PRACTICAL LOSS RESERVING METHOD
WITH STOCHASTIC DEVELOPMENT FACTORS

BY MARY V. KELLY

BIOGRAPHY

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ABSTRACT

The paper presents a theoretical framework for measuring the inherent statistical variability of the loss development process. Chain ladder loss development factors are assumed to follow a LogNormal, Log Gamma or Log Inverse Gaussian distribution. From this, the conditional distribution of ultimate losses for each accident year (conditional on the amount reserved at the end of development year zero) is developed, and its parameters are estimated. By use of simulation, the distribution for the usual loss development triangle can also be calculated. Approximation formulae for the tail behaviour of the distribution of ultimate losses of the trapezium are presented.

In modelling the conditional distribution of ultimate losses, one can obtain an unbiased estimate of the mean of the total outstanding reserve, percentiles of the conditional distribution of reserves, which are required for solvency issues, and confidence intervals, which provide a measure of accuracy about the point estimate. The use of asymptotic formulae allows for statements about the probability of ruin without the need to simulate the distribution of losses.

INTRODUCTION

A paper by Hayne (1986) explored the properties of the lognormal distribution and its possible use as a measure of statistical variation inherent in loss development factor models. This paper extends Hayne's work in the calculation of the distribution of ultimate incurred claims using the traditional *chain ladder* method with stochastic loss development factor models. Two other distributions; the loggamma (Hogg and Klugman, 1984) and the log inverse Gaussian, possess characteristics similar to that of the lognormal. The development of the stochastic loss development factor model under these distributional assumptions is also considered.

This paper presents the framework of the stochastic loss development factor method, and the appropriate data requirements. The basic properties of the three distributions are presented. These results are required to calculate the distribution (and estimate its parameters) of ultimate claims and the sum of ultimate losses over all relevant accident years. Further conclusions concerning the tail behaviour of these distributions allow for the development of asymptotic formulae that estimate the upper percentiles of the distribution of total ultimate losses over all years. Finally, some numerical work using the stochastic loss development factor model is presented. One trapezium of data, a long tailed line presented by Zehnwirth (1989) and Sundt (1989) is presented.

Contrary to current actuarial beliefs, I believe that percentiles provide a better estimate of uncertainty concerning the loss reserving process than does the confidence interval about the mean expected ultimate loss. The confidence interval about the mean, which varies inversely with the square root of the number of observations, measures the precision about a point estimate. Given that the assumptions are correct, the precision will increase as more data are available. The amount of data available however does not affect the percentiles. Given that the process assumptions are valid the distribution characteristics of total claims are the major force driving insolvencies.

MODEL

Given claims data for a particular line (or other homogeneous grouping) of insurance, the run-off triangle, more correctly referred to as a trapezium can be represented as follows:

Acc't			Development Year				
Yr	0	1	2	3	...	$n-1$	n
1	x_{10}	x_{11}	x_{12}	x_{13}	$\cdots$	$x_{1,n-1}$	x_{1n}
2	x_{20}	x_{21}	x_{22}	x_{23}	$\cdots$	$x_{2,n-1}$	x_{2n}
$\vdots$							
$r-n+1$	$x_{r-n+1,0}$	$x_{r-n+1,1}$	$x_{r-n+1,2}$	$x_{r-n+1,3}$	$\cdots$	$x_{r-n+1,n-1}$	
$r-n+2$	$x_{r-n+2,0}$	$x_{r-n+2,1}$	$x_{r-n+2,2}$	$x_{r-n+2,3}$	$\cdots$		
$\vdots$							
r	x_{r0}						

The $x_{\{ij\}}$ in this model usually are aggregate loss dollars, either paid or incurred. It is assumed that development after n years is negligible. The data for the first $r-n$ years are fully developed losses, there is no expected change in the level reserved. Although the heading of the trapezium indicates that the data are collected by accident year, the models developed in this paper can be applied data recorded by other collection methods.

By dividing the elements of column k by the corresponding elements of column $k-1$ of the trapezium, it is possible to construct a trapezium of observed development factors, d_{ik} from the original trapezium of data.

Acc't.	Development between Years					
Yr	$0-1$	$1-2$	$2-3$	...	$n{-}2-n{-}1$	$n{-}1-n$
1	d_{11}	d_{12}	d_{13}	...	$d_{1,n-1}$	d_{1n}
2	d_{21}	d_{22}	d_{23}	...	$d_{2,n-1}$	d_{2n}
$\vdots$						
$r-n+1$	$d_{r-n+1,1}$	$d_{r-n+1,2}$	$d_{r-n+1,3}$	...	$d_{r-n+1,n-1}$	
$r-n+2$	$d_{r-n+2,1}$	$d_{r-n+2,2}$	$d_{r-n+2,3}$	...		
$\vdots$						
$r-1$	$d_{r-1,1}$					
r						

The stochastic loss development factor model assumes that the emergence of cumulative ultimate losses is represented by the following rectangle:

Acc't.	Development Year				
Yr	0	1	...	$n{-}1$	n
1	I_1	$I_1 D_{11}$	...	$I_1 \prod_{k=1}^{n-1} D_{1k}$	$I_1 \prod_{k=1}^{n} D_{1k}$
2	I_2	$I_2 D_{21}$	...	$I_2 \prod_{k=1}^{n-1} D_{2k}$	$I_2 \prod_{k=1}^{n} D_{2k}$
$\vdots$					
$r-n+1$	I_{r-n+1}	$I_{r-n+1} D_{r-n+1,1}$	...	$I_{r-n+1} \prod_{k=1}^{n-1} D_{r-n+1,k}$	$I_{r-n+1} \prod_{k=1}^{n} D_{r-n+1,k}$
$r-n+2$	I_{r-n+2}	$I_{r-n+2} D_{r-n+2,1}$	...	$I_{r-n+2} \prod_{k=1}^{n-1} D_{r-n+2,k}$	$I_{r-n+2} \prod_{k=1}^{n} D_{r-n+2,k}$
$\vdots$					
r	I_r	$I_r D_{r1}$	...	$I_r \prod_{k=1}^{n-1} D_{rk}$	$I_r \prod_{k=1}^{n} D_{rk}$

Only the upper trapezium is observable at time r. Let $U_i = I_i \prod_{k=1}^{n} D_{ik}$ represent the ultimate claims arising from accident year i. The total ultimate claims arising from the trapezium is given by $S_r = \sum_{i=1}^{r} U_i$. If the original trapezium of data consists of paid data, then the outstanding future liability is given by

$$S_r - \left(\sum_{k=1}^{r-n} x_{kn} + \sum_{k=r-n+1}^{r} x_{k,r-k}\right).$$

I_i is a random variable representing the initial reporting for accident year i, and D_{ij} is also a random variate which represents the growth in those claims assignable to accident year i between development years j and $j+1$.

I_i is a function of several exogenous variables, including earned exposure, social inflation, regulatory and legislative changes, residual or involuntary market mechanisms and written or earned premiums (Canadian Institute of Actuaries, 1990). This model assumes that I_i is determined outside of the model, and so it is treated as a deterministic input. Implicit in this assumption is that the loss development factor process is independent of the determination of the initial losses, I_i. Obviously, this is a simplification.

This model attempts to account for the pure statistical variation inherent in the loss development process. Since the D_{ij} variates are sensitive to exogenous forces such as changes in the claims handling mechanism of the company or reserve strengthening and deterioration, the assumption is made that there has not been any alterations in these external influences. This restriction is not as limiting as it appears. Data can be transformed to account for these changes, and the stochastic loss development factor procedure can be applied to the transformed data. Berquist and Sherman (1977) discuss two such transformation techniques.

Let D_{ij}, for $i = 1, \ldots, r$, be independently and identically distributed random variates with distribution function $F(d; \boldsymbol{\theta}_j)$ where $F(d; \boldsymbol{\theta}_j)$ is chosen in such a way that the distributions of the transformed variables $Y_{ij} = \ln D_{ij}$ are closed under convolution.

If the assumption is made that D_{ij} is independent of D_{ik}, then the calculation of the conditional distribution of ultimate accident year claims $U_i|I_i = I_i \prod_{k=1}^{n} D_{ik}$ is straightforward. The I_i values are given by the first column of the data trapezium, that is $x_{i0} = I_i$. For each column, the parameters $\boldsymbol{\theta}_j$ are estimated by $\hat{\boldsymbol{\theta}}_j(d_{1j}, d_{2j}, \ldots, d_{r-j,j})$, a function of the observed development factors.

What is ultimately required are estimates of the variates, $U_1, \ldots, U_r$, along the last column of the rectangle. The intermediate fitted values calculated for the upper trapezium can be used to assess the validity of the assumptions and the fit of the model.

The total incurred loss $S_r|(I_1, I_2, \ldots, I_r) = \sum_{k=1}^{r} U_k|I_k$ is also a random variable. Its distribution and inherent variability reflect the uncertainty of the development process only. It is this variable that is used for reserving purposes. The mean of this distribution less the amount paid-to-date is the expected liabilities accruing to the company. The mean is easily calculated as $\mathbf{E}(S_r|(I_1, I_2, \ldots, I_r)) = \sum_{k=1}^{r} \mathbf{E}(U_k|I_k)$.

Remaining characteristics of $S_r|(I_1, I_2, \ldots, I_r)$, such as percentiles, tail behaviour and moments, are not as easy to calculate; either simulation or approximation methods must be used to obtain the r-fold convolution of the distributions of the variates $I_1|U_1, \ldots, I_r|U_r$. Efron (1982) provides an overview on the use of the parametric bootstrap. This can be used to produce the distribution of ultimate claims. Simulation algorithms are outlined in Appendix B. Tail approximations are listed in this paper.

