TITLE: OPTIMIZATION OF EXCESS PORTFOLIOS

AUTHOR: Philip E. Heckman, A.C.A.S., M.A.A.A.

B. S. Purdue University
PH.D. University of Chicago
Member, American Statistical Association
Previous publication in
  Physical Review
  Nuovo Cimento
  Canadian Journal of Physics
Actuarial publications:
  "Credibility and Solvency, CAS 1980
    Discussion Paper Program
  "The 1979 NCCI Remarriage Table", PCAS 1981

Phillip N. Norton, B.A.

B.A. University of Rochester
Ph.D. Candidate, University of Chicago
  Department of Statistics
Member, American Statistical Association

REVIEWER: Lyle W. DeGarmo

Lyle DeGarmo is Actuary for American Agricultural Insurance Company. He holds a BS and a MS degree in Mathematics from Iowa State University. He received his ACAS in 1974 and became a member of the American Academy of Actuaries in 1979.

Optimization of Excess Portfolios

I. Introduction

In confronting the difficult problems of choosing retentions and limits for a program of excess insurance or reinsurance coverages, practitioners have always, quite necessarily, placed heavy reliance on experience and intuition. The complex, intricate nature of the undertaking gives little choice. Our purpose here is to bring some very powerful new technology to bear on this problem; to suggest an explicit rationale for solution and to show how the potent combination of Fourier methods with the collective risk model can be applied to carry out the actual computations.

Our considerations will have equal relevance to large-account risk management and to the planning of excess reinsurance cessions. We shall treat the two cases in parallel, from the risk manager/cession planner's point of view, calling attention to differences where they are relevant. In each case, the problem resolves into two levels: first - what degree of risk to tolerate in the enterprise's operations, or, more likely, how much to spend on excess coverage; second - once this is decided, how to get the most coverage for one's money or to achieve an acceptable degree of risk at least cost. On the first level, we have a classic example of a management decision, whose irreducible anguish will not yield to any analysis, even the most brilliant. On the second level we have a complex of technical decisions which will yield readily if only the relevant variables can be quantified and sufficiently powerful techniques brought to bear.

The analogy with securities portfolio analysis is inescapable, and we may exploit it here to the full. As a risk surrogate, we propose the variance of retained losses, mainly because it is the simplest quantity with qualitatively correct behavior. Identification of a constraint variable depends on whether the cost of excess coverage is viewed as a cash flow item or as an investment. In the former case the gross cost is more likely relevant as a constraint imposed by higher management. In the latter case, the cost net of expected ceded losses is more relevant. In either case we shall assume that the cost can be modeled as a constant policy (treaty) fee plus a multiple of expected ceded losses plus a risk loading proportional to the root mean square (r.m.s.) deviation of the ceded losses. Conversations with reinsurance actuaries have provided convincing evidence that this last item is necessary for a consistent behavioral picture of reinsurance and excess pricing, which is largely an activity of underwriters.

Thus we shall consider the following detailed problem: We have a risk portfolio consisting of exposures (assumed independent) in several casualty lines. For each independent coverage, we wish to purchase specific excess insurance and then to cap the exposure on that coverage with an aggregate excess treaty (stop loss). We shall assume that the specific and aggregate coverages are taken by different carriers to be sure that the risk loadings do not consider the correlation between loss payments for the two coverages. We shall further assume that the stop-loss coverage is offered only with a fixed coinsurance provision to avoid moral hazard and to assure year-round vigilance on the part of the insured. The retention and limit for each specific and aggregate coverage are assumed to be freely adjustable within reasonable ranges, and by making these adjustments we shall attempt to achieve the most efficient use of resources devoted to excess coverage: The smallest variance of total retained losses at a given level of cost for the entire portfolio. Given a schedule of retention and limits agains total cost, management can then concentrate on the financially relevant aspect o the problem, what resources to devote to the control of risk in the portfolio with confidence that the funds, once allocated, will be applied in a manner ver nearly the most efficient possible.

The remainder of this paper is devoted to establishing the premise that suc methods can and should be used, exploiting existing technology. We shall rel heavily on recent work (1) of our colleague, Glenn Meyers, and one of us, also 1 be reported on at this meeting. Without the techniques described in that paper our present discussion would be merely academic. In Section II, we shall discus the problem of optimization under equality constraints, show how the solution carried out in the present instance and attempt to convey some intuition as 1 the meaning of the solution. Section III will show how the terms of the proble can be realized in the context of the Collective Risk Model, and will discus input data requirements and ensuing computations, carried out with the aid of rather elaborate Fortran program.

Section IV gives details of a single example: casualty excess above sel insurance for workers' compensation. Associated exhibits show the input a output information. We conclude with a discussion of the scope and limitatio of the method's applicability.

The calculations associated with this method are extensive and laborious and a presented in terse and skeletal form in two appendices, the first devoted to t optimization formulae, the second to their realization in the context of t Collective Risk Model.

II. Optimization

Constrained optimizations are an important class of problems whose actuarial uses have been pointed out previously in the literature (2, 3, 4). In the present application, we have a problem of the following form: Minimize the risk function,

$$(1) \quad \mathcal{R}_{\mu} = \sum_{\ell=1}^{L} V_\ell(R_\ell, T_\ell | r_\ell, t_\ell) + \mu\left[\sum_{\ell=1}^{L} C_\ell(R_\ell, T_\ell | r_\ell, t_\ell) - C\right],$$

where

l	identifies the coverage,
V_l	is half the variance of the retained loss for coverage, l,
C_l	is the cost of insurance for coverage, l,
R	is the stop-loss retention,
Γ	is the stop loss limit (thickness of the covered layer),
r	is the specific retention,
t	is the specific limit,
C	is the constraint variable, the total cost of excess coverage for the entire portolio,
	is an auxiliary parameter, the Lagrange Multiplier (LM), introduced to facilitate enforcement of the constraint.

To avoid clutter, we shall suppress arguments and indices wherever ambiguity will not result.

