TITLE: PRICING EXCESS-OF-LOSS CASUALTY WORKING COVER REINSURANCE TREATIES

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I. INTRODUCTION

An excess-of-loss reinsurance treaty provides the primary insurance company (cedant) with reinsurance protection covering a certain layer of loss for a specified category of individual (direct) insurance policies. Hence, for each loss event (occurrence) coming within the terms of the treaty, the reinsurer reimburses the cedant for the dollars of loss in excess of a certain fixed retention up to some maximum amount of liability per occurrence. For example, if the cedant's retention is $100,000 and the reinsurer's limit of liability is $400,000, then the reinsurer covers losses in the layer $100,000 up to $500,000; in reinsurance terminology, this is the layer $400,000 excess of $100,000. The reimbursement generally takes place at the time that the cedant reimburses the injured party. Allocated loss adjustment expenses are usually shared pro rata according to the loss shares, although in a few treaties they may be included in with the loss amounts before the retention and reinsurance limit are applied.

In this paper, casualty coverage will mean either third party liability coverage or worker's compensation coverage, although on certain treaties it may be broader. For example, for automobile insurance, first party coverage may be included within the terms of the excess treaty along with the third party coverage;

in any case, the total loss covered per occurrence is added together before application of the retention and the reinsurer's limit.

A working cover is a treaty on which the reinsurer expects to pay some losses; reinsurance underwriters say that the cover is substantially exposed by the primary insurance policy limits. Typically, layers below $1,000,000 per occurrence for casualty coverage are considered to be working covers. For a more complete discussion of this coverage, see Reinarz (1969), The Insurance Institute of London (1976) or Barile (1978).

An excess-of-loss casualty working cover is typically a large, risky contract. The annual reinsurance premium is usually six figures and quite often is millions of dollars. Although losses are expected, the number of losses to the treaty and their sizes are highly uncertain. Each cedant's insurance portfolio is unique, so there are no simple standard reinsurance rates. Industrywide average increased limits factors might be used as a starting point for pricing; however, competition and uncertainty force the reinsurer to be more sophisticated in his analysis of each proposal. A further complication is that the reinsurer usually has much less information to work with than does his primary insurance colleague. The reinsurer is provided with often vague and incomplete estimates of past and future exposure, of underlying coverage, of aggregate ground-up direct losses, and with some details about the very few historical large losses which are known. The final price will be reached by competitive bidding and by negotiation over particular contract terms. To compete, the reinsurer must work within severe time and manpower constraints to estimate a price which he believes to be adequate and which he can justify to the cedant.

Pricing excess-of-loss casualty working covers with any degree of accuracy is a complex and difficult underwriting and actuarial problem. We believe that the general theoretical pricing problem will remain insolvable: there will always be more questions than there are answers. However, in the spirit of a "Call for Papers", we offer a progress report on our work to date, knowing that we have only the beginnings of a truly satisfying practical solution. We will illustrate the actuarial problem by pricing two relatively simple and representative treaties. The approach is mathematical/actuarial; underwriting considerations are only briefly and incompletely mentioned, although these are very important. Some general solution criteria are presented and some tentative partial solutions are discussed. Although the point of view is that of a reinsurance actuary, we believe that the general approach may be of interest to other actuaries and that some of the particular techniques will be immediately useful to our primary insurance colleagues.

Any complicated procedure such as the one presented in this paper develops over time from the work and ideas of many people.

We wish to acknowledge the help of a few who have contributed to this development: Ralph Cellars, Howard Friedman, Charles Hachemeister, Mark Kleiman, Stephen Orlich, James Stanard and Edward Weissner.

II. TWO TREATY PROPOSALS

Reinsurers often receive proposals for which historical data are virtually non-existent. Such is the case when a newly formed or an about-to-be-formed primary company seeks reinsurance coverage or when an existing company writes a new insurance line or a new territory. There may be some vaguely analogous historical data, general industry information and some underwriting guesses about next year's primary exposure, coverage, rates and gross premium. An example is that of a new doctors' mutual offering professional liability coverage to the members of the medical society in state A.

<u>Example A</u>: A Doctors' Mutual Insurance Company

Proposal

1. reinsured layer: \$750,000 excess of \$250,000 per occurrence; no annual aggregate reinsurance limit; allocated loss adjustment expense shared pro rata according to loss share.