Ultimate Losses in the Lognormal Model

The lognormal distribution has been used to model many natural occurring events including entomological phenomena, or the incubation period of diseases. Business applications modelled include insurance claim size severities and random walk models for stock prices. A thorough treatment of the lognormal distribution and its properties can be found in Crow and Shimizu (1988) and Johnson and Kotz (1970).

If X is a normal variate with mean μ and variance σ^2, then $Y = e^X$ is said to have a lognormal distribution with probability density function

$$f(y) = \frac{1}{(2\pi)^{0.5}\sigma y} \exp\left(\frac{-(\ln y - \mu)^2}{2\sigma^2}\right) \qquad x \geq 0. \tag{1}$$

The r^{th} moment of the lognormal distribution is calculated from the moment generating distribution of the normal distribution, since

$$\mathbf{E}(Y^r) = \mathbf{E}((e^X)^r) = M_X(r) = \exp[r\mu + \frac{1}{2}r^2\sigma^2]. \tag{2}$$

The lognormal distribution is closed under multiplication. This is a consequence of the additive property of the normal distribution which states that if $X_1, \ldots, X_n$ are independent normal variates with mean μ_i and variance σ_i^2, then the random variate $T_n = \sum_{i=1}^n (X_i + b_i)$ for $b_i \in \Re$ is also normally distributed with mean $\mu = \sum_{i=1}^n (\mu_i + b_i)$ and variance $\sigma^2 = \sum_{i=1}^n \sigma_i^2$.

If $Y_1, \ldots, Y_n$ are independent lognormal variates such that $Y_i = e^{X_i}$ then the distribution of $P_n = \prod_{j=1}^n (\alpha_j Y_j)$ for $\alpha_j \geq 0$ is lognormal with parameters

$$\mu = \sum_{j=1}^n (\mu_j + \ln \alpha_j) \quad \text{and} \quad \sigma^2 = \sum_{j=1}^n \sigma_j^2. \tag{3}$$

Hayne (1986) introduces a loss development model in which each column of development factors, D_{ij} for $i = 1, \ldots, r$, follows an independent lognormal distribution with parameters μ_j and σ_j^2. This assumption of independence makes the estimation procedure tractable. From (3), the conditional distribution of ultimate claims $U_i|I_i = I_i \prod_{j=1}^n D_{ij}$ is also lognormally distributed with parameters $\mu_{(i)} = \ln I_i + \sum_{j=1}^n \mu_j$ and $\sigma^2 = \sum_{j=1}^n \sigma_j^2$. Using this with (1), gives the following probability density function

$$g_i(u_i) = \frac{1}{(2\pi)^{0.5}\sigma u_i} \exp\left(\frac{-(\ln u_i - \mu_{(i)})^2}{2\sigma^2}\right) \qquad u_i \geq 0. \tag{4}$$

From (2), the expected conditional ultimate claims arising from accident year i is

$$\mathbf{E}(U_i|I_i) = I_i \exp(\sum_{j=1}^n \mu_j + \frac{1}{2}\sum_{j=1}^n \sigma_j^2).$$

Estimation in the lognormal case is straightforward. Assuming that the realization of the process produces the loss development trapezium presented in the previous section, then under the independence assumption, d_{ij} for $i = 1, \ldots, r-j$ represent a random sample from a lognormal distribution with parameters μ_j and σ_j^2 for $j = 1, \ldots, n$.

Using maximum likelihood estimation, an estimate of μ_j is given by

$$\hat{\mu}_j = \frac{1}{r-j}\sum_{i=1}^{r-j} \ln d_{ij} \tag{5}$$

and the maximum likelihood estimate of σ_j^2 is

$$\hat{\sigma^2}_j = \frac{SS_j}{r-j} \tag{6}$$

where $SS_j = \sum_{i=1}^{r-j}(\ln d_{ij} - \hat{\mu}_j)^2$. The maximum likelihood estimate of μ_j is unbiased, while the estimate for σ_j^2 is asymptotically unbiased. These estimators are required to calculate the estimated losses accruing to the insurance company.

Theorem 1 *The uniformly minimum variance unbiased estimator of* $\mathbf{E}(U_i|I_i)$ *is*

$$\hat{\mu}_{(i)} = x_{i0}\exp(\sum_{j=1}^{n}\hat{\mu}_j)\prod_{j=1}^{n} {}_0F_1(\frac{(r-j)-1}{2};\frac{(r-j)-1}{4(r-j)}SS_j)$$

where ${}_0F_1(\eta;z) = \sum_{t=0}^{\infty}\frac{z^t}{t!}\cdot\frac{\Gamma(\eta)}{\Gamma(\eta+t)}$ *is the generalized hypergeometric function.*

Proof: See Appendix A.

In practice, each generalized hypergeometric function will converge within five or six terms.

Given $\hat{\mu}_{(1)},\ldots,\hat{\mu}_{(r)}$ the expected ultimate losses from the trapezium is $\hat{\mu}^{LN} = \sum_{k=1}^{r}\hat{\mu}_{(k)}$. The simulation algorithm, outlined in Appendix B, is used to calculate the other characteristics of the distribution.

Ultimate Losses in the Loggamma Model

In this model, assume that the loss development factors follow a loggamma distribution. This model along with the log inverse Gaussian (IG) model, which are special cases of the log generalized gamma distribution, do not share many of the favourable characteristics of the lognormal model.

- The support for these distributions is defined only for development factors greater than one. These models can only be used with cumulative data where recoverables received do not exceed the amount paid out at any age. It is possible to shift the observed development factors ($d'_{ij} = 1+d_{ij}$) so that these models can be used with incremental data or with cumulative data displaying negative development.
- Since the closure of these models is subject to restrictions on one parameter, estimation procedures that compute the parameter estimates for the whole trapezium simultaneously are required. In the lognormal model, parameter estimates for each column can be calculated separately.

The loggamma distribution, as introduced in Hogg and Klugman (1984), is an artificial variate constructed from the gamma distribution. The gamma distribution is used in life testing situations and to model meteorological precipitation. Johnson and Kotz (1970) and Lawless (1982) both provide an overview of the uses and characteristics of this distribution.

The probability distribution function of a gamma variate with parameters α and λ is

$$f(x) = \frac{x^{\alpha-1}e^{-\lambda x}\lambda^{\alpha}}{\Gamma(\alpha)} \qquad x \geq 0 \tag{7}$$

with moment generating function

$$M_X(t) = \mathbf{E}(e^{tX}) = [\frac{\lambda}{\lambda - t}]^\alpha \qquad t < \lambda. \tag{8}$$

From this it can be seen that the gamma distribution is closed under addition only if λ is constant between variates. Specifically, if $X_1, X_2, \ldots, X_n$ are independent random variates such that $X_i \sim \text{gamma}(\alpha_i, \lambda)$ for $i = 1, \ldots, r$ then $\sum_{i=1}^n X_i$ is also a gamma random variate with parameters $\alpha = \sum_{i=1}^n \alpha_i$ and λ.

The gamma distribution is not closed under shifts of location, and so the loggamma is not closed under multiplication by a scalar. If $Z = X + \delta$, where $X \sim \text{gamma}(\alpha, \lambda)$, then Z is said to have a shifted gamma distribution with probability density function

$$h(z) = \frac{(z-\delta)^{\alpha-1} e^{-\lambda(z-\delta)} \lambda^\alpha}{\Gamma(\alpha)} \qquad z \geq \delta \tag{9}$$

and corresponding moment generating function

$$M_Z(t) = e^{t\delta} [\frac{\lambda}{\lambda - t}]^\alpha \qquad t < \lambda. \tag{10}$$

If $Y = e^X$, then Y is said to have a loggamma distribution with probability distribution function

$$g(y) = f(\ln y)\frac{1}{y} = \frac{(\ln y)^{\alpha-1} y^{-\lambda-1} \lambda^\alpha}{\Gamma(\alpha)} \qquad y \geq 1. \tag{11}$$

For the loss development trapezium, assume each column of variables D_{ij}, $i = 1, \ldots, r$ represent an independent and identically distributed sample of loggamma variates with index parameter α_j and scale parameter λ. Using (9) and (11) the distribution of $U_i | I_i = I_i \prod_{j=1}^n D_{ij}$ is a shifted loggamma distribution with probability density function

$$g_i(u_i) = \frac{u_i^{-\lambda-1} I_i^\lambda \, (\ln u_i - \ln I_i)^{\alpha-1} \lambda^\alpha}{\Gamma(\alpha)} \qquad u_i \geq I_i \tag{12}$$

where $\alpha = \sum_{j=1}^n \alpha_j$.

From (10) and (12), the expected value of ultimate claims for the i^{th} accident year is given by

$$\mathbf{E}(U_i | I_i) = I_i [\frac{\lambda}{\lambda - 1}]^\alpha \qquad \lambda > 1. \tag{13}$$

If $\lambda \leq 1$, then the mean of the distribution is undefined.