Before proceeding, we owe the reader a word on Lagrange multipliers, objects which often seem wispy, and ethereal, appearing out of nowhere for no apparent reason, and eventually dominating the problem at hand. The most familiar example of an LM comes from statistical physics: temperature is a parameter defined for a mechanical system in thermal equilibrium and arises in the problem of maximizing entropy at constant total energy. The notion of equilibrium is a very helpful one in understanding what is going on. In the present problem, the LM, μ , is the ratio of the incremental reduction (increase) in risk to the incremental associated increase (reduction) in cost due to adjustment of one or another of the variables. When this ratio is the same for all variables, equilibrium is achieved; and the variable of constraint, C, can be expressed as a function of the LM only. Changing the definition of the risk function, say, by using a monotonic function of the variance in place of the variance itself, merely results in a remapping of the LM, leaving the practical results, the optimal retentions and limits, unchanged. This lack of need to fret over the detailed definition of risk (as along as we accept that greater variance always entails greater risk) is a clue to the nature of the mysterious difference between technical and managerial problems. In this instance the LM intervenes between the two, absolving the technician from fretting about the managerial context and the manager from fretting over the technical details. One expects that a good many knotty business problems would yield to a similar analysis under sufficiently keen scrutiny.

The technical problem that is left is simply that of balancing the portfolio: achieving optimal balance between specific and aggregate within coverage and also balancing across coverages. There is a sizeable literature, mostly European, on the problem of balancing specific and aggregate cessions for optimal risk control, some of which we shall cite here for the benefit of the curious reader (5, 6, 7, 8, 9, 10, 11). A principal finding of these workers is that, for a fixed amount of ceded expected losses, stop-loss (aggregate excess) reinsurance gives a greater reduction in the variance of retained losses. This is not too surprising and accords well enough with our intuition. If reinsurance and excess underwriters were indifferent to risk and bound to a rigid schedule of rates, it would settle part of our problem outright: all specific excess treaties could be consigned to the ashcan. However, life is not so simple, and stop-loss coverage is approached by underwriters with great circumspection for several reasons. First the reinsurer's risk is much greater since the ceded losses can fluctuate wildly from year to year. Second, the technical means of calculating expected losses reliably have not been readily at hand. Finally, the underwriter must be convinced of the ceding carrier's good faith: that his vigilance will not be relaxed for the remainder of the treaty year after the stop-loss threshold is exceeded.

This stop-loss coverage is made available, if at all, under conditions and with price loadings that make the choice of specific and aggregate retentions less than straightforward. This is why we have included explicit risk loadings in the reinsurance pricing model.

Let us now return to our concrete optimization problem as posed in equation (1). We may easily find the conditions for a stationary value of the risk function. Differentiating with respect to the LM, μ, gives

$$\frac{\partial \mathcal{R}_\mu}{\partial \mu} = \sum_{\ell=1}^{L} C_\ell - C = 0,$$

the equation of constraint. Since we have assumed that the various coverages are independent, differentiating with respect to the retentions and limits gives a separate set of four equations for each coverage:

(2) $$\frac{\partial V_\ell}{\partial R_\ell} + \mu \frac{\partial C_\ell}{\partial R_\ell} = 0,$$

and similarly for T, r, and t. Thus the various coverages are connected only by the LM which expresses the constraint. As remarked before, the task of optimization amounts to choosing a value of μ and adjusting retentions and limits to bring all the marginal benefit-cost ratios into equality with this value so that

(3) $$\mu = -\frac{(\partial V_\ell/\partial R_\ell)}{(\partial C_\ell/\partial R_\ell)} = -\frac{(\partial V_\ell/\partial T_\ell)}{(\partial C_\ell/\partial T_\ell)} = -\frac{(\partial V_\ell/\partial r_\ell)}{(\partial C_\ell/\partial r_\ell)} = -\frac{(\partial V_\ell/\partial t_\ell)}{(\partial C_\ell/\partial t_\ell)},$$

for l = 1, 2, . . . L. Once the optimal retentions and limits are determined, the cost and the risk surrogate can be computed. Repeating this for several values of μ gives us the "efficient frontier", as it is known in the theory of securities portfolios. It is this curve in the space of risk vs cost which is most helpful to the managerial decision-maker. Associated with each point on the curve is a mass of detail - the characteristics of the equilibrium portfolio - with which he need not be concerned in reaching his decision. Such details are passed along as instructions to the technicians who do the actual portfolio handling.

It is clear that this scheme, if it can be brought off, is a very valuable risk management tool. In the succeeding section we shall discuss the technical means of making it operational.

III. Realization

A. The Collective Risk Model (CRM)

Clearly, the choice of retention and limit for specific excess will have an effect on the distribution of aggregate losses: this is a commonplace of reinsurance underwriting. Any model we choose should take this into account automatically and without any guessing. Such a model is available and has been well discussed in the actuarial literature. This model is founded on the following assumptions:

1. The number of claims on a policy (treaty) is a random variable of known distribution. (We shall assume a negative binomial.)
2. The claim amounts are independent and identically distributed (i.i.d.) random variables, with values drawn from a known severity distribution.

Thus the aggregate loss variable is modeled as a random sum, and its distribution is expressible as the probability average over the claim count, $\tilde{n}$, of the $\tilde{n}$ -fold convolution of the assumed severity distribution. This conceptual simplicity turns to computational facility when Fourier methods are applied to the problem. In the Fourier representation, convolutions become simple products and the probability sum a power series in the characteristic function (CF) of the severity, $\phi_{\tilde{x}}(t) = E[e^{itx}]$. It is easy to see that, if two random variables, $\tilde{x}_1$ and $\tilde{x}_2$, are independent, then the CF of their sum, $\tilde{x} = \tilde{x}_1 + \tilde{x}_2$,

$$\phi_{\tilde{x}}(t) = E[e^{it\tilde{x}}] = E[e^{it(\tilde{x}_1 + \tilde{x}_2)}] =$$
$$= E[e^{it\tilde{x}_1}]\, E[e^{it\tilde{x}_2}] = \phi_{\tilde{x}_1}(t) \cdot \phi_{\tilde{x}_2}(t),$$

the product of CF's of the components.