2. underlying coverage: professional liability claims-made coverage for limits of \$1,000,000/\$3,000,000 per claimant/ annual aggregate per doctor using the standard ISO policy form.
3. coverage period: beginning July 1, 1980 and continuous until terminated.
4. reinsurance rate: the offer is 25% of the gross direct earned premium with a 20% ceding commission and brokerage fee (thus, the net rate is 20%).

Information

5. exposure estimate of 500 doctors; no class breakdown.
6. class definitions - identical with ISO classes.
7. list of claims-made rates to be charged by doctor class for \$1M/\$3M limits.
8. summary of calendar/accident year 1974 - 1978 aggregate known losses and earned premiums for state A doctors covered by the BIG Insurance Company.
9. details about the five known losses paid or presently reserved for more than \$100,000 in state A for accident years 1974 - 1978.

10. a booklet describing the organization and financial structure of the doctors' mutual, together with biographies of the principal managers, claims-persons and attorneys and a statement of a get-tough attitude toward defending professional liability claims.
11. other miscellaneous letters and memos stating why this is an especially attractive deal for the reinsurer and the doctors.

It should be apparent that most of this information is only indirectly useful for pricing the reinsurance coverage. The offered rate must be analyzed using analogous industry information. There is great uncertainty regarding the potential loss situation.

At the opposite extreme is the treaty proposal for which there is a great wealth of historical information. This is sometimes the case when a treaty has been in place for many years with only minor changes, such as increasing the primary retention over time to parallel the inflation in individual loss amounts. If a reinsurer has been on the treaty for a few years, his underwriting and claims-persons have gotten to know the primary company people and have audited the treaty accounts. Thus, there is less uncertainty regarding the potential loss situation. A much simplified example of this situation is considered (only one line of business).

Example B: **P&C** Insurance Company

Proposal

1. reinsured layer: $400,000 excess of $100,000 per occurrence; no annual aggregate reinsurance limit; allocated loss adjustment expense shared pro rata according to loss share.
2. underlying coverage: general liability premises/ operations coverage, mainly in state B, written at various limits for bodily injury and property damage liability.
3. coverage period: beginning January 1, 1980 and continuous until terminated.
4. reinsurance rate: the net rate is to be negotiated as a percentage of gross direct earned premium.

Information

5. estimate of 1980 gross direct earned premium.
6. estimate of 1980 premium by policy limit.
7. summary of calendar/accident year 1969 - 1978 aggregate known losses as of 6/30/79 and gross earned premiums for P&C's general liability coverage insurance portfolio.

8. list of rate changes and effective dates for this line of business for 1969 through present and information that no change is contemplated through 1980.

9. detailed listings of all 358 general liability losses occurring since 1969 which were valued greater than $25,000 as of 6/30/75, 6/30/76, . . . or 6/30/79. At each evaluation, the information listed for each loss includes the following:

   a) identification number

   b) accident year (occurrence)

   c) amount of loss paid

   d) amount of loss outstanding

   e) policy limits

The evaluation of these two treaty proposals will illustrate the pricing procedure. Note that for example A we are to evaluate an offered rate, while for example B we are to propose a net rate and negotiate.

Before proceeding with the details, we believe it necessary to discuss some general pricing philosophy.

III. PRICING PHILOSOPHY

An insurance contract may be thought of as a financial stochastic process - a random pattern of pay-ins and payouts over time. The financial repercussions of a casualty excess-of-loss treaty may continue for 20 years or more. Thus, a reinsurer must consider the many aspects of this financial process to be able to estimate prices which are reasonably consistent with broad corporate policy. An actuarial goal is to combine all the contract financial parameters and all the corporate (underwriting) decision-making criteria into one comprehensive premium calculation principle or function - a black box which for each particular treaty produces the final premium or, more realistically, a negotiable premium range. Such a black box will not be purely mathematical, but will require substantial subjective input.

Present actuarial knowledge is short of this utopian goal. However, actuaries and underwriters have identified certain major contract parameters and decision-making criteria which should be considered when evaluating a particular contract. See Pratt (1964), Reinarz (1969), Bühlmann (1970), Gerber (1974) and Freifelder (1976) among others for discussions of premium calculation principles.