Estimation of the parameters of the model, $\alpha_1, \alpha_2, \ldots, \alpha_n$ and λ is numerically intensive. From (11), the log likelihood function for the trapezium of development factors is given by

$$\ln L(\alpha_{\{j\}}, \lambda; \mathrm{d}_{\{ij\}}) = \sum_{j=1}^{n} \sum_{i=1}^{r-j} [-\lambda \ln d_{ij} + (\alpha_j - 1)\ln(\ln d_{ij}) + \alpha_j \ln \lambda - \ln \Gamma(\alpha_j)].$$

Maximizing this equation produces the following $n + 1$ equations

$$\hat{\lambda} = \frac{\sum_{j=1}^n (r-j)\hat{\alpha}_j}{d_{..}} \tag{14}$$

$$\psi(\hat{\alpha}_j) = \ln \hat{\lambda} + \tilde{d}_{.j} \ \text{ for } j = 1, \ldots, n \tag{15}$$

where $\psi(x)$, the digamma function, is $\frac{d \ln \Gamma(x)}{dx}$, and

$$\tilde{d}_{.j} = \frac{\sum_{i=1}^{r-j} \ln(\ln d_{ij})}{r-j} \quad \text{and} \quad d_{..} = \sum_{j=1}^{n} \sum_{i=1}^{r-j} \ln d_{ij}.$$

These $n+1$ values are sufficient for the parameters. The geometric mean of the j^{th} column of the log values of the development factors is given by $e^{\tilde{d}_{.j}}$.

Tables of exact values for the digamma function can be found in Abramowitz and Stegun (1965). Alternatively, using the asymptotic distribution of $\psi(x)$,

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} + \frac{691}{32\,760x^{12}} - \frac{1}{12x^{14}} + \frac{255}{28\,936x^{16}} + - \cdots \tag{16}$$

along with the relationship

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{17}$$

Bowman and Shenton (1984, p 32) give the following algorithm for calculating $\psi(x)$ for values of x between 0.01 and 4000.

Range of x	Strategy
$[.01, 40]$	Use the first 10 terms of (16) with argument $t = 40 + (x - [x])$. $\psi(x)$ can be derived from $\psi(t)$ by (17)
$(40, 60]$	Use the first 10 terms of (16)
$(60, 200]$	Use the first 8 terms of (16)
$(200, 500]$	Use the first 6 terms of (16)
$(500, 4000]$	Use the first 5 terms of (16)

Numeric methods are required to simultaneously estimate $(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n, \hat{\lambda})$ from (14) and (15). Kennedy and Gentle (1980) provide many algorithms for solving non-linear systems of equations which do not require the calculation of the derivatives of the system of equations. If good starting values for of the unknown parameters can be found and the matrix of derivatives can be easily calculated, a Newton-Raphson procedure (c.f. Burden and Faires, 1988) converges quickly.

Under the loggamma model, substituting for (14) into (15) produces the following vector of n functions:

$$\mathbf{F}(\hat{\alpha}) = (f_1(\hat{\alpha}), f_2(\hat{\alpha}), \ldots, f_n(\hat{\alpha}))^T$$

where

$$f_k(\hat{\alpha}) = f_k(\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n) = \psi(\hat{\alpha}_k) - \ln[\frac{\sum_{j=1}^{n}(r-j)\hat{\alpha}_j}{d_{..}}] - \tilde{d}_{.k}$$

for $k = 1, \ldots, n$. The matrix of derivatives is given by

$$\mathbf{J}(\hat{\alpha}) = \psi'(\hat{\alpha}) - \frac{1}{\sum_{j=1}^{n}(r-j)\hat{\alpha}_j}\mathbf{R}$$

where

$$\psi'(\hat{\alpha}) = \begin{pmatrix} \psi'(\hat{\alpha}_1) & 0 & \dots & 0 \\ 0 & \psi'(\hat{\alpha}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \psi'(\hat{\alpha}_n) \end{pmatrix}$$

and

$$\mathbf{R} = \begin{pmatrix} r-1 & r-1 & \dots & r-1 \\ r-2 & r-2 & \dots & r-2 \\ \vdots & \vdots & \ddots & \vdots \\ r-n & r-n & \dots & r-n \end{pmatrix}$$

Values of the trigamma function, $\psi'(x) = \frac{d\psi(x)}{dx}$, can be found by differentiating (16) and (17) and using Bowman and Shenton's algorithm.

The m^{th} iteration of the Newton-Raphson procedure yields the following vector of parameter estimates

$$(\hat{\alpha}_{1,m}, \hat{\alpha}_{2,m}, \dots, \hat{\alpha}_{n,m})^T = (\hat{\alpha}_{1,m-1}, \hat{\alpha}_{2,m-1}, \dots, \hat{\alpha}_{n,m-1})^T - \mathbf{F}(\hat{\alpha}_{m-1}) \cdot \mathbf{J}^{-1}(\hat{\alpha}_{m-1}).$$

Good initial values for $\hat{\alpha}_j$, for $j = 1, \dots, n$ are often difficult to find. Using the method of moments for each column j, gives the following values

$$\hat{\alpha}_{j0} = \frac{\frac{1}{r-j}(\sum_{i=1}^{r-j} \ln d_{ij})^2}{\sum_{i=1}^{r-j} (\ln d_{ij})^2 - \frac{1}{r-j}(\sum_{i=1}^{r-j} \ln d_{ij})^2}.$$

In practice, the final parameter estimates from the log IG model may provide better initial values. This is not surprising since the gamma and the IG distributions are both members of the same family of distributions, and possess many similar characteristics.

Once convergence has been obtained, $\hat{\lambda}$ can be calculated from (14). An estimate of the expected ultimate losses for the trapezium is given by

$$\hat{\mu}^{LG} = [\frac{\hat{\lambda}}{\hat{\lambda}-1}]^{\hat{\alpha}} \cdot \sum_{i=1}^{r} x_{i0} \qquad \text{where} \quad \hat{\alpha} = \sum_{j=1}^{n} \hat{\alpha}_j.$$

Since this estimate is almost certain to be biased, a better estimate is given by the sample mean of the simulation results. That is, if $S_1^*, S_2^*, \dots, S_{1000}^*$ are the observations generated (see Appendix B for details), then $\hat{\mu}^{LG} = \frac{1}{1000}\sum_{k=1}^{1000} S_k^*$.

Ultimate Losses in the Log Inverse Gaussian Model

The IG distribution is the law governing the time to passage for a particle subject to Brownian motion. Wald (1947) showed that the distribution is the limiting form of the sample size in certain sequential probability tests. A detailed analysis of the distribution is contained in Chhikara and Folks (1989), Johnson and Kotz (1970) contains an overview of the distribution and its properties. Like the loggamma distribution, the log IG distribution is an artificial variate constructed to take advantage of the additive nature of the IG distribution.

If X is an IG random variate, the probability density function of X is given by

$$f(x) = \mu \, (\frac{\beta}{2\pi})^{0.5} \, x^{-1.5} \, \exp[\frac{-\beta(x-\mu)^2}{2x}] \qquad x \geq 0. \tag{18}$$

The corresponding moment generating function is

$$M_X(t) = \exp[\mu\beta - \mu\beta(1-\frac{2}{\beta}t)^{0.5}] \qquad t \leq \frac{\beta}{2} \tag{19}$$

which is found by rewriting the integral $\int_0^\infty e^{tx} f(x)dx$ in terms of an IG variate with parameters $\beta' = \beta - 2t$ and $\mu' = \mu(1-\frac{2}{\beta}t)^{0.5}$. By (19), it can be shown that the IG distribution is closed under addition if β remains constant between variates.

The distribution is not closed under shifts of location. If $Z = X + \delta$, for $\delta \geq 0$, and $X \sim \mathrm{IG}(\mu, \beta)$ then Z is said to have a shifted IG distribution with probability density function

$$h(z) = \mu \, (\frac{\beta}{2\pi})^{0.5} \, (z-\delta)^{-1.5} \, \exp[\frac{-\beta(z-\delta-\mu)^2}{2(z-\delta)}] \qquad z \geq \delta \tag{20}$$

and moment generating function

$$M_Z(t) = \exp[\mu\beta + t\delta - \mu\beta(1-\frac{2}{\beta}t)^{0.5}]. \tag{21}$$

Using (21), if $X_1, \ldots, X_n$ are IG variates with parameters μ_i and β then the distribution of $T_n = \sum_{k=1}^n (X_k + \delta_k)$ is a shifted IG variate with parameters $\mu = \sum_{k=1}^n \mu_k$ and β and threshold parameter $\delta = \sum_{k=1}^n \delta_k$.

Let $Y = e^X$. Then Y is a log IG random variate with probability density function

$$g(y) = f(\ln y)\frac{1}{y} = \mu \, (\frac{\beta}{2\pi})^{0.5} \, (\ln y)^{-1.5} \, y^{-(1+0.5\beta)} \exp[\mu\beta - \frac{\beta\mu^2}{2\ln y}] \qquad y \geq 1. \tag{22}$$

Assume that each column of development factors D_{ij}, $i = 1, \ldots, r$ follows an independent log IG distribution with parameters μ_j and β. From (20) and (22) the distribution of $U_i | I_i = I_i \prod_{j=1}^n D_{ij}$ is a shifted log IG distribution with probability density function

$$g_i(u_i) = \mu \, (\frac{\beta}{2\pi})^{0.5} \, I_i^{0.5\beta} (\ln u_i - \ln I_i)^{-1.5} \, u_i^{-(1+0.5\beta)} \exp[\beta\mu - \frac{\beta\mu^2}{2(\ln u_i - \ln I_i)}] \qquad u_i \geq I_i \tag{23}$$

where $\mu = \sum_{k=1}^n \mu_k$.