Amazing computational benefits ensue if we simply evaluate the characteristic function of the severity distribution, carry out all computations in the Fourier representation, and perform the inverse Fourier transform (FT) only at the end to obtain the relevant moment distributions. Most interesting claim count distributions lead to an exact sum (the probability generating function evaluated at the value of the CF). The only real computation challenge in all this - a substantial one - is the numerics for the Fourier inversion. Plainly, the worker who overcomes the natural human fear and loathing of Fourier integrals will have a powerful tool at his disposal.

For the interested reader, reference (1) gives a detailed account of the development of the model and of the numerics needed for reliable inversion of the aggregate CF to form the cumulative probability distribution and the first moment distribution ($M(z) = E[(\tilde{X} - z)_+]$, where $(r)_+ = r$ if $r>0$, zero otherwise). The work reported in reference (1) led to a Fortran program, tagged the "Aggregate Loss System" (ALS) which is the basis of our computational approach to the present problem. In addition to the basic collective risk model, this program allows for overall scale uncertainty in severity, treated as the smearing of a random scaling factor over a normalized gamma distribution. Claim counts are assumed to follow a negative binomial distribution, which includes the Poisson as a limit and the regular binomial as a continuation to negative variances. Severity input is in the form of a histogram density (or a piecewise linear cumulative distribution) with allowance for a point mass probability at the upper limit (policy limits).

For present purposes, we have extended this program to modify the severity CF to describe the effect of specific excess coverage and have installed a routine for computing the second moment distribution, $S(x) = E[\frac{1}{2}(\tilde{X}-x)_+^2]$. This function is needed in the calculation of aggregate loss variances. The modified program also calculates some auxiliary functions which arise in differentiating with respect to the specific retention and limit. These are described in the appendices. The entire modified ALS is embedded in an iteration scheme which computes the marginal ratios and adjusts the retentions and limits to bring them into equality with the selected Lagrange multiplier.

So as to keep matters in perspective, we shall conclude our discussion of the CRM by remarking briefly on some of its drawbacks.

- Input values relating to uncertainties in parameter estimates (negative binomial variance, scale uncertainty variance) are difficult to estimate themselves, so that one must often resort to rules of thumb. (This problem is being addressed in another of this session's call papers (12).)
- More seriously, the CRM is a static model, which does not describe how losses occur and develop in time. The input severity distribution is expected to reflect ultimate losses, fully developed and valued by a method consistent with the problem at hand. For instance, valuing losses with payment lag discounts under fixed dollar retentions puts some strain on our present conceptual framework unless retentions are indexed and an interest rate chosen to parallel the inflation rate.

It is impossible to overstress the importance of the severity input in this approach and the closeness of the attention it deserves. In fact some of the technology for really doing a job of it has yet to be fully developed. Certainly the dynamic aspects of the problem deserve the keenest scrutiny.

B. <u>The Excess Pricing Model</u>

Our model for the specific and aggregate excess coverage prices is a hybrid one, intended to bring together information obtained by shopping in the excess markets with knowledge of the behavior of losses in the covered business. Far from attempting a prescriptive model of excess pricing - certainly a vain undertaking - we shall use the aggregate loss functions and the severity and claim count information only to interpolate on a schedule of fresh quotes for different retentions and limits. The purpose in this is to be able to consider a continuous range of parameters in optimizing the portfolio. The model is as simple as possible, given the facts we have to deal with. For line of coverage, l, we have for the cost of coverage:

$$C_\ell = (1-\alpha_\ell)\left[B_\ell + \Gamma_\ell \cdot \Lambda_\ell(R_\ell,T_\ell|r_\ell,t_\ell) + \Delta_\ell\sqrt{\mathcal{U}_\ell(R_\ell,T_\ell|r_\ell,t_\ell)}\right] + \beta_\ell + \gamma_\ell \cdot \lambda_\ell(r_\ell,t_\ell) + \delta_\ell\sqrt{u_\ell(r_\ell,t_\ell)} \tag{4}$$

where (suppressing the line indices and arguments),

- α is the stop-loss coinsurance factor (usually prescribed by the excess underwriter);
- B is the stop-loss policy/treaty fee;
- Λ is expected payments under the stop-loss;
- $\mathcal{U}$ is half the variance of payments under the stop-loss;
- β is the specific excess policy/treaty fee;
- λ is the expected value of ceded losses under specific excess;
- u is half the variance of such losses.

The policy fees and the coefficients, Γ, Δ, γ, and δ, are adjustable t(allow interpolation on the actual quotes. In Appendix A, the above quanti· ties are related to the distribution functions of the problem. In ou illustrative runs, described in the next section, we simply assume values o the coefficients. In making this scheme operational, one of the firs priorities should be a detailed test of the pricing model under realisti conditions.

C. <u>Iterations</u>

Our actual approach to the optimization problem is to return to the ris function of equation (1), which we attempt to minimize for a fixed value o the LM, μ. The choice of μ in this context reflects the relative impor tance placed on cost as opposed to risk control. The risk function and it first derivatives are computed directly and the values used in a search do the gradient from a prescribed set of starting values for the retentio an limits. The decision as to the step size is made by calculating the chang in the derivatives at nearby points, assuming that the risk function has parabolic shape along the current gradient, and aiming for the minimum o the parabola. The step size is constrained to half the current paramete value, or less. At each step, the value of the risk function is teste against the previous value. If it has increased, the last step is revoke and tried again with half the step size (but in the same direction). Thi last condition enables the algorithm to turn sharp corners in the paramete space (per actual observation). The search terminates when the next ste size for each parameter is less than a prescribed tolerance. This algorithm is a provisional solution to the problem, and we expect that a more efficient and reliable one can and, we hope, will be found. An element of trial and error is unavoidable due to the peculiarities of the risk function, which contains near-singularities in the stop-loss parameters at multiples of the specific retention and modified policy limit (if any). In the risk function, these are, at worst, sudden changes in slope, which lead to sudden leaps in the first derivatives, which in turn render second derivatives almost uninterpretable. Thus a Newton-Raphson scheme, relying on the entire matrix of second partial derivatives was tried and rejected as too erratic. More reflection than we have had time for will be required to devise an entirely reliable and efficient algorithm. One simple possibility may be to reparametrize the stop-loss retention and limit as multiples of the specific retention. This would constrain variations in the parameter space so as to avoid the most violent singularities. Clearly, the matter is far from settled.