We believe that a reinsurer should consider the following items for each treaty either explicitly or implicitly:

1. The potential distribution of the aggregate loss to be ultimately paid by the reinsurer. Although the whole (past and future) coverage period should be considered, most important is the potential distribution of the aggregate loss arising from the next coverage year. The potential distribution of the aggregate loss is based upon the reinsurer's subjective evaluation of the situation and is difficult to specify in detail. Consequently, only certain major characteristics are estimated, such as the expected value, the variance or standard deviation, and certain percentiles, such as the 90th, 95th and 99th.

2. The potential distribution of the cash flow. The overall pattern over time is of interest, but more easily understood is the present value of the cash flow generated by the next coverage year. This random variable is distributed according to various price assumptions and the reinsurer's subjective assessment of the potential distributions of aggregate loss, payout patterns and investment rates-of-return. Since the loss payout varies by line of business, consideration of the potential distribution of this present value for each treaty may provide a more reasonable basis of comparison than does item (1).

3. Various corporate parameters and decision-making criteria. These include the following:

a) the potential distributions of aggregate loss and/or present value of cash flow estimated on the rest of the reinsurer's contract portfolio.

b) the reinsurer's financial surplus, both the current evaluation and the potential distribution of future values due to reserve changes and losses arising from the rest of the contract portfolio.

c) the reinsurer's financial assets and investment opportunities.

d) various corporate goals, e.g., "growth and profits with honor" (David J. Grady, address at the March 7, 1979 Casualty Actuaries of New York meeting).

e) the reinsurer's attitude toward the trade-off of risk versus rate-of-return on each contract and on his whole reinsurance portfolio.

Items (a) - (e) are meant to indicate some of the considerations which might define a utility function for corporate decision-making. For any typical treaty evaluation, it may be possible to localize our attention and only reflect these global considerations indirectly. However, in the long run they may not be ignored.

Other more ambiguous items which a reinsurer might consider include:

4. The surplus necessary to "support" the treaty from the reinsurer's point-of-view. The seller of any insurance or reinsurance contract exposes part of his surplus or net worth to

the risk that the loss will exceed the pure premium. Although it seems reasonable that some amount of surplus might be allocated to support any contract, there is yet no satisfactory theoretical functional definition. Note that this "supporting surplus" per treaty may not sum to the reinsurer's total surplus; he may be interested in surplus allocation on a relative basis: Does treaty A need more "supporting surplus" than treaty B?

5. The potential distribution of rate-of-return on the "supporting surplus" for this treaty relative to the rates-of-return on other treaties in the reinsurer's contract portfolio.

It should be apparent that neither we nor anyone else has a premium calculation principle which explicitly considers all these items. They are listed here to illustrate the complexity of the problem of accurately pricing reinsurance treaties. (Indeed, we would argue that it is almost as difficult to price any other large insurance contract or group of contracts.) We believe that thoughtful reinsurance underwriters do evaluate treaty proposals along these or similar lines. To model this process reasonably well is difficult but not impossible, since there are many good theoretical models and estimation techniques available to the modern actuary.

Of all the items, item (1), the potential distribution of aggregate loss to the reinsurer, is the least ambiguous and the most important. Thus, the remainder of this paper concentrates upon the estimation of this distribution for excess-of-loss casualty working covers. We will describe a reasonable mathematical model for this distribution and an estimation procedure for parameterizing the model.

IV. AN AGGREGATE LOSS MODEL

This section describes a mathematical model for the aggregate losses to be paid out on a particular insurance contract. The general insurance loss model will then be specialized for an excess-of-loss reinsurance treaty. The model is based upon the concepts of collective risk theory developed by Bühlmann and others: for example, see Bühlmann (1969) and Beard, Pentikäinen and Pesonen (1977). The model is designed to allow the observer to account for and quantify his uncertainty regarding the "true" distribution of aggregate loss for a particular insurance contract(s). This uncertainty arises from many sources; among them are:

1. Any particular probability model is inexact.
2. Any parameters estimated from sample data are random; that is, subject to sampling errors.
3. The historical loss data may not be at final settlement values, but are themselves random estimates.
4. The proper adjustments for inflation over time are unknown.

5. The underlying insured population for the coverage period to be evaluated is different from the past population.
6. There are often data errors and analytical blunders.

The model will be developed from a subjective Bayesian viewpoint; the particularization of the model is determined from the viewpoint of an observer at a particular time with particular information. An honest competent reinsurer and an honest competent cedant would most likely have different final parameterized models for any given treaty. For a further discussion of subjective or "personal" probability, see Savage (1954) and Raiffa (1968).

