E. (2001), Modern Valuation Techniques, available at www.sias.org.uk/papers/mvt.pdf

Neftci, S. (2000), *Mathematics of Financial Derivatives*, Academic Press.

Schmidli, H., 1996, Martingales and insurance risk. In: Obretenov A. (ed.) *Lecture notes of the 8-th international summer school on probability and mathematical statistics* (Varna). Science Culture Technology Publishing, Singapore, 155-188.

Shannon, C. E., 1948, A Mathematical Theory of Information, *The Bell System Technical Journal*, 27, 379-423 and 623-656.

Stutzer, M. (1996), A Simple Nonparametric Approach to Derivative Security Valuation, *Journal of Finance*, Vol. 51, pp. 1633-1652.

Sondermann D., 1991, Reinsurance in arbitrage-free markets. *Insurance: Mathematics and Economics*, Vol. 10, 191-202.

Rubinstein, M. (1994), Implied Binominal Trees, *Journal of Finance*, Vol. 49, pp. 771-818.

	Loss Number	Loss Severity	Invest. Return	Risk-free Rate
Type	Poisson	LogNormal	Normal	Constant
μ	4.0	1.0	4%, 5%, 6%	3%
σ	-	8.0	3%,4%,5%,7%	

Table 1 Distribution types and parameter values referring to numerical examples discussed in Section 7.

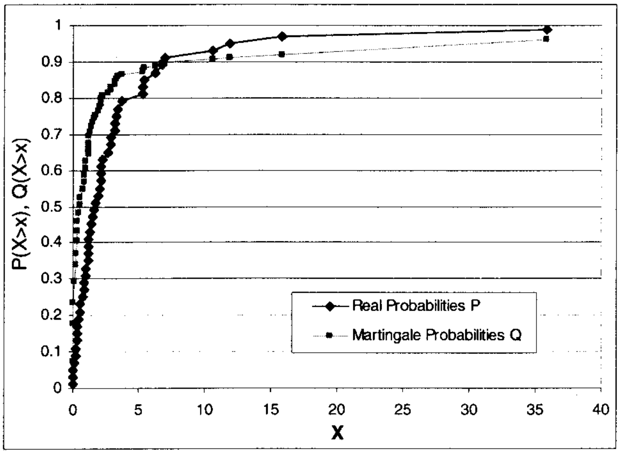

Figure 1 Real probabilities P and corresponding equivalent martingale probabilities Q referring to an investment return μ_I =6% with volatility σ_I =5% (see text).

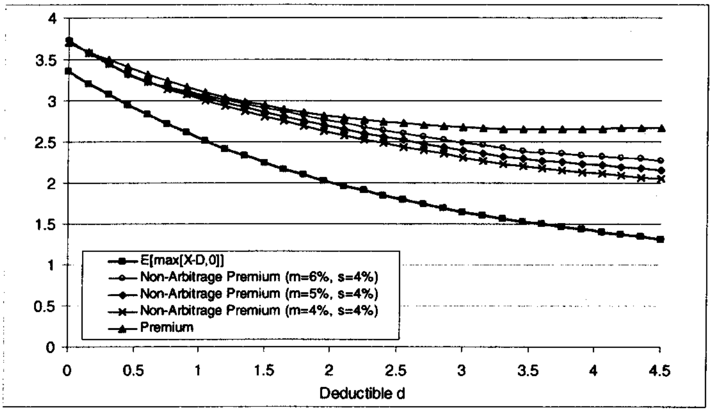

Figure 2 Stop-loss structure. Expected loss, premium and non-arbitrage premiums for different investment opportunities as a function of the deductible *d*.

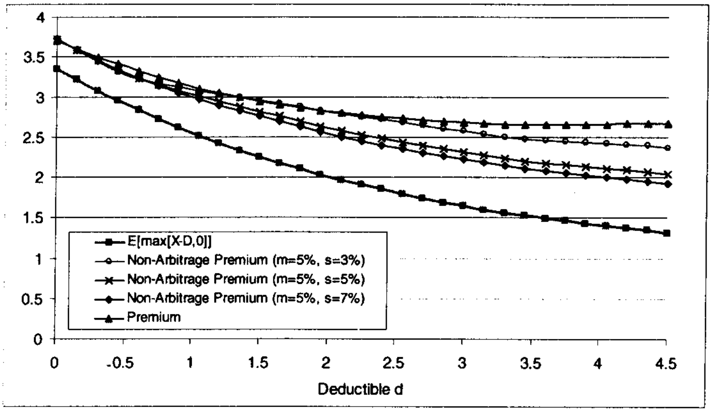

Figure 3 Stop-loss structure. Expected loss, premium and non-arbitrage premiums for different investment opportunities as a function of the deductible *d*.