IV. *Results and Discussion*

The reader would certainly have a right to dismiss all the foregoing as a elaborate impracticality if we had not achieved successful calculations f presentation here. This we have done, by the skin of our teeth; and the art analytic excess plan design may be considered launched and afloat, if not y seaworthy. Our example, while entirely realistic, is a simple one which bypass some computational difficulties and leaves us with plenty of work yet to do. this section, we shall describe the example and the results, concluding wi caveats and reminders of what the scheme does and does not do.

A. Example: Workers' Compensation

Exhibit I gives a complete account of the input data for the probl studied. This is a simple risk management problem: designing an efficie combination of specific and aggregate excess covers for a workers' compens tion self-insurance program on a modest-sized account: total expected loss of $500,000. It is assumed that stop-loss coverage will be provided onl under a ten-percent coinsurance provision. Further, as discussed above, t excess pricing reflects the underwriters' risk aversion by adding amoun proportional to the standard deviation of ceded losses. The coefficient f specific coverage is 0.1/ 2; that for stop-loss is 0.2/ 2, reflecting t greater perceived uncertainty associated with this coverage. Expen loadings are 10% of expected losses. The cumulative probability distrib tion of the severity is presented and in the computations is assum piecewise linear between the tabulated points.

B. Solutions: The Efficient Frontier

From the starting values of the retentions and limits given in Exhibit I, for a given value of the LM, the program searches the parameter space for a minimum of the effective risk function. The results of this search can be expressed in terms of the cost of coverage and the agreed risk surrogate, the standard deviation (or r.m.s. error) of retained losses. Plotted in Cartesian coordinates these solution points trace a curve, the 'efficient frontier', which is graphed in Exhibit II and tabulated with the associated plan parameters (in thousands of dollars) in Exhibit III. The frontier descends at a decreasing rate from the trivial full-retention solution, which entails zero cost and the full risk of about $267,000 (a little more than half the expected losses), to the full coverage point with a cost of about $569,000 and zero risk. Computed solution points are marked on the graph with crosses. The dotted portion of the curve is conjectural, lying in a region where our algorithm became 'sticky' and yielded no solutions - another area for further study. The starting point of the iterations is marked by a circle and the risk-equivalent efficient solution by a square. The solid line connecting the two represents the cost of suboptimality, about $27,500 out of the total cost of about $160,000. The multipliers associated with the solutions are shown for the benefit of the curious in the last column of Exhibit III.

C. Conclusions

The solutions presented above were obtained via TSO on an IBM 370-168-at the terminal, on a while-you-wait basis. They are certaintly plausible, being mutually consistent and in accord with intuition. Further, if correlations between fluctuations in distinct lines of business are unimportant, there is no obstacle to extending the same treatment to multiple lines of coverage. As remarked before, the algorithm for minimizing the risk function needs to be made more general and flexible, not to mention more efficient. When these goals are achieved, we shall have in hand a risk management tool of major utility. The next step would be to include the possibility of a single global stop-loss above all the specific coverages. This would open up the study of risk-control aspects of retrospective plan design. The deepest problems with this approach, using static distribution models, have, as we all know, to do with the time element of the loss payouts (and sometimes of the premium collections), an aspect which has come to the fore in this period of high inflation/interest rates and slack insurance markets. There is no simple, elegant solution to this latter problem. We shall simply have to keep it in mind and work on it while doing our best with what we have.

D. Acknowledgments

We should like to thank our colleagues, Glenn Meyers, Yakov Avichai, Ed Seligman and Nat Schenker for many helpful discussions and suggestions.

Appendix A

Optimization Formulas

1. Individual Claim Variable (suppress line indices)

a. Uncensored Loss

Define a random variable (RV), $\tilde{x}$, the amount of an individual loss subject to primary policy limits (if any) but not altered by excess retentions and limits. We define the following:

$$\Pr[\tilde{x} > x] = g(x)$$

$$E[(\tilde{x}-x)_+] = m(x) = \int_x^\infty dy\, g(y)$$

$$E[\tfrac{1}{2}(\tilde{x}-x)_+^2] = S(x) = \int_x^\infty dy\, m(y)$$

(The "wedge function" $(x)_+$ was defined in the main text.)

b. Loss Ceded under Specific Excess

$$\tilde{y}^{r,t} = (\tilde{x}-r)_+ - (\tilde{x}-r-t)_+ ;$$

$$E[\tilde{y}^{r,t}] = m(r) - m(r+t) ;$$

$$(\tilde{y}^{r,t})^2 = (\tilde{x}-r)_+^2 - (\tilde{x}-r-t)_+^2 - 2t(\tilde{x}-r-t)_+ ;$$

$$E[\tfrac{1}{2}(\tilde{y}^{r,t})^2] = S(r) - S(r+t) - t\, m(r+t) .$$

2. Total Losses

a. Ceded under Specific Excess

Define the claim count variable $\tilde{n}$ drawn from a negative binomial distribution with mean, ν, and variance $\nu + c\nu^2$ (c is the "contagion" parameter).

$$\tilde{Y}^{r,t} = \sum_{k=1}^{\tilde{n}} \tilde{y}_k^{r,t}, \ (i.i.d.) \ ;$$

$$E[\tilde{Y}^{r,t}] = \nu[m(r) - m(r+t)] = \lambda(r,t);$$

$$E[\tfrac{1}{2}(\tilde{Y}^{r,t})^2] = \nu[s(r) - s(r+t) - t\,m(r+t)] + \tfrac{1}{2}(1+c)\nu^2[m(r) - m(r+t)]^2.$$