The collective risk model describing the distribution of aggregate loss consists of many possible particular probability models, each of which is given a "weight" based upon its subjective likelihood. In this way, the total uncertainty regarding the particular outcome which will be realized is broken down into two pieces: 1) the uncertainty regarding the "best" particular model, sometimes called the parameter risk, and 2) the uncertainty regarding the actual loss value to be realized even when the particular probability model is known, sometimes called process risk. See Freifelder (1976) or Miccolis (1977) for further discussions of these actuarial concepts.

We will use the term "parameter" in a broader sense than is customary. A "parameter" will consist of a complete specification of a particular probability model such as the lognormal, or group of models, together with their usual parameters. Our uncertainty as to which parameter is "best" will be defined by a subjective probability distribution on the set of possible parameters.

It is easier to start with the case where the parameter is known (the particular model is specified). Let the random variable L denote the aggregate loss to be paid out on a given insurance contract for a particular coverage year. We begin by assuming that the total coverage (exposure) can be split into independent homogeneous coverage groups in the following manner. Suppose that L can be written as:

$$(4.1) \qquad L = L_1 + L_2 + \ldots + L_k$$

where L_i = random variable denoting the aggregate loss for group i, i = 1, 2, . . ., k.

Further, suppose that each L_i can be written as:

$$(4.2) \qquad L_i = X_{i1} + X_{i2} + \ldots + X_{iN_i}$$

where N_i = random variable denoting the number of losses (occurrences) for group i.

X_{ij} = random variable denoting the size (loss amount) of the j^{th} loss for group i.

Groups may be defined by any grouping of insureds or coverage which our power of analysis can reasonably and credibly separate. Examples of groups could be:

1. distinct groups of classes of insureds or coverages.
2. similar insureds grouped by distinct policy limit.
3. the overall coverage time period split into sub-periods.

For example A, our groups will be defined by year of coverage and ISO doctor class (the older seven class scheme). For example B, our groups will be defined by combined bodily injury and property damage policy limit.

Let $F(x|\theta) = \text{Prob}[L \leq x|\theta]$ be a particular c.d.f. (cumulative distribution function) for L with known parameter θ. Think of θ as being a comprehensive parameter (vector) containing all the parameters necessary to specify the particular c.d.f.'s for the L_i's, N_i's and X_{ij}'s. Now make the following assumptions:

<u>Assumption 1</u>: Given θ, the L_i's are stochastically independent.

<u>Assumption 2</u>: Given θ, the X_{ij}'s are stochastically independent of the N_i's.

<u>Assumption 3</u>: Given θ, for fixed i, the X_{ij}'s are stochastically independent and identically distributed.

These assumptions split the total coverage into independent homogeneous coverage groups.

The model with known parameter θ has very nice properties. The first property is that $F(x|\theta)$ is the convolution of the c.d.f.'s for individual groups:

$$(4.3) \qquad F(x|\theta) = F_1(x|\theta)*F_2(x|\theta)* \cdots *F_k(x|\theta)$$

where $F_i(x|\theta) = \text{Prob}[L_i \leq x|\theta]$ for $i = 1, 2, \ldots, k$.

From this it follows that the cumulants of L given θ are straightforward sums of the cumulants of the L_i's given θ:

$$(4.4) \qquad K_m(L|\theta) = \sum_i K_m(L_i|\theta)$$

where $K_m(L|\theta)$ is the m^{th} derivative of the logarithm of the moment generating function of L evaluated at 0 (if it exists). Likewise for the $K_m(L_i|\theta)$'s.

See Kendall and Stuart (1966), pp. 157ff, for a discussion of cumulants. In particular, the first three cumulants add:

$$(4.5) \qquad \begin{aligned} K_1(L|\theta) &= E[L|\theta] = \sum_i E[L_i|\theta] \\ K_2(L|\theta) &= \mathrm{Var}[L|\theta] = \sum_i \mathrm{Var}[L_i|\theta] \\ K_3(L|\theta) &= \mu_3(L|\theta) = \sum_i \mu_3(L_i|\theta) \end{aligned}$$

where $\mu_m(L|\theta) = E[(L - E[L|\theta])^m|\theta]$

Because of assumptions 2 and 3, each $F_i(x|\theta)$ can be written in terms of the c.d.f.'s of N_i and X_i, where X_i is the common loss amount random variable for group i:

$$(4.6) \qquad F_i(x|\theta) = \sum_n \text{Prob}[N_i = n|\theta] \cdot G_i^{*n}(x|\theta)$$

where $G_i(x|\theta) = \text{Prob}[X_i \le x|\theta]$ for $i = 1, 2, \ldots, k$.