From (21) and (23), the mean of ultimate losses for accident year i is given by

$$\mathbf{E}(U_i | I_i) = I_i \exp[\beta(1 - (1-\frac{2}{\beta})^{0.5}) \sum_{k=1}^n \mu_k]. \tag{24}$$

This is defined for $\beta > 2$.

As in the loggamma model, estimation of the parameters is numerically intensive. Assuming that the development factors $\mathbf{d}_{\{ij\}}$ are realised, using (22) the log likelihood function for the trapezium is

$$\ln L(\boldsymbol{\mu}_{\{j\}}, \beta; \mathbf{d}_{\{ij\}}) = \sum_{j=1}^{n} \sum_{i=1}^{r-j} [\ln \mu_j + \frac{1}{2} \ln \beta - \frac{1}{2} \beta \frac{(\ln d_{ij} - \mu_j)^2}{\ln d_{ij}}].$$

Maximum likelihood estimation yields the following $n+1$ equations

$$\frac{1}{\hat{\beta}} = \frac{\sum_{j=1}^{n} \sum_{i=1}^{r-j} [(\ln d_{ij} - \hat{\mu}_j)^2 / \ln d_{ij}]}{rn - 0.5n(n+1)} \tag{25}$$

and for each $j = 1, \ldots, n$

$$\hat{\mu}_j^2 \sum_{i=1}^{r-j} \frac{1}{\ln d_{ij}} - (r-j)\hat{\mu}_j = \frac{r-j}{\hat{\beta}}. \tag{26}$$

These $n+1$ equations can be solved iteratively. Let $\mu_{j0} = \frac{1}{r-j} \sum_{i=1}^{r-j} \ln d_{ij}$, which is the uniformly minimum variance unbiased estimate of μ in the single sample case. For $m = 1, \ldots, N$, where N is the number of iterations until convergence, the following two-step algorithm will provide values for the maximum likelihood estimates.

1. Using (25) calculate $\hat{\beta}_m$ as a function of $(\hat{\mu}_{1,m-1}, \ldots, \hat{\mu}_{n,m-1})$.
2. Let $\hat{\mu}_{j,m}$, for $j = 1, \ldots, n$, be the admissible root of (26) calculated using $\hat{\beta}_m$, then $\hat{\mu}_{j,m} = (r-j + [(r-j)(r-j + 4\hat{\beta}_m^{-1} \sum_{i=1}^{r-j} (\ln d_{ij})^{-1})]^{0.5})(2 \sum_{i=1}^{r-j} (\ln d_{ij})^{-1})^{-1}$.

Using (24) an estimate of the expected liability for the trapezium is given by

$$\hat{\mu}^{LIG} = \sum_{i=1}^{r} x_{i0} \cdot \exp[\hat{\beta}\hat{\mu}(1 - (1 - \frac{2}{\hat{\beta}})^{0.5})] \qquad \text{where } \hat{\mu} = \sum_{j=1}^{n} \hat{\mu}_j.$$

Since the estimates are not unbiased, it is recommended that the expected ultimate losses for the trapezium be calculated from the sample mean of the simulation outlined in Appendix B.

TAIL APPROXIMATIONS FOR ULTIMATE LOSSES

Upper percentiles of the distribution of ultimate claims, which are used to estimate the probability of ruin, can be calculated numerically using the simulation techniques discussed in Appendix B, or by approximations based on the tail characteristics of the distributions of claims for each accident year.

Embrechts and Veraverbeke (1982) have defined a system which classifies continuous distributions according to their behaviour in the tail. Distributions are defined to be light tailed, medium tailed or heavy tailed, which is also known as subexponential. Embrechts, Goldie and Veraverbeke (1979) and Embrechts and Goldie (1980) provide asymptotic formulae for the tail behaviour of sums of random variables possessing subexponential distributions. These results will be used to approximate the tail behaviour of the distribution of ultimate claims, $S_r|(I_1, I_2, \ldots, I_r) = \sum_{i=1}^{r} U_i|I_i$ under the three distributional assumptions.

This first requires proving that the conditional total claims for a given accident year follow a subexponential distribution. Under the assumption of lognormal development factors, the distribution of $U_i|I_i$ is also lognormal, which is a subexponential distribution as shown by Embrechts and Veraverbeke.

Willmot (1986) proves that if Y is a random variate such that $Y = e^X$ where X has a density function $F(x)$ characterized by

$$1 - F(x) \underset{x\to\infty}{\sim} Cx^{\beta}e^{-\delta x} \tag{27}$$

for $C, \delta > 0$ and $\beta \in \Re$, then the distribution of Y is subexponential. $A(x) \underset{x\to\infty}{\sim} B(x)$ is defined as $\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1$.

Under the loggamma assumption, the distribution of $U_i|I_i$ is a shifted loggamma distribution, and so $X_i = \ln U_i|I_i$ is distributed as a shifted gamma variate. Using L'Hôpital's rule with (9) gives

$$\lim_{x\to\infty} \frac{f_i(x)}{1 - F_i(x)} = \lim_{x\to\infty} \frac{f_i'(x)}{-f_i(x)} = \lambda.$$

Therefore as $x \to \infty$,

$$1 - F_i(x) \sim \frac{\lambda^{\alpha-1} I_i^{\lambda}}{\Gamma(\alpha)}(x - \ln I_i)^{\alpha-1} e^{-\lambda x}$$

for $\alpha = \sum_{j=1}^{n} \alpha_j$, which is of the form (27), since $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} h(x - c)$ for any constant c. Therefore $U_i|I_i$ has a subexponential distribution.

Under the log IG distribution, $U_i|I_i$, $i = 1, \ldots, r$ follows a shifted log IG distribution, and so $X_i = \ln U_i|I_i$ is a shifted IG variate with probability distribution function as defined by (20). The tail behaviour of X_i is given by

$$1 - F_i(x) \sim \mu\,(\frac{2}{\beta\pi})^{0.5}\,(x - \ln I_i)^{-1.5} \exp[\frac{\beta[(x - \ln I_i) - \mu)^2}{2(x - \ln I_i)}]$$

for $\mu = \sum_{j=1}^{n} \mu_j$, since

$$\lim_{x\to\infty} \frac{f_i(x)}{1 - F_i(x)} = \lim_{x\to\infty} \frac{f_i'(x)}{-f_i(x)} = \frac{\beta}{2}.$$

As $x \to \infty$, $1 - F_i(x)$ is of the same form as (27), and so the distribution of $U_i|I_i$ under the IG model is subexponential.

Some terminology common to the analysis of tail behaviour is required for proofs for the approximations.

- A distribution is said to dominate another distribution in the tail if $1 - F(y) = o(1 - G(y))$ as $y \to \infty$, where $o(h)$ is defined as $\lim_{h\to 0} \frac{o(h)}{h} = 0$.
- Two distributions F and G will have the same tail behaviour, denoted by $(1 - F(y)) \sim (1 - G(y))$ as $y \to \infty$ if $\lim_{y\to\infty} \frac{1-F(y)}{1-G(y)} = 1$.
- F and G are tail equivalent, written $(1 - F(y)) \sim c\,(1 - G(y))$ as $y \to \infty$ for some finite $c > 0$ if $\lim_{y\to\infty} \frac{1-F(y)}{1-G(y)} = c$. The previous definition is simply a special case of tail equivalence.
- And finally, two distributions F and G are said to be *max sum equivalent*, written $F \sim_M G$, if $1 - (F \star G)(y) \sim (1 - F(y)) + (1 - G(y))$ as $y \to \infty$, where $\star$ denotes the convolution of two distributions functions.

The behaviour of the tail of the distribution of $S_r|(I_1, I_2, \ldots, I_r)$ under the lognormal assumption requires the following two lemmas.

Lemma 1 (Embrechts, Goldie and Veraverbeke, 1979) *Let F and G be two subexponential distributions with support $y > 0$, and $H = F \star G$. If $1 - F(y) = o(1 - G(y))$ as $y \to \infty$, then H is also subexponential and $(1 - H(y)) \sim (1 - G(y))$ as $y \to \infty$.*

Lemma 2 (Embrechts, Goldie and Veraverbeke, 1979) *If G is a subexponential distribution, then for every positive integer n, $\lim_{y\to\infty} \frac{1-G^{(n)}(y)}{1-G(y)} = n$, where $G^{(n)}(y)$ is the n-fold convolution of G with itself.*

Theorem 2 *Let I_M represent the unique largest initial reporting of losses. Under the lognormal model, the tail behaviour of $S_r|(I_1, I_2, \ldots, I_r)$ is given by*

$$1 - H_r(s_r) \sim (\frac{\sigma^2}{2\pi})^{0.5}(\ln s_r - \ln I_M - \mu)^{-1} \exp[-\frac{1}{2}\frac{(\ln s_r - \ln I_M - \mu)^2}{\sigma^2}] \tag{28}$$

for $s_r \to \infty$ where $\sigma^2 = \sum_{j=1}^{n} \sigma_j^2$ and $\mu = \sum_{j=1}^{n} \mu_j$.

Proof: From Embrechts, Goldie and Veraverbeke, the tail behaviour of a random variate with probability density function (4) is given by

$$\begin{aligned} 1 - G_i(y) &\sim (\frac{\sigma^2}{2\pi})^{0.5}(\ln y - \ln I_i - \mu)^{-1} \exp[-\frac{1}{2}\frac{(\ln y - \ln I_i - \mu)^2}{\sigma^2}] \\ &= t_i(y) \qquad\qquad y \to \infty \end{aligned}$$

where $\sigma^2 = \sum_{j=1}^{n} \sigma_j^2$ and $\mu = \sum_{j=1}^{n} \mu_j$.

