Surplus Allocation for the Internal Rate of Return Model: Resolving the Unresolved Issue

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Abstract

In this paper, it is shown that with a certain definition of risk-based discounted loss reserves and a certain method of surplus allocation, there is an amount of premium for a contract which has the following properties:

(1.) It is the amount of premium required for the contract to neither help nor hurt the insurer's risk-return relation.
(2.) It produces an internal rate of return equal to the insurer's target return.

If the insurer gets more than this amount of premium, then the insurer can get more return with the same risk by increasing the percentage of the premium for the overall book which is in the segment. Conversely, if the insurer gets less than this amount of premium, the insurer can increase its return by decreasing the percentage of the overall premium which is in the segment. The amount of premium is equal to the risk-based premium in "Pricing to Optimize an Insurer's Risk-Return Relation," (PCAS 1996).

The above property 1 of risk-based premium is proven by Theorem 2 of the 1996 PCAS paper and not by the present paper. The present paper proves property 2.

1. INTRODUCTION

The problem of relating pricing to the risk-return relation has been discussed in many recent actuarial papers. Surplus allocation is not described in these papers as something that can be done in a theoretically justifiable way. Actually, though, surplus allocation has been used in a way that has been proven by a theorem (Theorem 2 of [1]) to derive the amount of premium for any contract which will neither improve nor worsen the insurer's risk-return relation. Certain estimates have to be used of course, e.g. covariances and expected losses. The precise mathematical relationship between this premium and the risk-return relation is specified by the statement of Theorem 2. This theorem, and the corresponding definition of risk-based premium, are very rarely mentioned in recent papers relating pricing to the risk-return relation.

Since the internal rate of return (IRR) model has been a part of the CAS exam syllabus for years, and since it is a widely used method in insurance and other industries, it may be possible to explain the method of [1] to a larger group of readers by relating it to the IRR method.

The IRR model can be used to measure the rate of return for an insurance contract or a segment of business, but only if the method used for allocating surplus can be related to the insurer's risk-return relation. The model is generally presented without a theoretically justifiable method of allocating surplus. But if an arbitrary method of allocation (such as allocating in proportion to expected losses) is used, the results are almost meaningless. The purpose of this paper is to complete the IRR method.

Incidentally, there are several actuarial papers which argue that surplus allocation doesn't make sense because risk is not additive, or because in the real world all of surplus is available to support all risks. Actually, just as a function f(x) associates each number x with another number, surplus allocation is a mathematical function which associates each member of a set of risks with a portion of surplus. This function can be used as a part of a chain of reasoning in order to prove a theorem, as was done in Theorem 2 of [1].

Although surplus allocation was used in deriving the properties of risk-based premium in [1], the derivation doesn't actually require any mention of surplus allocation. When risk-based premium is related to the IRR method in section 4 of this paper, surplus allocation is used because that is the traditional way of explaining the IRR method.

The approach presented here:

(1.) addresses the risk-return relation in a fundamental way

(2.) addresses the problems of the time value of money and the discounting of losses

(3.) addresses the problem of loss reserve risk

(4.) assigns a risk-based premium to the sum of two contracts or segments which equals the sum of the individual risk-based premiums

The premium derived in this paper by the IRR method is the same as that determined by the method in [1] The method in [1] is simpler to apply, but the IRR model has the advantage of being widely used and understood. It is on the CAS syllabus and has also been used by non-actuaries for many years. In order to relate the method in [1] to the IRR method, explanations will be given of both methods. However, since both methods are explained at length in the literature (see [2],[3],[4]), the explanations will be brief and informal. This could actually be an advantage, since it could make the presentation more lively and readable. The part of the paper which is new is the derivation of the equivalence of the two methods, given certain assumptions and conditions.