A consequence of (4.6) is that the first three moments of L_i given θ may be written:

$$(4.7) \qquad \begin{aligned} E[L_i|\theta] &= E[N_i|\theta] \cdot E[X_i|\theta] \\ \text{Var}[L_i|\theta] &= E[N_i|\theta] \cdot \text{Var}[X_i|\theta] + \text{Var}[N_i|\theta] \cdot E[X_i|\theta]^2 \\ \mu_3(L_i|\theta) &= E[N_i|\theta] \cdot \mu_3(X_i|\theta) + \mu_3(N_i|\theta) \cdot E[X_i|\theta]^3 \\ &\quad + 3 \cdot \text{Var}[N_i|\theta] \cdot E[X_i|\theta] \cdot \text{Var}[X_i|\theta] \end{aligned}$$

The scheme will be to develop parameterized models for the N_i's and X_i's, calculate their first three moments given θ, and then use (4.7) to calculate the first three moments of the L_i's and use (4.5) to calculate the first three moments of L given θ.

The collective risk model is obtained by deleting the restriction that θ is known. Instead, assume that the set Ω of possible parameters is known and that we can specify a subjective probability distribution $U(\theta)$ on Ω which gives the subjective likelihood of each subset of Ω. Bühlmann (1970) calls $U(\theta)$ a structure function. For simplicity, assume that Ω is finite so that $U(\theta)$ is a discrete probability:

Assumption 4: Ω is the finite set of possible parameters and $U(\theta)$ is the likelihood of the parameter θ.

Ω and $U(\theta)$ specify the observer's uncertainty regarding the "best" parameter.

With Ω and $U(\theta)$ specified, the unconditional c.d.f. $F(x)$ of L is the weighted sum of the conditional c.d.f.'s $F(x|\theta)$:

$$F(x) = \sum_{\theta} F(x|\theta) \cdot U(\theta) \tag{4.8}$$

Likewise, for each $F_i(x)$, the c.d.f. of L_i.

A consequence of (4.8) is (Bühlmann (1970), p. 66):

$$E[L^m] = \sum_{\theta} E[L^m|\theta] \cdot U(\theta) \qquad \text{for } m = 0, 1, 2, \ldots \tag{4.9}$$

Likewise, for each L_i.

With θ unknown, assumptions (1) - (3) may no longer hold, for the uncertainty regarding θ may simultaneously affect the model at all levels. For example, the c.d.f.'s of the L_i's are usually subjectively derived from historical data altered by loss development and inflationary trend assumptions. The assumptions made simultaneously about each L_i and L_j are usually not independent, i.e., the particular parameters for the c.d.f. of L_i are correlated with the particular parameters for the c.d.f. of L_j. Symbolically:

$$
\begin{aligned}
E[L_iL_j] &= \sum_\theta E[L_iL_j|\theta]\cdot U(\theta) \\
&= \sum_\theta E[L_i|\theta]\cdot E[L_j|\theta]\cdot U(\theta) \qquad (4.10)\\
&\neq (\sum_\theta E[L_i|\theta]\cdot U(\theta))\cdot(\sum_\theta E[L_j|\theta]\cdot U(\theta))
\end{aligned}
$$

When θ is unknown, equations (4.3) - (4.7) usually no longer hold. In particular, equation (4.5) now holds only for the first moment:

$$
\begin{aligned}
E[L] &= \sum_i E[L_i] \\
K_m(L) &\neq \sum_i K_m(L_i) \qquad \text{for } m \neq 1 \\
E[L^m] &\neq \sum_i E[L_i^m] \qquad \text{for } m \neq 1 \qquad (4.11)\\
\mu_m(L) &\neq \sum_i \mu_m(L_i) \qquad \text{for } m \neq 1
\end{aligned}
$$

Thus, the moments of L must now be evaluated directly from (4.9) by using (4.5) and (4.7); likewise for each L_i. For example, the second moment of L is now written:

$$
\begin{aligned}
E[L^2] &= \sum_{\theta} E[L^2|\theta]\cdot U(\theta) \\
\text{(4.12)} \qquad &= \sum_{\theta}\{Var[L|\theta] + E[L|\theta]^2\}\cdot U(\theta) \\
&= \sum_{\theta}\{(\sum_{i} Var[L_i|\theta])+(\sum_{i} E[L_i|\theta])^2\}\cdot U(\theta)
\end{aligned}
$$

Continue the expansion using formula (4.7).