Let G_i be the cumulative distribution function associated with $U_i|I_i$ with corresponding tail behaviour t_i as defined above. For $i \neq M$

$$\begin{aligned} \lim_{y\to\infty} \frac{1 - G_i(y)}{1 - G_M(y)} &= \lim_{y\to\infty} \frac{1 - G_i(y)}{t_i(y)} \frac{t_M(y)}{1 - G_M(y)} \frac{t_i(y)}{t_M(y)} \\ &= \lim_{y\to\infty} \frac{t_i(y)}{t_M(y)} \\ &= \lim_{y\to\infty} [\frac{\ln y - \ln I_M - \mu}{\ln y - \ln I_i - \mu}] \exp[-\frac{1}{2\sigma^2}((\ln y - \ln I_i - \mu)^2 - (\ln y - \ln I_M - \mu)^2)] \\ &= \kappa \lim_{y\to\infty} y^{[(\ln I_i - \ln I_M)/\sigma^2]} \qquad \text{for some constant } \kappa \\ &= 0. \end{aligned}$$

Therefore $1 - G_i(y) = o(1 - G_M(y))$ for every i, as $y \to \infty$. Using Lemma 1 inductively, it is easy to see that if I_M is the unique maximum, then it will dominate the r-fold convolution of variates $\mathbf{U}_i|\mathbf{I}_i$. Therefore the tail behaviour of H_r is the same as the tail behaviour of G_M, and so

$$1 - H_r(s_r) \sim 1 - G_M(s_r) \sim t_M(s_r) \qquad\qquad s_r \to \infty.$$

Q.E.D.

If I_M is not unique, say there exists k accident years all with initial reporting I_M, then combining Lemma 2 with (28) yields

$$1 - H_r(s_r) \sim k(\frac{\sigma^2}{2\pi})^{0.5}(\ln s_r - \ln I_M - \mu)^{-1} \exp[-\frac{1}{2}\frac{(\ln s_r - \ln I_M - \mu)^2}{\sigma^2}] \qquad s_r \to \infty$$

where $\sigma^2 = \sum_{j=1}^{n} \sigma_j^2$ and $\mu = \sum_{j=1}^{n} \mu_j$.

For the loggamma and the log IG models, the following two lemmas are required.

Lemma 3 (Embrechts and Goldie, 1980) *Let F and G be two subexponential distributions which are tail equivalent, then $H = F \star G$, is also subexponential.*

Lemma 4 (Embrechts and Goldie, 1980) *If F and G are both subexponential distributions, and $H = F \star G$, then the following three statements are equivalent:*

1. *H is subexponential.*

2. *$F \sim_M G$.*

3. *$pF + (1-p)G$ is subexponential for $0 < p < 1$.*

Theorem 3 *Under the loggamma model, the tail behaviour of $S_r|(I_1, I_2, \ldots, I_r)$ is given by*

$$1 - H_r(s_r) \sim \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{r} (\frac{s_r}{I_k})^{-\lambda} (\ln s_r - \ln I_k)^{\alpha-1} \lambda^{\alpha-1} \qquad s_r \to \infty \tag{29}$$

where $\alpha = \sum_{j=1}^{n} \alpha_j$.

Proof: The tail behaviour of a shifted loggamma distribution is given by

$$t_k(y) = \frac{y^{-\lambda} I_k^{\lambda} [\ln y - \ln I_k]^{\alpha-1} \lambda^{\alpha-1}}{\Gamma(\alpha)} \qquad y \to \infty$$

where $\alpha = \sum_{j=1}^{n} \alpha_j$.

To show this, note that from (12)

$$\begin{aligned} \lim_{y\to\infty} \frac{y g_k(y)}{1 - G_k(y)} &= \lim_{y\to\infty} \frac{g_k(y) + y g_k'(y)}{-g_k(y)} \\ &= \lim_{y\to\infty} -1 - [\frac{\alpha - 1}{\ln y - \ln I_k} - \lambda - 1] \\ &= \lambda. \end{aligned}$$

Therefore $t_k(y) = \frac{y}{\lambda} g_k(y)$.

Trivially, if $r = 1$, then (29) is true. For $r = 2$, note that G_1 and G_2 are tail equivalent because

$$\lim_{y\to\infty} \frac{1 - G_1(y)}{1 - G_2(y)} = \lim_{y\to\infty} \frac{t_1(y)}{t_2(y)} = \lim_{y\to\infty} \frac{I_1^{\lambda}}{I_2^{\lambda}} \frac{[\ln y - \ln I_1]^{\alpha-1}}{[\ln y - \ln I_2]^{\alpha-1}} = (\frac{I_1}{I_2})^{\lambda}.$$

Since G_1 and G_2 are subexponential and tail equivalent, by Lemma 3, $H_2 = G_1 \star G_2$ is also subexponential. By Lemma 4,

$$\begin{aligned} 1 - H_2(y) &= 1 - (G_1 \star G_2)(y) \\ &\underset{y\to\infty}{\sim} (1 - G_1(y)) + (1 - G_2(y)) \\ &\underset{y\to\infty}{\sim} t_1(y) + t_2(y) \\ &= \frac{y^{-\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{2} I_k^{\lambda}[\ln y - \ln I_k]^{\alpha-1} \qquad y \to \infty. \end{aligned}$$

Therefore the theorem holds true for $r = 2$.

Assume that (29) holds true for $r = m-1$. That is H_{m-1} is subexponential with tail behaviour given by

$$1 - H_{m-1}(y) \underset{y\to\infty}{\sim} \frac{y^{-\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{m-1} I_k^{\lambda}[\ln y - \ln I_k]^{\alpha-1}.$$

To extend to $r = m$, first note that H_{m-1} and G_m are tail equivalent because

$$\lim_{y\to\infty} \frac{1 - H_{m-1}(y)}{1 - G_m(y)} = \lim_{y\to\infty} \frac{\sum_{k=1}^{m-1} I_k^{\lambda}[\ln y - \ln I_k]^{\alpha-1}}{I_m^{\lambda}[\ln y - \ln I_m]^{\alpha-1}} = \sum_{k=1}^{m-1} (\frac{I_k}{I_m})^{\lambda}.$$

Since H_{m-1} and G_m are tail equivalent, then by Lemmas 3 and 4

$$\begin{aligned} 1 - H_m(y) &= 1 - (H_{m-1} \star G_m)(y) \\ &\underset{y\to\infty}{\sim} (1 - H_{m-1}(y)) + (1 - G_m(y)) \\ &\underset{y\to\infty}{\sim} \frac{y^{-\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{m} I_k^{\lambda}[\ln y - \ln I_k]^{\alpha-1}. \end{aligned}$$

Therefore by mathematical induction, (29) holds true for $r = 1, 2, 3, \ldots$.

Q.E.D.

Theorem 4 *Under the log IG assumption, the tail behaviour of $S_r|(I_1, I_2, \ldots, I_r)$ is given by*

$$1 - H_r(s_r) \sim \sum_{k=1}^{r} \mu(\frac{s_r}{I_k})^{-\frac{\beta}{2}}(\frac{2}{\beta\pi})^{0.5}(\ln s_r - \ln I_k)^{-1.5}\exp[\mu\beta - \frac{\beta\mu^2}{2(\ln s_r - \ln I_k)}] \tag{30}$$

where $\mu = \sum_{j=1}^{n} \mu_j$ as $s_r \to \infty$.

Proof: The proof for (30) precedes in the same manner as the proof for (29). Using (23), it can be shown that the tail behaviour of a shifted log IG variate is

$$\begin{aligned} 1 - G_k(y) &\underset{y\to\infty}{\sim} \frac{2}{\beta} y g_k(y) \\ &= \mu(\frac{y}{I_k})^{-\frac{\beta}{2}}(\frac{2}{\beta\pi})^{0.5}(\ln y - \ln I_k)^{-1.5}\exp[\mu\beta - \frac{\beta\mu^2}{2(\ln y - \ln I_k)}] \end{aligned}$$

as $y \to \infty$ for $\mu = \sum_{k=1}^{n} \mu_k$.

Tail equivalence follows because

$$\begin{aligned}
\lim_{y\to\infty} \frac{1-H_{r-1}(y)}{1-G_r(y)} &= \lim_{y\to\infty} \frac{\sum_{k=1}^{r-1} I_k^{0.5\beta}(\ln y - \ln I_k)^{-1.5}\exp[-(\beta\mu^2)/(2(\ln y - \ln I_k))]}{I_r^{0.5\beta}(\ln y - \ln I_r)^{-1.5}\exp[(\beta\mu^2)/(2(\ln y - \ln I_r))]} \\
&= \sum_{k=1}^{r-1} (\frac{I_k}{I_r})^{0.5\beta} \lim_{y\to\infty} (\frac{\ln y - \ln I_k}{\ln y - \ln I_r})^{-1.5} \exp[\frac{\beta\mu^2/(\ln I_r - \ln I_k)}{2(\ln y - \ln I_k)(\ln y - \ln I_r)}] \\
&= \sum_{k=1}^{r-1} (\frac{I_k}{I_r})^{0.5\beta}.
\end{aligned}$$

Q.E.D.