The function u(r,t), defined in the main text, is

$$u(r,t) = \nu[s(r) - s(r+t) - t\,m(r+t)] + \tfrac{1}{2}c\nu^2[m(r) - m(r+t)]^2$$

b. Retained after Specific Excess

$$\tilde{X}^{r,t} = \sum_{k=1}^{\tilde{n}} (\tilde{x}_k - \tilde{y}_k^{r,t}) \ ;$$

$$Pr[\tilde{X}^{r,t} > x] = G(x|r,t) \ ;$$

$$E[(\tilde{X}^{r,t}-x)_+] = M(x|r,t) = \int_x^\infty dy\, G(y|r,t);$$

$$E[\tfrac{1}{2}(\tilde{X}^{r,t}-x_+)^2] = S(x|r,t) = \int_x^\infty dy\, M(y|r,t);$$

$$E[\tilde{X}^{r,t}] = \nu\,(m(0)-m(r)-m(r+t));$$

$$E[\tfrac{1}{2}(\tilde{X}^{r,t})^2] =$$

$$= E\left[\tfrac{1}{2}\left(\sum_{k=1}^{\tilde{n}}\{\tilde{x}_k^{r,t}-(\tilde{x}_k^{r,t}-r)_+ + (\tilde{x}_k^{r,t}-r-t)_+\}\right)^2\right]$$

$$= \nu\left[S(0)-S(r)+S(r+t)-r(m(r)-m(r+t))\right]$$

$$+ \tfrac{1}{2}(1+c)\nu^2[m(0)-m(r)+m(r+t)]^2.$$

c. <u>Ceded under Stop-Loss</u>

$$\tilde{Y}_{R,T}^{r,t} = (1-\alpha)[(\tilde{X}_{R,T}^{r,t}-R)_+ - (\tilde{X}_{R,T}^{r,t}-R-T)_+];$$

$$\frac{1}{1-\alpha}E[\tilde{Y}_{R,T}^{r,t}] = M(R)-M(R+T) = \Lambda(R,T|r,t);$$

$$\frac{1}{(1-\alpha)^2}\,\frac{1}{2}Var[\tilde{Y}_{R,T}^{r,t}] = S(R)-S(R+T)-T\,M(R+T)$$

$$-\frac{1}{2}[M(R)-M(R+T)]^2 = U(R,T|r,t).$$

d. <u>Retained after Stop-Loss</u>

$$\tilde{X}_{R,T}^{r,t} = \tilde{X}^{r,t} - \tilde{Y}_{R,T}^{r,t};$$

$$E[\tilde{X}_{R,T}^{r,t}] = M(0)-(1-\alpha)[M(R)-M(R+T)];$$

$$\frac{1}{2}Var[\tilde{X}_{R,T}^{r,t}] = S(0)-(1-\alpha^2)[S(R)-S(R+T)] -$$

$$-(1-\alpha)R\left[M(R)-M(R+T)\right] + \alpha(1-\alpha)TM(R+T)$$

$$-\frac{1}{2}\left[M(0)-(1-\alpha)(M(R)-M(R+T))\right]^2$$

$$= V(R,T|r,t).$$

3. Derivatives

Derivatives with respect to R and T pose no problem, following, as they do, from the definitions, However, derivatives of the aggregate functions with respect to the specific variables, r and t are troublesome. We shall not present the solution until Appendix B. For the present we adopt the following definitions:

$$\frac{\partial S}{\partial r} = \nu(g(r)-g(r+t))\hat{M};$$

$$\frac{\partial S}{\partial t} = -\nu g(r+t)M^{+};$$

$$\frac{\partial M}{\partial r} = \nu(g(r)-g(r+t))\hat{G};$$

$$\frac{\partial M}{\partial t} = -\nu g(r+t)G^{+}.$$

The new M's and G's will be defined in the next Appendix. We shall no present the formulas need to calculate the marginal benefit/cost ratios.

a. <u>Marginal Benefits</u>

$$-\frac{\partial V}{\partial r} = \nu\left(g(r) - g(r+t)\right) \times$$
$$\times \Big\{ (1-\alpha)\left[\hat{G}(R) - \hat{G}(R+T)\right]\left[R - M(0) + (1-\alpha)\left(M(R) - M(R+T)\right)\right]$$
$$+ \left[M(0) - (1-\alpha)\left(M(R) - M(R+T)\right)\right] - \left[\hat{M}(0) - (1-\alpha)\left(\hat{M}(R) - \hat{M}(R+T)\right)\right]$$
$$- \alpha(1-\alpha)\, T\, \hat{G}(R+T) \Big\};$$

$$-\frac{\partial V}{\partial t} = -\nu g(r+t) \times$$
$$\times \Big\{ (1-\alpha)\left[G^{+}(R) - G^{+}(R+T)\right]\left[R - M(0) + (1-\alpha)\left(M(R) - M(R+T)\right)\right]$$
$$+ \left[M(0) - (1-\alpha)\left(M(R) - M(R+T)\right)\right] - \left[M^{+}(0) - (1-\alpha)\left(M^{+}(R) - M^{+}(R+T)\right)\right]$$
$$- \alpha(1-\alpha)\, T\, G^{+}(R+T) \Big\};$$

$$-\frac{\partial V}{\partial R} = -(1-\alpha)\Big\{ \left[G(R) - G(R+T)\right]\left[R - M(0) + (1-\alpha)\left(M(R) - M(R+T)\right)\right]$$
$$+ \alpha\left[M(R) - M(R+T) - T\, G(R+T)\right] \Big\};$$

$$-\frac{\partial V}{\partial T} = (1-\alpha)\Big\{ G(R+T)\left[R - M(0) + (1-\alpha)\left(M(R) - M(R+T)\right)\right]$$
$$+ M(R+T) + \alpha T\, G(R+T) \Big\}.$$

b. Marginal Costs

$$\frac{\partial C}{\partial r} = \nu\left(g(r)-g(r+t)\right)\Big\{(1-\alpha)\Big[\Gamma\left(\hat{G}(R)-\hat{G}(R+T)\right)+$$

$$+\frac{\Delta}{2\sqrt{U}}\left(\hat{M}(R)-\hat{M}(R+T)-T\hat{G}(R+T)-[\hat{G}(R)-\hat{G}(R+T)][M(R)-M(R+\right.$$

$$-\gamma-\frac{\delta}{2\sqrt{U}}\left[\frac{m(r)-m(r+t)-tg(r+t)}{g(r)-g(r+t)}+c\nu\left(m(r)-m(r+t)\right)\right]\Big\};$$

$$\frac{\partial C}{\partial t} = -\nu g(r+t)\Big\{(1-\alpha)\Big[\Gamma\left(G^{+}(R)-G^{+}(R+T)\right)+$$