2. THE IRR MODEL

The IRR model is a method of estimating the rate of return from the point of view of the suppliers of surplus. Suppose for example that an investor supplied \$100 million to establish a new insurer, and that \$200 million in premium was written the next day. Also suppose that twenty years later the insurer was sold for \$800 million. Ignoring taxes, the return r to the investor satisfies the equation \$100 million $(1+r)^{20}$ = \$800 million. Therefore, the return r equals 10.96%.

Suppose that, beginning at the time of the above initial investment, each dollar of surplus is thought of as being assigned to either an insurance policy currently in effect, a loss reserve liability, or some other risk. Suppose that a cash flow consisting of premium, losses, expenses, outflows of surplus, and inflows of surplus, is assigned to each policy in such a way that the following is satisfied: the total of all the cash flows minus the outstanding liabilities immediately prior to the time at which the above insurer is sold for \$800 million produces an \$800 million surplus. It is then possible to express the input of \$100 million, and the payback of \$800 million twenty years later, as the total result of the individual cash flows assigned to each policy. Based on the individual cash flows, the overall return of 10.96% could then be expressed as a weighted average of individual returns for each policy. The individual return for a policy is called its internal rate of return. The following example is taken from [2].

An Equity Flow Illustration

A simplified illustration of an insurance internal rate of return model should clarify the relationships between premium, loss, investment, and equity flows. There are no taxes or expenses in this heuristic example. Actual Internal Rate of Return models, of course, must realistically mirror all cash flows.

Suppose an insurer

- collects $1,000 of premium on January 1, 1989,
- pays two claims of $500 each on January 1, 1990, and January 1, 1991,
- wants a 2:1 ratio of undiscounted reserves to surplus, and
- earns 10% on its financial investments.

The internal rate of return analysis models the cash flows to and from investors. The cash transactions among the insurer, its policyholders, claimants, financial markets, and taxing authorities are relevant only in so far as they affect the cash flows to and from investors.

Reviewing each of these transactions should clarify the equity flows. On January 1, 1989, the insurer collects $1,000 in premium and sets up a $1,000 reserve, first as an unearned premium reserve and then as a loss reserve. Since the insurer desires a 2:1 reserves to surplus ratio, equity holders must supply $500 of surplus. The combined $1,500 is invested in the capital markets (e.g., stocks or bonds).

At 10% per annum interest, the $1,500 in financial assets earns $150 during 1989, for a total of $1,650 on December 31, 1989. On January 1, 1990, the insurer pays $500 in losses, reducing the loss reserve from $1,000 to $500, so the required surplus is now $250.

The $500 paid loss reduces the assets from $1,650 to $1,150. Assets of $500 must be kept for the second anticipated loss payment, and $250 must be held as surplus. This leaves $400 that can be returned to the equity holders. Similar analysis leads to the $325 cash flow to the equity holders on January 1, 1991.

Thus, the investors supplied \$500 on 1/1/89, and received \$400 on 1/1/90 and \$325 on 1/1/91. Solving the following equation for v

$$\$500 = (\$400)(v) + (\$325)(v^2)$$

yields v = 0.769, or r = 30%. (V is the discount factor and r is the annual interest rate, so v = 1/(1+r).)

The internal rate of return to investors is 30%. If the cost of equity capital is less than 30%, the insurer has a financial incentive to write the policy.

This concludes Feldblum's example. My only attempt to improve this simplified illustration is the following. The \$1,000 reserve set up on January 1, 1989 is an unearned premium reserve and by the end of 1989 there is a \$1,000 loss reserve. In between, the sum of the unearned premium reserve and the loss reserve is always \$1,000.

Feldblum doesn't claim that the method of surplus allocation in the illustration can be directly related to the risk-return relation. Allocating in proportion to expected losses doesn't distinguish between the riskiness of unearned premium, loss reserves, property risks, casualty risks, catastrophe covers, excess layers, and ground-up layers, for example. Different methods of surplus allocation could be judgmentally applied to different types of contracts, but from a theoretical risk-return perspective a certain use of covariance is required. This will be explained in the next section.