Combining equations (28), (29) and (30) with their corresponding parameter estimates allows for estimation of tail probabilities of $S_r | I_1, I_2, \ldots, I_r$ for specific values of s_r under the three distributional assumptions. Since these equations are asymptotic approximations, their accuracy improves further out into the tail.

NUMERICAL EXAMPLE

This trapezium of data is from Zehnwirth (1989) and discussed by Sundt (1989). The line which generated this data is automobile bodily injury liability, which is a relatively long tailed distribution.

Acc't	Development Year								
Yr	0	1	2	3	4	5	6	7	8
71	568 891	2 148 049	3 425 871	4 160 541	4 840 910	5 058 131	5 205 931	5 263 030	5 327 859
72	428 753	1 399 393	2 355 291	3 451 062	3 961 134	4 452 987	4 695 982	4 995 827	
73	458 252	1 447 324	2 864 930	3 818 152	4 699 285	4 978 063	5 175 219		
74	355 229	1 304 036	2 596 936	3 344 939	3 892 227	4 166 594			
75	282 419	970 751	2 129 544	3 032 994	3 662 977				
76	267 600	1 312 390	2 528 827	3 367 532					
77	560 307	1 500 309	2 686 208						
78	360 171	1 371 944							
79	445 545								

This trapezium produces the following set of observed development factors. The assumption is made that the losses are fully developed at the end of the eighth year.

Accident	Development Year							
Year	1	2	3	4	5	6	7	8
1971	3.7759	1.5949	1.2144	1.1635	1.0449	1.0292	1.0110	1.0123
1972	3.2639	1.6831	1.4652	1.1478	1.1242	1.0546	1.0639	
1973	3.1584	1.9795	1.3327	1.2308	1.0593	1.0396		
1974	3.7610	1.9915	1.2880	1.1636	1.0705			
1975	3.4373	2.1937	1.4242	1.2077				
1976	4.9043	1.9269	1.3317					
1977	2.6777	1.7904						
1978	3.8091							

Using the algorithms presented in the previous section produces the following parameter estimates.

Development Period	Lognormal μ_j	Lognormal σ_j^2	Loggamma α_j	Loggamma λ	Log IG μ_j	Log IG β
0−1	1.2636	0.2155	94.2400	74.8081	1.2567	69.7551
1−2	0.6262	0.0719	46.7075	74.8081	0.6230	69.7551
2−3	0.2928	0.0230	21.8887	74.8081	0.2925	69.7551
3−4	0.1674	0.0035	12.8737	74.8081	0.1768	69.7551
4−5	0.0717	0.0030	5.5049	74.8081	0.0752	69.7551
5−6	0.0403	0.0003	3.4054	74.8081	0.0489	69.7551
6−7	0.0364	0.0013	2.4230	74.8081	0.0280	69.7551
7−8	0.0122	0.0013	1.3745	74.8081	0.0207	69.7551

Since there exists only one observation in the last development year, it is not possible to measure $\hat{\sigma}_8^2$ for the lognormal model. Instead $\hat{\sigma}_7^2$ is used as an estimate of σ_8^2. The parameter estimates of $\hat{\mu}_7$, $\hat{\mu}_8$ and $\hat{\sigma}_7^2$ are extremely unstable due to the scarcity of data in the latest development years. Because of this, the model is very sensitive to fluctuations in the most developed data.

Estimation in the loggamma and the log IG model requires the use of iterative algorithms. In both models, convergence was reached in less than ten iterations.

To test the validity of the three models, the empirical cumulative distribution functions of the observed development factors are compared to the cumulative distribution functions of the three fitted models. Applying the Kolmogorov-Smirnov test (c.f. Silvey, 1975) along the first four columns of the development factor trapezium, yields the following statistics.

Development Period	# Elements n	Lognormal D_n	Loggamma D_n	Log IG D_n
0−1	8	0.2417	0.1434	0.2228
1−2	7	0.2317	0.2229	0.2657
2−3	6	0.3295	0.1742	0.2578
3−4	5	0.6000	0.2517	0.3809

The lognormal model would be rejected for the 3−4 development period. For this development period, the log IG is not significant at the 1% level, but it is at the 5% level. From this it can be seen that the loggamma distribution provides the consistently best fit of the observed loss development factors over the first 4 development years. The log IG model generally outperforms the lognormal model.

Figure 1 displays the quantile plots for the first two years of observed loss development factors under the three distributional assumptions. With so few data points, it is conceivable that other models may also fit the data. There does not appear to be any evidence suggesting that the models are inappropriate. As with the Kolmogorov-Smirnov test, it would seem that the loggamma model provides a better fit to the data than both the lognormal and the log IG models.

The parameters calculated give the following expected ultimate losses for each accident year. As a benchmark, two *chain ladder* estimates are given. *Chain Ladder* 1 is calculated by applying the age-to-age development factors to the first column of data. *Chain Ladder* 2 is the traditional *chain ladder* algorithm

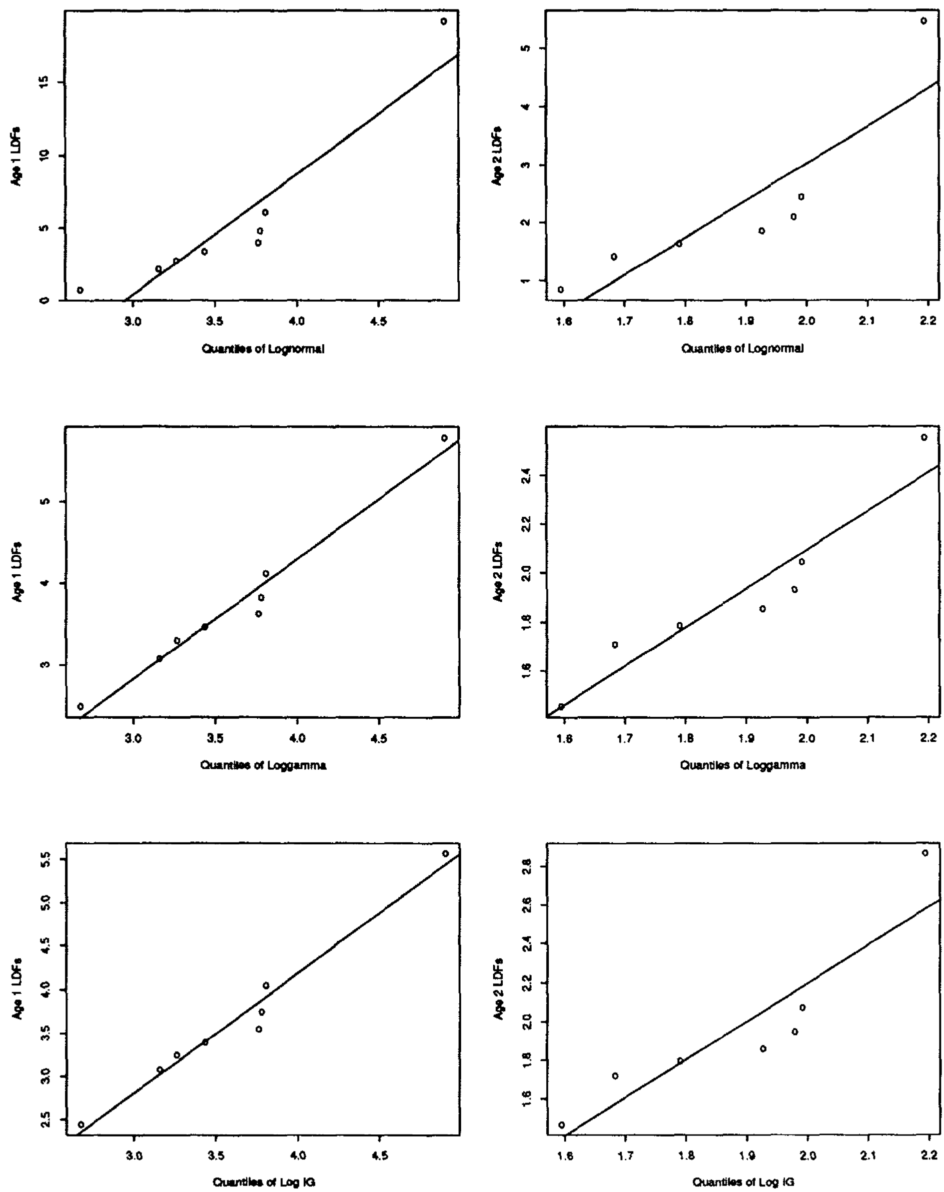

Figure 1: QQ Plots for the First Two Columns of Loss Development Factors Under the Three Distributional Assumptions

where the age-to-age development factors are applied to the latest diagonal of data. The age-to-age factors used are the mean loss development factors calculated for each column of the trapezium of data.

Acc't Year	Chain Ladder 1	Chain Ladder 2	Lognormal	Loggamma	Log IG
1971	7 159 109	5 327 859	7 157 330	7 182 137	7 215 595
1972	5 395 567	5 057 258	5 394 226	5 412 922	5 438 138
1973	5 766 792	5 435 070	5 765 359	5 785 341	5 812 292
1974	4 470 317	4 556 012	4 469 206	4 484 696	4 505 588
1974	3 554 052	4 304 386	3 553 169	3 565 484	3 582 094
1976	3 367 565	4 680 189	3 366 728	3 378 397	3 394 136
1977	7 051 085	5 012 683	7 049 333	7 073 765	7 106 719
1978	4 532 509	4 813 100	4 531 382	4 547 088	4 568 271
1979	5 606 883	5 607 066	5 605 489	5 624 918	5 651 122
TOTAL	46 903 879	44 793 623	46 892 222	47 054 748	47 273 955

The levels of ultimate incurred losses for the three stochastic loss development models are less than the amount calculated by the traditional *chain ladder* model for four of the accident years, and greater than the *chain ladder* results for four of the years. Only for the most recent accident year is the traditional *chain ladder* estimate within the range of the stochastic loss development factor model results. This is not a fair comparison in that the *Chain Ladder* 2 estimate makes use of more data, since it estimates ultimate losses from the latest diagonal of the trapezium. The marked differences between this model and the remaining models suggest that perhaps the age-to-age development factors are not random samples within each column, but follow some pattern. Independence may not exist between the columns of the loss reserving matrix. On examination of the observed loss development factors, it is difficult to judge whether any patterns exist.