$$+\frac{\Delta}{2\sqrt{U}}\left(M^{+}(R)-M^{+}(R+T)-TG^{+}(R+T)-[G^{+}(R)-G^{+}(R+T)][M(R)-M(R+T)]\right)\Big]$$

$$-\gamma-\frac{\delta}{2\sqrt{U}}\left[t+c\nu\left(m(r)-m(r+t)\right)\right]\Big\};$$

$$\frac{\partial C}{\partial R} = -(1-\alpha)\Big\{\Gamma\left(G(R)-G(R+T)\right)+$$

$$+\frac{\Delta}{2\sqrt{U}}\left[M(R)-M(R+T)-TG(R+T)-\left(G(R)-G(R+T)\right)\left(M(R)-M(R+T)\right)\right]\Big\}$$

$$\frac{\partial C}{\partial T} = (1-\alpha)G(R+T)\Big\{\Gamma+\frac{\Delta}{2\sqrt{U}}\left[T-\left(M(R)-M(R+T)\right)\right]\Big\}.$$

Appendix B

Characteristic Functions and Fourier Integrals

1. Severity CF under Specific Excess

If Pr [$\tilde{x} \le x$] = $f(x) = 1 - g(x)$,

the uncensored CF,

$$\phi_{\tilde{x}}(\tau) = \int_0^\infty e^{i\tau x}\, df(x),$$

and the censored CF,

$$\begin{aligned}\phi_{\tilde{x}^{r,t}}(\tau) &= \int_0^r e^{i\tau x}\, df(x) + [f(r+t) - f(r)]\, e^{i\tau r} \\ &\quad + \int_{r+t}^\infty e^{i\tau(x-t)}\, df(x) \\ &= \phi(\tau | r, t).\end{aligned}$$

The derivatives,

$$\begin{aligned}\frac{\partial \phi}{\partial r} &= i\tau\, [f(r+t) - f(r)]\, e^{i\tau r} \\ &= i\tau\, [g(r) - g(r+t)]\, e^{i\tau r};\end{aligned}$$

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= -i\tau \int_{r+t}^\infty e^{i\tau(x-t)}\, df(x) \\ &= -i\tau\, g(r+t)\, e^{i\tau r}\, \phi^+(\tau | r, t).\end{aligned}$$

2. Aggregate CF for Negative Binomial

The CF of total loss retained after specific excess is given by

$$\Phi(\tau|r,t) = (1+c\nu)^{-\frac{1}{c}} \sum_{n=0}^{\infty} \binom{n+\frac{1}{c}-1}{n} \left(\frac{c\nu}{1+c\nu}\right)^{n} [\phi(\tau|r,t)]^{n}$$
$$= [1+c\nu(1-\phi(\tau|r,t))]^{-\frac{1}{c}}$$

Its derivatives are

$$\frac{\partial\Phi}{\partial r} = i\tau\nu\left(g(r)-g(r+t)\right) e^{i\tau r}[1+c\nu(1-\phi)]^{-\frac{1}{c}-1}$$

$$\frac{\partial\Phi}{\partial t} = -i\tau\nu g(r+t) e^{i\tau r} \phi^{+} [1+c\nu(1-\phi)]^{-\frac{1}{c}-1}$$

It is convenient to define the auxiliary CF's;

$$\hat{\Phi}(\tau|r,t) = e^{i\tau r}[1+c\nu(1-\phi)]^{-\frac{1}{c}-1}$$

$$\Phi^{+}(\tau|r,t) = e^{i\tau r}\phi^{+}[1+c\nu(1-\phi)]^{-\frac{1}{c}-1}$$

which are useful in calculating the auxiliary functions defined in Appendix A.

3. Fourier Integrals

a. Scale Uncertainty

Following the treatment of reference (1), we introduce a RV, $\tilde{\beta}$, to describe overall uncertainty in the scale of the loss size. This could arise, for instance, from uncertainty in trend projections, or in loss developments, or both. For convenience we assume that $\tilde{\beta}$ is a gamma RV with parameters chosen so that expected losses are left unchanged. Thus, if the observed losses are $\tilde{X}^{r,t}/\tilde{\beta}$ we require $E[\tilde{X}^{r,t}/\tilde{\beta}] = E[\tilde{X}^{r,t}]$, that is $E[\tilde{\beta}^{-1}] = 1$. We further define $b = Var[\tilde{\beta}^{-1}]$ and $\rho = 1 + 1/b$. We shall have use for the relationship,

$$E[(\tilde{X}^{r,t}/\tilde{\beta})^2] = (1+b)E[(\tilde{X}^{r,t})^2]$$

$$= \frac{\rho}{\rho-1} E[(\tilde{X}^{r,t})^2].$$

From previous definitions,

$$Pr[\tilde{X}^{r,t}/\tilde{\beta} > x] = Pr[\tilde{X}^{r,t} > \tilde{\beta}x] \quad (\tilde{\beta}>0)$$

$$= E[G(\tilde{\beta}x|r,t)].$$

b. <u>Aggregate Loss Functions</u>

Using Fourier's theorem and the properties of probability distributions, we can derive

$$G(\tilde{\beta}x|r,t) = \frac{1}{2} - \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \frac{e^{-i\tau\tilde{\beta}x}}{-i\tau} \Phi(\tau|r,t).$$

Taking the expectation on $\tilde{\beta}$, we find that the Fourier kernel, $e^{-i\tau x}$, in the absence of scale uncertainty, becomes the CF of $\tilde{\beta}$ evaluated at argument $-\tau x$, whence, if $\tilde{\beta}$ is a gamma RV,

$$\bar{G}(x|r,t) = E\left[G(\tilde{\beta}x|r,t)\right] = \frac{1}{2} - \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \frac{\left(1+i\frac{\tau x}{\rho}\right)^{-\rho-1}}{-i\tau} \Phi(\tau|r,t).$$

As the scale uncerainty disappears, b $\rightarrow 0$, and $\rho \rightarrow \infty$, restoring the original Fourier kernel. We may take some strain off the numerics used to evaluate this integral by subtracting from the CF the probability of no claims and adjusting the constant term to compensate, whence

$$\bar{G}(x|r,t) = \frac{1}{2}\left[1-(1+c\nu)^{-\frac{1}{c}}\right] - \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \frac{\left(1+i\frac{\tau x}{\rho}\right)^{-\rho-1}}{-i\tau} \left[\Phi(\tau|r,t) - (1+c\nu)^{-\frac{1}{c}}\right].$$