3. RISK-BASED PREMIUM

What follows is an informal explanation of the derivation in [1] of the properties of risk-based premium. In the discussions below of an insurer's risk-return relation over a one year time period, "return" refers to the increase in surplus, using the definition of surplus below. (The term "risk-based discounted" is used in the definition and will be explained later.)

surplus = market value of assets - risk-based discounted loss and loss adjustment reserves – market value of other liabilities (3.1)

At any given time, the return in the coming year is a random variable. The variance of this random variable is what we refer to by the term "risk". The expression "optimizing the risk-return relation" is used in the same way that Markowitz [5] used it, i.e., maximizing return with a given risk or minimizing risk with a given return. (Markowitz was awarded the Nobel Prize several years ago for his work on optimizing the risk-return relation of asset portfolios.)

For an insurance contract, or for a segment of business, the risk-based premium can be expressed as follows: (The term "loss" will be used for "loss and loss adjustment expense.")

risk-based premium = expense provision + risk-based discounted losses + risk-based profit margin. (3.2)

The above expense provision is equal to expected expenses discounted at a risk-free rate. The starting time T for discounting recognizes the delay in premium collection. Expenses are considered to be predictable enough so that the risk-free rate is appropriate. The risk-based discounted loss provision is equal to the sum of the discounted values, using a risk-free rate and the above time T, of

(a.) the expected loss payout during the year

(b.) the expected discounted loss reserve at year-end, discounted as of year-end at a "risk-based" (not "risk-free") discount rate

A risk-free rate is used to discount (a) and (b) above because the risk arising from the fact that (a) and (b) may differ from the actual results is theoretically correctly compensated by the risk-based profit margin (see (3.2) above).

The phrase "contract or segment of business" will be replaced below by "contract", since the covariance method used below has the following property: the risk-based premium for a segment equals the sum of the risk-based premiums of the contracts in the segment. At the inception of an insurance policy, the payout of losses during the year that the contract is effective, and the estimated risk-based discounted loss reserve for the contract at the end of the year, are unknown. The effect of the contract on surplus at the end of the year, i.e., the difference in end of year surplus with and without the contract, can be thought of as a random variable X at inception. The insurer's return, i.e., the increase in surplus during the year, is also a random variable. Call it Y.

Assuming that the contract premium equals the risk-based premium, the expected effect of the contract on surplus at the end of the year is equal to the accumulated value, at the risk-free interest rate, of the risk-based profit margin. This is true because the expense provision portion of the formula (3.2) above pays the expenses, and the risk-based discounted losses portion pays the losses during the year and also accumulates at risk-free interest, to the expected value of the risk-based discounted loss reserves at the end of the year. Therefore, by formulas (3.1) and (3.2), above, the effect of the contract equals the accumulated value of the risk-based profit margin.

The random variables X and Y were defined above. If

$$\mathrm{Cov}(X,Y)/\mathrm{Var}(Y)=E(X)/E(Y) \tag{3.3}$$

then, according to Theorem 2 of [1], the contract neither improves nor worsens the risk-return relation, in a certain sense. This was defined above in the abstract. Note that E(X), above, equals the accumulated value of the risk-based profit margin.

One of the components of risk-based premium is the expected value of the risk-based discounted loss reserves at the end of the year. This expected value is greater than the expected value of the loss reserves discounted at the risk-free rate corresponding to the duration of the loss reserves. This is how the risk-based premium provides a reward for the risk of loss reserve variability. The risk-based discount rate is therefore less than the risk-free rate.

At the end of each year following the effective period of a contract, if the matching assets for the risk-based discounted loss reserves are invested at the risk-free rate, their expected value at the end of the following year will be greater than the expected discounted liability. This is because the risk-based discount rate is less than the risk-free rate. Assume, for example, that the loss payout is exactly equal to the expected loss payout. At the moments that loss payments are made, both the discounted loss reserve and the matching assets are reduced by the same amount. At other times, the matching assets are growing at the risk-free rate and the discounted liability is growing at the lower risk-based discount rate.