Once the parameter estimates are calculated, the distribution of ultimate claims under the three assumptions can be simulated. The loggamma model produced the least varying data set, as can be seen in Figure 2. The box-and-whisker plot graphically compares the mean and variability of the simulated distributions.

The sum of the accident year expected values for the lognormal distribution was lower than the simulated mean for the sum, but within one-half standard deviation of the mean for the simulation. The two calculated values for the loggamma were remarkably close - within $5000 of each other. The log IG performed poorly. The sum of the accident year calculated expected values was approximately two standard deviations larger than the mean of the simulated distribution.

The upper percentiles of the distributions can be used to build a provision for adverse deviation into the loss reserves. Since the loggamma model appears to have provided the best fit of the observed development factors, it is reasonable to use the percentiles from this distribution. Based on the degree of conservatism desired by the actuary, the company may decide to set the level of estimated ultimate loss at $49 500 000, $51 000 000 or $54 500 000 for this line of insurance instead of the expected value of $47 000 000. These amounts correspond to the 80^{th}, 90^{th} and 95^{th} percentiles of the distribution of ultimate claims respectively. The loss reserve would be then calculated as the estimated ultimate loss minus the amount paid-to-date.

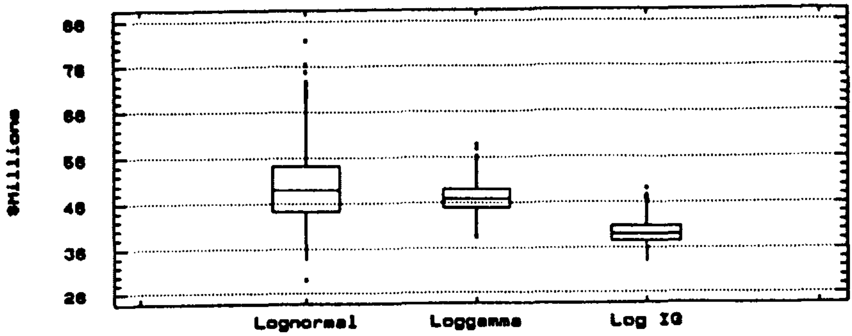

Figure 2: Mean and Standard Deviation from Simulation of Three Models

The upper percentiles of all three distributions are required to assess the accuracy of the asymptotic tail approximations. The asymptotic formulae in the previous section provide estimates of the upper percentiles of ultimate claims. Numerically, these are easy to apply once the parameter estimates have been calculated. In practice, the usefulness of such formulae depends upon whether accuracy is realised for moderate values of the random variate. This accuracy depends upon the functional form of $g_i(x)$ and the corresponding parameter estimates. Since all three distributions are unimodal, it is reasonable to expect that the formulae should be useful for moderate values of S_r and that the accuracy will be a function of the parameter values. For further discussion, see Willmot (1989).

Therefore it is extremely difficult to measure the usefulness of the tail approximations without first evaluating the entire distribution of S_r. Comparing the tail probabilities of the simulation, $1 - H_r(s_r)$ to the asymptotic formulae $t_r(s_r)$ under all three distributions shows that the approximations are less than satisfactory for the estimates of the parameter values. For example in the lognormal model, the following results based on a simulation size of 2000 are realised.

Lognormal Model		
s_r	$1 - H_r(s_r)$	$t_r(s_r)$
59 858 058	0.1	0.000079
63 803 376	0.05	0.000050
64 488 708	0.02	0.000029
71 257 439	0.01	0.000021

Simulated tail values corresponding to $1 - H_r(s_r) = 0.1, 0.05, 0.02$ and 0.01 under both the loggamma and log IG assumptions were compared to the corresponding tail formulae. The approximations performed less favourably than the lognormal model. All calculated values of $t_r(s_r)$ were less than 10^{-15} for both models.

CONCLUDING REMARKS

This paper provides a framework for measuring the variation inherent in the loss development process. Like every model, its usefulness is limited to the the extent that the underlying assumptions are valid. Further development of the stochastic loss development factor model can be roughly divided into two groups: theoretical considerations and practical improvements, of which the former is much more imposing.

Paramount of the theoretical difficulties is the assumption of independence between development years. In a trapesium of paid data this assumption is questionable at best, and for incurred losses, it is not reasonable to assume that this year's development on claims is independent of last year's development. However to build a dependent structure into a development projection method would destroy the simplicity of the model, which is its main advantage. Given the general scarcity of data in loss reserving trapeziums, adding a dependence structure could introduce identifiability problems.

A second flaw is that these models still contain too many parameters given the amount of data available, and thus are extremely sensitive to fluctuations in data. It is difficult to differentiate between competing distributional models because of the large number of parameters compared to the size of the data sets. Except for cases of extreme poor fit, goodness-of-fit tests will offer little information about the validity of the model.

This framework is not resistant to changes in exogenous forces. A necessary step to improve the performance of the models is to adjust the data first for known and measurable external influences

The intermediate values of the loss development trapezium have not been used directly to measure the appropriateness of the stochastic loss development models. Comparing the observed values to the fitted values under the three models would provide insight as to where the underlying assumptions are violated.

From the previous section, it is clear that the tail approximations performed poorly for the calculated parameter values. Further investigation is needed to discover the parameter values for which the asymptotic formulae are useful. Willmot (1989) discusses considerations for the IG distribution within a compound mixture framework.

I believe the ability to simulate the distribution of ultimate claims and the approximations for upper probabilities of the distributions are the most relevant contributions of this framework. Increasing concern about solvency issues makes it imperative that actuaries draw *quantitative* as well as *qualitative* conclusions about the sufficiency of loss reserves to cover future losses.

APPENDIX A

Proof Of Theorem 1:

For each D_{ik}, $(\hat{\mu}_k, SS_k)$ are independent and completely and jointly sufficient. Since each of the D_{ik} is also assumed independent, it follows that $(\sum_{j=1}^{n} \hat{\mu}_j, \sum_{j=1}^{n} SS_j)$ are independent and completely and jointly sufficient for the parameters of $\prod_{j=1}^{n} D_{ij}$.

Assume that there exists an unbiased estimator $g(\sum_{j=1}^{n} \hat{\mu}_j, \sum_{j=1}^{n} SS_j)$ of the mean of $U_i|I_i$. Then by the Lehmann-Scheffé theorem, if $\mathbf{E}(U_i|I_i)$ admits an unbiased estimator, then it must also be the uniformly minimum variance unbiased estimator.

From normal theory, for each j, $\hat{\mu}_j$ is lognormally distributed with parameters μ_j and $\frac{\sigma_j^2}{r-j}$, and $\frac{SS_j}{\sigma_j^2}$ follows a χ^2 distribution with $r-j-1$ degrees of freedom.

$$\mathbf{E}[I_i \cdot e^{\sum_{j=1}^{n} \hat{\mu}_j}] = I_i \mathbf{E}[\prod_{j=1}^{n} e^{(\hat{\mu}_j)}] = I_i \cdot e^{(\sum_{j=1}^{n} \mu_j + \frac{1}{2(r-j)}\sigma_j^2)}.$$

Using the technique discussed in Crow and Shimizu (1988, p 29), assume that there exists a function $h(\sum_{j=1}^{n} SS_j)$ such that $g(\sum_{j=1}^{n} \hat{\mu}_j, \sum_{j=1}^{n} SS_j) = I_i \cdot e^{(\sum_{j=1}^{n} \hat{\mu}_j)} \cdot h(\sum_{j=1}^{n} SS_j)$. Since $g(\sum_{j=1}^{n} \hat{\mu}_j, \sum_{j=1}^{n} SS_j)$ is unbiased,

$$\begin{aligned} \mathbf{E}[I_i \cdot e^{\sum_{j=1}^{n} \hat{\mu}_j} \cdot h(\sum_{j=1}^{n} SS_j)] &= \mathbf{E}(U_i|I_i) \\ I_i \cdot e^{\sum_{j=1}^{n} \mu_j + \frac{1}{2(r-j)}\sigma_j^2} \cdot \mathbf{E}[(h(\sum_{j=1}^{n} SS_j)]) &= I_i \cdot e^{(\sum_{j=1}^{n} \mu_j + 0.5\sigma_j^2)}. \end{aligned}$$

Therefore, using the MacLaurin series expansion,

$$\begin{aligned} \mathbf{E}[h(\sum_{j=1}^{n} SS_j)] &= e^{(\sum_{j=1}^{n} \frac{(r-j)-1}{2(r-j)}\sigma_j^2)} \\ &= \prod_{j=1}^{n}[\sum_{k=0}^{\infty} \frac{(\frac{(r-j)-1}{2(r-j)}\sigma_j^2)^k}{k!}]. \end{aligned} \tag{31}$$