This device becomes important when the expected number of claims, ν, is small. The higher moment distributions can be obtained by integrating this result from x = o and supplying the known values at x = o, whence

$$\bar{M}(x|r,t) = E[\tilde{X}^{r,t}/\tilde{\beta}] - \int_0^x dy\, \bar{G}(y|r,t),$$

and

$$\bar{M}(x|r,t) = \nu[m(0) - m(r) + m(r+t)] - \frac{x}{2}[1-(1+c\nu)^{-\frac{1}{c}}]$$
$$+ \int_{-\infty}^{\infty} \frac{d\tau}{2\pi}\left(\frac{1-(1+i\frac{\tau x}{\rho})^{-\rho}}{\tau^2}\right)[\Phi(\tau|r,t) - (1+c\nu)^{-\frac{1}{c}}];$$

likewise

$$\bar{S}(x|r,t) = \frac{\rho}{\rho-1}\nu\{s(0) - s(r) + s(r+t) - r[m(r) - m(r+t)]$$
$$+ \tfrac{1}{2}(1+c)\nu[m(0)-m(r)+m(r+t)]^2\}$$
$$- x\nu[m(0) - m(r) + m(r+t)] + \frac{x^2}{4}[1-(1+c\nu)^{-\frac{1}{c}}]$$
$$- \int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\left[\frac{x}{\tau^2} + \frac{\rho}{\rho-1}\left(\frac{1-(1+i\frac{\tau x}{\rho})^{-\rho+1}}{-i\tau^3}\right)\right][\Phi(\tau|r,t) - (1+c\nu)^{-\frac{1}{c}}].$$

c. <u>Auxiliary Functions</u>

We may use the differentiation formulas already derived, with the known properties of the distributions and integrals to derive the following:

$$\hat{G}(x|r,t) = \frac{1}{2} - \int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\,\frac{(1+i\frac{\tau x}{\rho})^{-\rho-1}}{-i\tau}\,\hat{\Phi}(\tau|r,t);$$

$$\hat{\bar{M}}(x|r,t) = \frac{\rho}{\rho-1}\left\{(1+c)\nu[m(0)-m(r)+m(r+t)]+r\right\}$$
$$-\frac{x}{2} + \frac{\rho}{\rho-1}\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\left(\frac{1-\left(1+i\frac{\tau x}{\rho}\right)^{-\rho+1}}{\tau^2}\right)\hat{\Phi}(\tau|r,t);$$

$$\bar{G}^+(x|r,t) = \frac{1}{2} - \int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\,\frac{\left(1+i\frac{\tau x}{\rho}\right)^{-\rho}}{-i\tau}\,\Phi^+(\tau|r,t);$$

$$\bar{M}^+(x|r,t) = \frac{\rho}{\rho-1}\left\{(1+c)\nu[m(0)-m(r)+m(r+t)]+r+\frac{m(r+t)}{g(r+t)}\right\}$$
$$-\frac{x}{2} + \frac{\rho}{\rho-1}\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\left(\frac{1-\left(1+i\frac{\tau x}{\rho}\right)^{-\rho+1}}{\tau^2}\right)\Phi^+(\tau|r,t).$$

We have enhanced the ALS to calculate $\bar{G}$, $\bar{M}$, $\bar{S}$, $\hat{\bar{G}}$, $\hat{\bar{M}}$, $\bar{G}^+$, and $\bar{M}^+$, rather than only the first two, but the numerics used in the actual calculation are basically those described in reference (1).

References

(1) Heckman, Philip E. and Glen G. Meyers, 'The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions', submitted to Proceedings of the Casualty Actuarial Society (1981).

(2) Ferrari, J. R., 'A Theoretical Portfolio Selection Approach for Insuring Property and Liability Lines'. PCASLIV, 1967. p. 33.

(3) Brubaker, R. E., 'A Constrained Profit Maximization Model for a Multi-Line Property/Liability Company'. Total Return Due a Property/Casualty Insurance Company, Casualty Actuarial Society, (1979). p. 28.

(4) Heckman, P. E., 'Credibility and Solvency', Pricing Property and Casualty Insurance Products, Casualty Actuarial Society, (1980), p. 116.

(5) Borch, Karl, 'An Attempt to Determine the Optimum Amount of Stop-Loss Reinsurance'. Proceedings of the XVIth International Congress of Actuaries. Brussels (1960).

(6) Vajda, Stefan, 'Minimum Variance Reinsurance'. ASTIN Bulletin II (1962). p. 257.

(7) Verbeck, H. G., 'On Optimal Reinsurance', ASTIN Bulletin IV (1966). p. 29.

(8) Borch, Karl 'Control of a Portfolio of Insurance Contracts', ibid. p. 59.

(9) Benktander, G., 'Some Aspects on Reinsurance Profits and Loadings', ASTIN Bulletin V, (1968). p. 314.

(10) Benktander, G., 'A Note on Optimal Reinsurance'. ASTIN Bulletin VIII, (1975). p. 154.

(11) Borch, Karl, 'Optimal Reinsurance Arrangements', ibid. p. 284.

(12) Meyers, Glenn and Nathaniel Schenker, "Parameter Uncertainty in the Collective Risk Model". Casualty Actuarial Society, Discussion Paper Program, 1982.