At the beginning of the second year after the inception of the policy, the end of the year matching assets minus the discounted loss reserve can be thought of as a random variable Z. If Cov(Y,Z)/Var(Y) is equal to E(Z)/E(Y), then, according to Theorem 2 of [1], the risk-based discounted loss reserve and matching assets neither improves nor worsens the risk-return relation for the year. It is possible to compute a discount rate before the inception of the contract such that Cov(Y,Z)/Var(Y) is equal to E(Z)/E(Y). Note that if the matching assets are not risk-free, that affects both Cov(Y,Z) and E(Z) and may have a slight effect on the risk-based discount rate. If risk-based discount rates are computed for each of the years until the loss reserve is expected to be fully paid, the risk-based premium for the contract is determined.

For practical purposes, the above derivation of risk-based discounting of losses for a contract can be simplified if certain estimates are used. For example a single risk-based discount rate can be used for all future years. This approach was used in [1]. Since risk-based premium is determined by the estimated expense payout, loss payout, risk-based profit margin, and risk-based discount rate, the explanation of risk-based premium has now been concluded. The following two examples are taken from [1].

Catastrophe Cover Risk Load

In this example, in order to estimate the value of a catastrophe cover to a ceding company, we will suppose that the ceding company re-assumes the cover, and we will estimate the required risk-based profit margin.

Assume that:

1. The probability of zero losses to the catastrophe cover is .96, and the probability that the losses will be $25 million is .04. Therefore, the variance of the losses is 24 trillion, and the expected losses are $1 million.
2. Property premium earned for the year is $100 million, and there is no casualty premium.
3. The standard deviation of pre-tax underwriting return is 15 million.
4. The expected pre-tax return from the entire underwriting portfolio is $8 million.
5. Taxes have the same proportional effect on the expected pre-tax returns on total premium and on the catastrophe cover, and on the standard deviations of returns.
6. The covariance between the catastrophe cover's losses and total property losses net of the cover is equal to .50 times the variance of the cover's losses.
7. The discount rate for losses is zero.

8. Total underwriting return, and the return on the catastrophe cover, are statistically independent of non-underwriting sources of surplus variability.

It follows from 1, 6 and 8 above, and from the fact that Cov(X,Y+Z) = Cov(X,Y) + Cov(X,Z), that the covariance with surplus of the pre-tax return on the catastrophe cover is 24 trillion + .50(24 trillion); i.e., 36 trillion. It follows from 3 and from Cov(X,Y+Z) = Cov(X,Y)+Cov(X,Z) that the corresponding covariance for total underwriting is (15 million)^2 , i.e., 225 trillion. Therefore, it follows from assumption 4 that the risk-based profit margin for the catastrophe cover should be such that the pre-tax return from re-assuming the catastrophe cover is given by (36/225)($8 million)= $1.28 million. (This is greater than the cover's expected losses.) If the cover costs more than $2.28 million, then it improves the insurer's risk-return relation to re-assume it. However, the cover may be necessary to maintain the insurer's rating and policyholder comfort.

<u>Required Profit Margin by Layer</u>

Suppose that for some insurer:

1. All premium is property premium.
2. The accident year expected property losses for the $500,000 excess of $500,000 layer, and the 0-$500,000 layer, respectively, are $10 million and $90 million. Expected losses excess of $1 million are zero.
3. The accident year property losses for each of the above layers are independent of all non-underwriting sources of surplus variation.
4. The discount rate is zero.
5. The coefficients of variation (ratios of standard deviations to means) of the higher and lower layers are .30 and .15, respectively.
6. The correlation between the two layers is .5.
7. Taxes have the same proportional effect on the returns of both layers.