Now for each j,

$$\begin{aligned} \mathbf{E}[(\frac{SS_j}{\sigma_j^2})^k] &= \int_0^{\infty} \frac{(\frac{SS_j}{\sigma_j^2})^{(\frac{2k+(r-j)-1}{2}-1)} \cdot e^{(-0.5 \cdot \frac{SS_j}{\sigma_j^2})}}{\Gamma(\frac{(r-j)-1}{2}) 2^{\frac{(r-j)-1}{2}}} d(\frac{SS_j}{\sigma_j^2}) \\ &= 2^k \frac{\Gamma(k + \frac{(r-j)-1}{2})}{\Gamma(\frac{(r-j)-1}{2})}. \end{aligned}$$

Rearranging the terms, gives

$$\mathbf{E}(SS_j^k) = 2^k (\sigma_j^2)^k \frac{\Gamma(k + \frac{(r-j)-1}{2})}{\Gamma(\frac{(r-j)-1}{2})}. \tag{32}$$

Suppose that $h(\sum_{j=1}^{n} SS_J)$ is of the form

$$h(\sum_{j=1}^{n} SS_j) = \prod_{j=1}^{n} {}_0F_1(\frac{(r-j)-1}{2}; \frac{(r-j)-1}{4(r-j)} SS_j),$$

then taking the expectation of both sides yields

$$\begin{aligned}
\mathbf{E}(h(\sum_{j=1}^{n} SS_j)) &= \mathbf{E}[\prod_{j=1}^{n} {}_0F_1(\frac{(r-j)-1}{2}; \frac{(r-j)-1}{4(r-j)} SS_j)] \\
&= \prod_{j=1}^{n} \mathbf{E}[\sum_{k=0}^{\infty} (\frac{(r-j)-1}{4(r-j)} SS_j)^k \cdot \frac{\Gamma(\frac{(r-j)-1}{2})}{\Gamma(\frac{(r-j)-1}{2}+k)\cdot k!}] \\
&= \prod_{j=1}^{n} [\sum_{k=0}^{\infty} \frac{\Gamma(\frac{(r-j)-1}{2})}{\Gamma(\frac{(r-j)-1}{2}+k)\cdot k!} (\frac{(r-j)-1}{4(r-j)})^k \mathbf{E}((SS_j)^k)].
\end{aligned}$$

And from (32),

$$\begin{aligned}
\mathbf{E}(h(\sum_{j=1}^{n} SS_j)) &= \prod_{j=1}^{n} [\sum_{k=0}^{\infty} \frac{\Gamma(\frac{(r-j)-1}{2})}{\Gamma(\frac{(r-j)-1}{2}+k)\cdot k!} (\frac{(r-j)-1}{4(r-j)})^k \cdot \frac{\Gamma(k+\frac{(r-j)-1}{2})}{\Gamma(\frac{(r-j)-1}{2})} 2^k (\sigma_j^2)^k] \\
&= \prod_{j=1}^{n} [\sum_{k=0}^{\infty} \frac{(\frac{(r-j)-1}{2(r-j)}\sigma_j^2)^k}{k!}].
\end{aligned}$$

Since the expectation of $h(\sum_{j=1}^{n} SS_j)$ equals (31), and given that under this model $x_{i0} = I_i$, therefore

$$\hat{\mu}_{(i)} = x_{i0} \exp(\sum_{j=1}^{n} \hat{\mu}_j) \prod_{j=1}^{n} {}_0F_1(\frac{(r-j)-1}{2}; \frac{(r-j)-1}{4(r-j)} SS_j).$$

Q.E.D

APPENDIX B

Simulation Algorithm for the Lognormal Model

1. Generate U_1 and U_2, two uniform $[0,1]$ variates.

2. Using the Box-Muller transformation (Hogg and Craig, 1978) calculate two independent standard normal variates

$$Z_1 = (-2\ln U_1)^{0.5}\cos(2\pi U_2) \quad \text{and} \quad Z_2 = (-2\ln U_1)^{0.5}\sin(2\pi U_2).$$

3. Two independent lognormal variates with parameters $\hat{\mu}$ and $\hat{\sigma}^2$ are given by

$$Y_1 = \exp(\hat{\sigma}\cdot Z_1 + \hat{\mu}) \quad \text{and} \quad Y_2 = \exp(\hat{\sigma}\cdot Z_2 + \hat{\mu}).$$

4. For each accident year k, $k = 1,\ldots,r$, repeat steps $1-3$, 500 times to generate 1000 conditional observations, $U_{k1}^*, U_{k2}^*, \cdots, U_{k,1000}^*$, for

$$\hat{\mu} = \ln x_{k0} + \sum_{j=1}^{n}\hat{\mu}_j \quad \text{and} \quad \hat{\sigma}^2 = \sum_{j=1}^{n}\hat{\sigma}_j^2$$

where $\hat{\mu}_j$ and $\hat{\sigma}_j^2$ are calculated by (5) and (6) respectively.

5. This gives 1000 sample elements from the conditional distribution of

$$S_r|(I_1, I_2, \ldots, I_r) = \sum_{k=1}^{r} U_k|I_k,$$

call them $S_1^*, S_2^*, \ldots, S_{1000}^*$ where $S_j^* = \sum_{k=1}^{r} U_{k,j}^*$.

6. These 1000 observations are then used to construct the empirical c.d.f of $S_r|(I_1, I_2, \ldots, I_r)$, from which various statistics, such as the mean of ultimate claims and percentiles of ultimate claims, are calculated.

Simulation Algorithm for the Loggamma Model

1. Uniformly fast rejection algorithms which will generate a gamma random variate, X, with parameters $\hat{\alpha}$ and $\lambda = 1$ are listed in Devroye. Separate algorithms are required for $\hat{\alpha} < 1$ and $\hat{\alpha} \geq 1$ since the shape of the distribution changes radically for values of α. For $\hat{\alpha} \geq 1$, Best's rejection algorithm (Devroye, 1986) is fast and simple to program. A modified version of Vaduva's algorithm (Devroye, 1986) is suitable for values of $\hat{\alpha} < 1$.

2. A shifted loggamma variate with parameters $\hat{\alpha}$ and $\hat{\lambda}$ and threshold δ is given by

$$Y = \exp[\frac{1}{\hat{\lambda}}X + \delta].$$

3. For each accident year k, $k = 1, \ldots, r$, repeat steps $1-3$, 1000 times to generate 1000 conditional observations, $U_{k1}^*, U_{k2}^*, \cdots, U_{k,1000}^*$, for $\delta = \ln x_{k0}$, where $\hat{\lambda}$ and $\hat{\alpha} = \sum_{j=1}^{n} \hat{\alpha}_j$ are calculated by the loggamma algorithm.

4. This will produce 1000 sample elements from the conditional distribution of

$$S_r|(I_1, I_2, \ldots, I_r) = \sum_{k=1}^{r} U_k|I_k,$$

call them $S_1^*, S_2^*, \ldots, S_{1000}^*$ where $S_j^* = \sum_{k=1}^{r} U_{k,j}^*$.

5. These 1000 observations are then used to construct the empirical c.d.f of $S_r|(I_1, I_2, \ldots, I_r)$, from which various statistics, such as the mean of ultimate claims and percentiles of ultimate claims, are calculated.

Simulation Algorithm for the Log IG Model

This algorithm for generating an IG variate can be found in Chhikara and Folks (1989) and Devroye (1986).

1. Generate U_1 and U_2, two uniform $[0,1]$ variates.

2. Using the Box-Muller transformation generate two independent $\chi^2_{(1)}$ variates.

$$Z_1 = ((-2\ln U_1)^{0.5}\cos(2\pi U_2))^2 \quad \text{and} \quad Z_2 = ((-2\ln U_1)^{0.5}\sin(2\pi U_2))^2.$$

3. For each Z_i, let $X_i = \hat{\mu} + \frac{Z_i - (Z_i^2 + 4\hat{\beta}\hat{\mu}Z_i)^{0.5}}{2\hat{\beta}}$.

4. For each X_i, generate U_s a uniform $[0,1]$ variate.

5. If $U_s \leq \frac{\hat{\mu}}{\hat{\mu}+X_i}$

   let $Y_i = \exp[X_i + \delta]$

   else

   let $Y_i = \exp[\frac{\hat{\mu}^2}{X_i} + \delta]$.

   $Y_i$is a shifted log IG variate with parameters $\hat{\mu}$ and $\hat{\beta}$ and threshold δ.

6. For each accident year k, $k = 1, \ldots, r$, repeat steps $1-5$, 500 times to generate 1000 conditional observations, $U_{k1}^*, U_{k2}^*, \cdots, U_{k,1000}^*$, where $\delta = \ln x_{k0}$ and $\hat{\beta}$ and $\hat{\mu} = \sum_{j=1}^{n} \hat{\mu}_j$ are calculated by the log IG algorithm.

7. This gives 1000 sample elements from the conditional distribution of

$$S_r|(I_1, I_2, \ldots, I_r) = \sum_{k=1}^{r} U_k|I_k,$$

call them $S_1^*, S_2^*, \ldots, S_{1000}^*$ where $S_j^* = \sum_{k=1}^{r} U_{k,j}^*$.

8. These 1000 observations are then used to construct the empirical c.d.f of $S_r|(I_1, I_2, \ldots, I_r)$, from which various statistics, such as the mean of ultimate claims and percentiles of ultimate claims, are calculated.

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