Exhibit I
Summary of Input Data

EXCESS PORTFOLIO SYSTEM:DISPLAY OF INPUT

LINE # 1:WORKERS COMP

EXPECTED LOSSES=	500000
EXPECTED CLAIMS=	506
CONTAGION PARAMETER=	0.0500
MIXING PARAMETER=	0.0250

SEVERITY DISTRIBUTION

LOSS AMOUNT	CUMULATIVE PROBABILITY
0.0	0.0
25.00	0.20230
50.00	0.48880
100.00	0.71960
150.00	0.78150
200.00	0.81090
250.00	0.82890
300.00	0.84270
400.00	0.86090
500.00	0.87410
750.00	0.89600
1000.00	0.90980
1500.00	0.92720
2000.00	0.93921
2500.00	0.94758
3000.00	0.95381
4000.00	0.96257
5000.00	0.96851
6000.00	0.97283
7000.00	0.97613
8000.00	0.97875
9000.00	0.98087
10000.00	0.98262
12500.00	0.98594
15000.00	0.98827
17250.00	0.98984
20000.00	0.99132
25000.00	0.99322
30000.00	0.99451
40000.00	0.99613
50000.00	0.99710
75000.00	0.99835
100000.00	0.99893
150000.00	0.99944
250000.00	0.99978
350000.00	0.99988
500000.00	0.99995
750000.00	0.99998
1000000.00	0.99999
1250000.00	0.99999
1500000.00	1.00000

REINSURANCE INFORMATION

	SPECIFIC	AGGREGATE
PRICING PARAMETERS		
COINSURANCE FACTOR	********	0.1000
TREATY FEE	$ 0.	$ 0.
LOSS MULTIPLE	1.1000	1.1000
RISK FACTOR	0.1000	0.2000
STARTING VALUES		
RETENTION	$ 50000.	$ 500000.
LIMIT	$ 500000.	$2000000.
STEP SIZE	$ 100.	$ 500.

Exhibit II

Efficient Frontier Implied By Data of Exhibit I

Retained Risk vs. Cost of Coverage

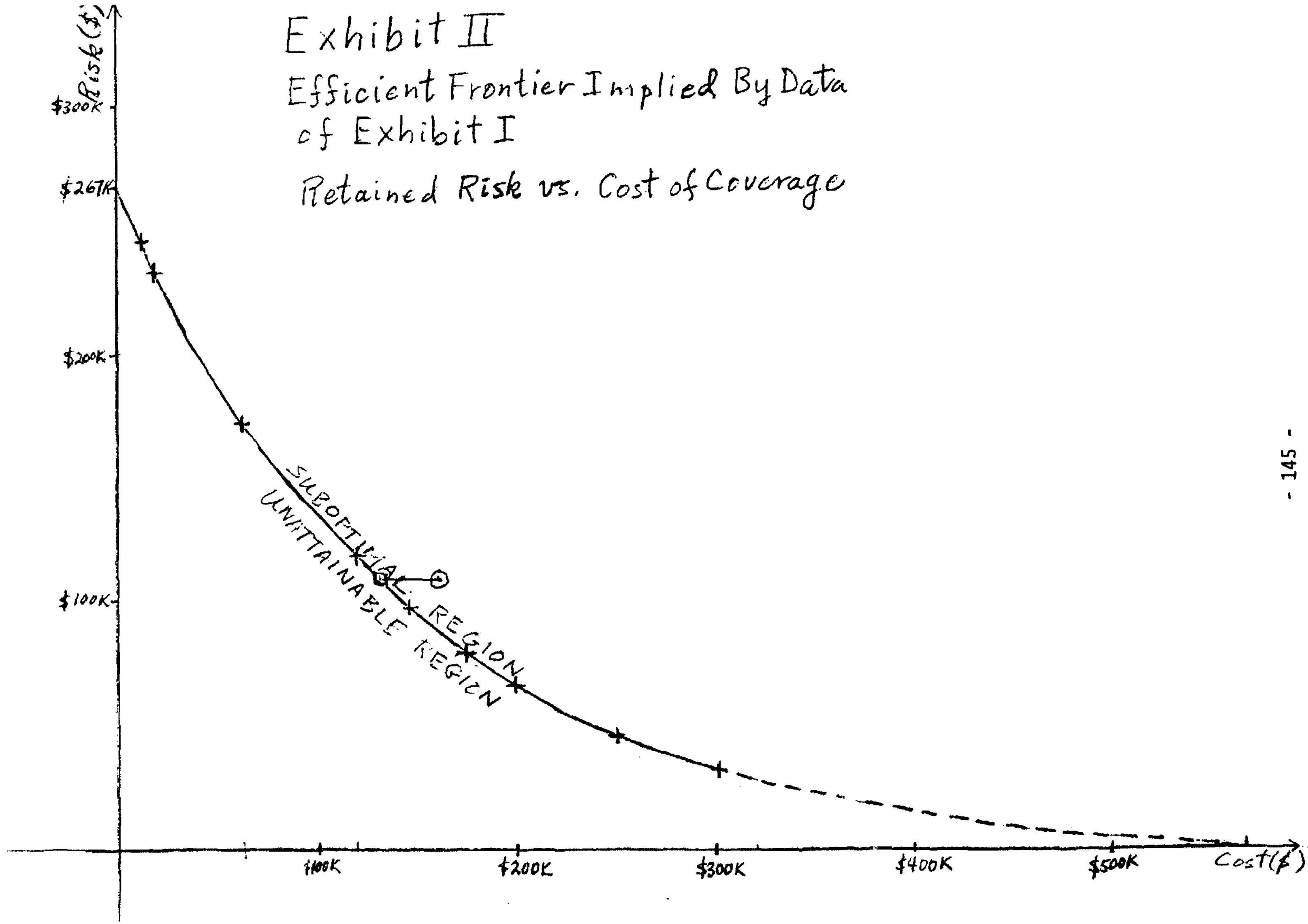

Exhibit III
Efficient Excess Plans Based on Exhibit I

		Specific Excess ÷ $1,000		Aggregate Excess (90%) ÷ $1,000		
Cost	Risk	Retention	Limit	Retention	Limit	LM
$ 0	$266,772	∞	-	-	-	∞
$ 12,032	$244,559	677	573*	1,447	7,702	700,000
$ 18,262	$233,330	526	602	1,239	7,628	400,000
$ 62,734	$171,319	311	582	706	6,355	200,000
$118,624	$118,030	188	537	515	4,207	100,000
$145,483	$ 97,764	172	541	455	4,525	70,000
$199,481	$ 66,257	101	550	368	5,601	40,000
$250,398	$ 45,273	57	577	304	8,834	20,000
$301,759	$ 31,502	28	584	250	9,659	10,000
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
$568,864	$ 0	0	∞	-	-	0

* This number must be considered rather soft since almost no probability exist above $1,250K. The solution is equivalent to unlimited specific excess.