Let a and b denote the standard deviations of the losses to the higher and lower layers, respectively. Let ρ denote the correlation. With the above assumptions, the pre-tax covariances with surplus for the higher and lower layers, respectively, are given by

$$
\begin{aligned}
a^2 + \rho ab &= ((10 \text{ million})(.30))^2 \\
&\quad + (.5)(10 \text{ million})(.30)(90 \text{ million})(.15) \\
&= 29.25 \text{ trillion, and} \\
b^2 + \rho ab &= ((90 \text{ million})(.15))^2 \\
&\quad + (.5)(10 \text{ million})(.30)(90 \text{ million})(.15) \\
&= 202.5 \text{ trillion}
\end{aligned}
$$

The allocated surplus for 0-$500,000 layer is 202.5/29.25 (i.e.., 6.9) times as great as the allocated surplus for the $500,000 excess of $500,000 layer. The expected losses are nine times as great for the lower layer. Therefore, the required profit margin, as a percentage of expected losses, is 1.3 (i.e., ((9)(29.25))/202.5) times as great for the higher layer as it is for the lower layer. This is expected due to the higher layer's larger coefficient of variation.

4. RISK-BASED PREMIUM AND THE IRR MODEL

An example is given below to show the following. Suppose that the risk-based premium of my model, for a certain contract corresponds to a certain expected rate of return for the insurer. Then, the expected rate of return for the contract, using the IRR model, also equals that target rate if the method of allocating surplus for the IRR model is the covariance method of my model.

Suppose the target rate of return is 15%. The risk-based premium for a contract equals expense provision + risk-based profit margin + risk-based discounted losses. Suppose that premium and expenses are paid at the end of the year, and the expected loss payout is \$100 at the end of each year for four years. Suppose expenses are \$70 and the risk-based profit margin is \$30. Suppose that the risk-based discount rate is 4%. It then follows that the risk-based discounted losses at the end of the year equal \$377.51 and the risk-based premium equals \$70 + \$30 + \$377.51, or \$477.51.

By the words surplus and return, I will mean them as defined in my model. The portion of surplus allocated to the contract for the first year will be called S_1 and, using (3.3), it equals

(Surplus)((Covariance(Contract's After-Tax Underwriting Return, Surplus))/(Variance of Surplus))

It is possible to estimate the taxes corresponding to underwriting return at the end of the year of a contract, and the taxes corresponding to the return on risk-based discounted loss reserves and matching assets in the following years. The effect on taxes of premium earned, expenses incurred, investment income from premium, and losses paid during the year, as well as the effect of loss reserves discounted at the beginning and the end of the year, can be used. In the case of risk-based discounted loss reserves and matching assets, the expected taxes are less than taxes on the matching assets. This is because loss reserves are discounted from a point in time one year later at the end of the year than at the beginning of the year, producing a loss for tax purposes. If the discount rate used for tax purposes equals the risk-free rate, but the tax law payout rate is faster than the actual payout rate, the tax effect is the same as the effect of using the actual payout rate and a certain discount rate which is lower than the risk-free rate.

Assume for simplicity that the insurer's assets earn 6% for the period of the coming year and that the tax produced by each type of return is 35% of the return. Then, the above \$30 risk-based profit margin and the above allocated surplus S_1 satisfy the equation

$$.15S_1 = .65(.06S_1+30)$$

since the 15% target return on allocated surplus is produced by the remainder, after 35% tax, of 6% investment income on allocated assets plus the \$30 risk-based profit margin. Solving the equation gives S_1 = \$175.68.

The surplus allocated for the second year equals

(Surplus)((Covariance(Contract's After Tax Loss Reserve Return, Surplus))/ (Variance of Surplus))

This allocated surplus will be called S_2. The amount of surplus allocated the next two years are defined similarly and will be called S_3 and S_4.

The risk-based discounted loss reserve corresponding to S_2 equals \$277.51 and satisfies the equation

$$.15S_2 = .65(.06S_2 + (.06 - .04)\$277.51) = .65(.06S_2 + 5.55)$$

since the 15% return is equal to the after-tax return from investment income from the allocated assets plus the after-tax return on loss reserves and matching assets. The matching assets earn 6% and the discounted reserves increase at a rate of 4%, i.e., the risk-based discounted rate. Solving the equation gives S_2 = \$32.50.