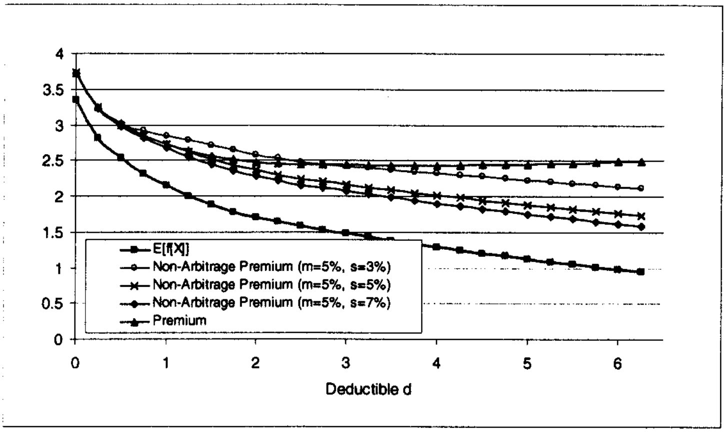

Figure 4 Excess-of-loss structure. Expected loss, premium and non-arbitrage premiums for different investment opportunities as a function of the deductible *d*.

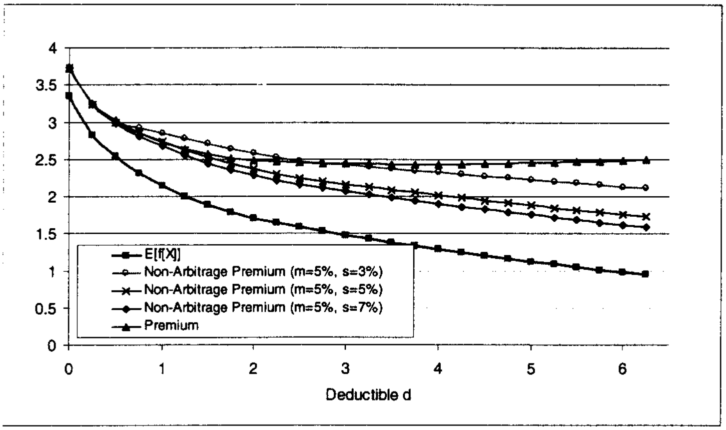

Figure 5 Excess-of-loss structure. Expected loss, premium and non-arbitrage premiums for different investment opportunities as a function of the deductible *d*.

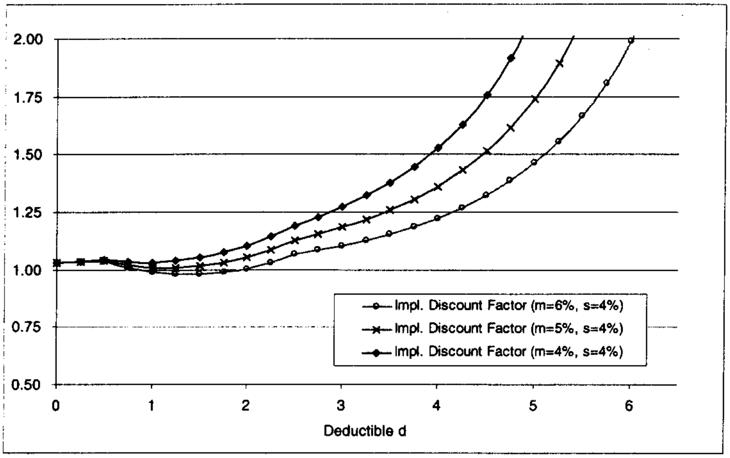

Figure 6 Implied discount factors (corresponding to Figure 5) as a function of the deductible *d*.

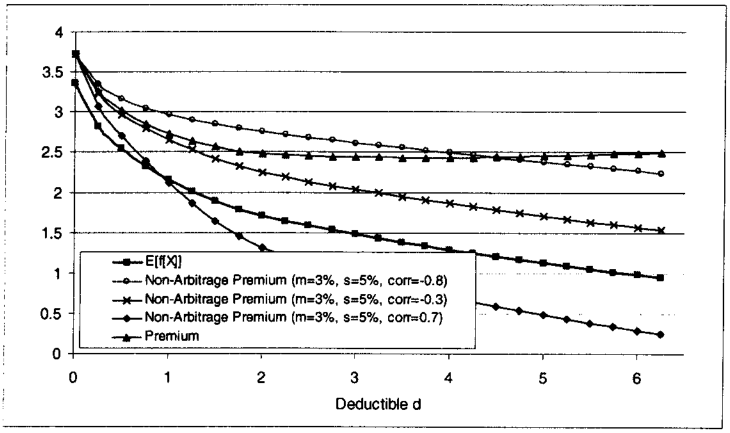

Figure 7 Effect of correlation between insurance and investment returns on non-arbitrage premiums for the excess-of loss structure.