Similarly, the risk-based discounted loss reserves corresponding to S_3 and S_4 are \$188.61 and \$96.15, respectively. Therefore,

$$.15S_3 = .65(.06S_3 + (.06 - .04)188.61) = .65(.06S_3 + 3.77)$$
$$.15S_4 = .65(.06S_4 + (.06 - .04)96.15) = .65(.06S_4 + 1.93)$$

So S_3 = \$22.09, and S_4 = \$11.26. It will now be shown that the return on the contract is 15% according to the IRR model, using the same allocation of surplus as above. It was shown above that

$$.15S_1 = .65(.06S_1 + 30) \quad (4.1)$$
$$.15S_2 = .65(.06S_2 + 5.55) \quad (4.2)$$
$$.15S_3 = .65(.06S_3 + 3.77) \quad (4.3)$$
$$.15S_4 = .65(.06S_4 + 1.93) \quad (4.4)$$

If S_1, S_2, S_3, and S_4 are added, respectively, to both sides of the four equations above, respectively, and each equation is divided on both sides by 1.15, we get

$$S_1 = (1/1.15)(S_1 + .65(.06S_1 + 30)) \quad (4.5)$$
$$S_2 = (1/1.15)(S_2 + .65(.06S_2 + 5.55)) \quad (4.6)$$
$$S_3 = (1/1.15)(S_3 + .65(.06S_3 + 3.77)) \quad (4.7)$$
$$S_4 = (1/1.15)(S_4 + .65(.06S_4 + 1.93)) \quad (4.8)$$

Therefore,

$$S_1 = (1/1.15)(S_1 - S_2 + .65(.06S_1 + 30)) + (1/1.15)S_2 \quad (4.9)$$
$$S_2 = (1/1.15)(S_2 - S_3 + .65(.06S_2 + 5.55)) + (1/1.15)S_3 \quad (4.10)$$
$$S_3 = (1/1.15)(S_3 - S_4 + .65(.06S_3 + 3.77)) + (1/1.15)S_4 \quad (4.11)$$

By substituting the expression which is equal to S_4 in equation (4.8) for S_4 in Equation (4.11), we get

$$S_3 = (1/1.15)(S_3 - S_4 + .65(.06S_3 + 3.77)) + (1/1.15)^2 (S_4 + .65(.06S_4 + 1.93))$$

By substituting the above expression for the term S_3 at the extreme right of Equation (4.10), we get

$$\begin{aligned} S_2 = {} & (1/1.15)(S_2 - S_3 + .65(.06S_2 + 5.55)) + \\ & (1/1.15)^2(S_3 - S_4 + .65(.06S_3 + 3.77)) + \\ & (1/1.15)^3 (S_4 + .65(.06S_4 + 1.93)) \end{aligned}$$

By substituting this expression for the term S_2 at the extreme right of Equation (4.9), we get

$$\begin{aligned} S_1 = {} & (1/1.15)(S_1 - S_2 + .65(.06S_1 + 30)) + \\ & (1/1.15)^2(S_2 - S_3 + .65(.06S_2 + 5.55)) + \\ & (1/1.15)^3(S_3 - S_4 + .65(.06S_3 + 3.77)) + \\ & (1/1.15)^4(S_4 + .65(.06S_4 + (1.93)) \end{aligned}$$

Therefore, S_1 is the discounted value, at a 15% return, of amounts at the end of years 1,2,3 and 4 which are each equal to the following: the sum of supporting surplus which is no longer needed at the end of the year plus the after-tax return during the year resulting from investment income from supporting surplus and from the contract. So, according to the IRR model, the rate of return on the contract is 15%. This completes the demonstration of the relationship between risk-based premium and the IRR model.

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