Likewise for each L_i.

This general collective risk model may be specialized to the case of an excess-of-loss reinsurance treaty. Suppose that the treaty covers group i losses in the layer from r_i (retention) up to b_i. The general model may be specialized in at least two different ways. The first interpretation views X_i as the excess portion of each loss. We drop the subscript i in the following:

<u>Model 1</u> Notation:

N = random variable denoting total number of non-zero losses ground-up.

X = random variable denoting that part between r and b of each ground-up loss.

S = random variable denoting the ground-up loss amount.

Given that a loss has occurred, X and S are related by:

$$X = \begin{cases} 0 & \text{if} \quad S \leq r \\ S - r & \text{if } r < S < b \\ b - r & \text{if } b \leq S \end{cases} \tag{4.13}$$

Thus, the c.d.f.'s of S and X given θ are related by:

$$G_X(x|\theta) = \begin{cases} G_S(r|\theta) & \text{if} \quad x \leq 0 \\ G_S(x+r|\theta) & \text{if} \quad 0 < x < b - r \\ 1 & \text{if} \quad b - r \leq x \end{cases} \tag{4.14}$$

If N is to denote the number of excess losses, then use the second specialization:

<u>Model 2</u> Notation:

N = random variable denoting the number of excess loss occurrences.

X = random variable denoting the size of an excess loss, given that an excess loss has occurred.

$\mathbb{N}$ = random variable denoting the total number of non-zero ground-up losses, called "base number".

S = random variable denoting the ground-up loss amount.

With known parameter θ, the c.d.f.'s of N and BN are related by:

$$\text{(4.15)} \quad \text{Prob}[N=n|\theta] = \sum_{m \geq n} \{\text{Prob}[BN=m|\theta] \cdot \binom{m}{n} \times (1 - G_S(r|\theta))^n \cdot G_S(r|\theta)^{m-n}\}$$

where $G_S(r|\theta) = \text{Prob}[S \leq r|\theta]$

In particular, it is easy to show that:

$$\text{(4.16)} \quad E[N|\theta] = E[BN|\theta] \cdot (1 - G_S(r|\theta))$$

Likewise, the c.d.f.'s of X and S for Model 2 are related by:

$$\text{(4.17)} \quad G_X(x|\theta) = \begin{cases} 0 & \text{if } x \leq 0 \\ G_S(x+r|\theta) \cdot (1 - G_S(r|\theta))^{-1} & \text{if } 0 < x < b - r \\ 1 & \text{if } b - r \leq x \end{cases}$$

Model 1 is easier to work with since the definition of N remains the same when different retentions are considered. But, it is easy to trade back and forth between the two models and, most importantly, they both yield identical answers for the distribution of L. We prefer to use Model 1, so hereafter N will be the number of non-zero ground-up losses.

The next three sections show how this general model may be used to evaluate the loss potentials of particular treaties. To do so, we must:

1. specify the homogeneous groups.
2. specify the set of possible parameters Ω and the subjective likelihood $U(\theta)$, of each θ in Ω.
3. calculate (using a computer package) the moments and approximate various percentiles of L from the moments of the N_i's and X_i's given the θ's.

V. PARAMETER ESTIMATION: EXAMPLE A

The most difficult part of this aggregate loss evaluation procedure is estimating the parameters to be used in the models. The estimation for A Doctors' Mutual Insurance Company, example A, will illustrate the case where there are no credible historical loss data directly related to the exposure. In this case, general industry information must be used together with substantial judgement. In general, in this situation we presently estimate three parameters based upon low, medium and high loss frequency and severity assumptions (We purposely use the word "medium" to avoid the statistical theoretic connotations of words such as "mean" and "median".) For example A, the estimates will be based upon Insurance Services Office ratemaking data and further modified by judgement based upon the NAIC Medical Malpractice Closed Claim Surveys (1977) and (1978).

The groups for example A are selected to be the seven doctors classes in the old ISO class plan because we believe there are sufficient data to separate these classes for loss frequency and severity. The complete parameter matrix is displayed in Table 5A. It looks formidable but is really quite simple; much of it is repetitive and based upon standardized judgement. Each class is represented by three rows: the low θ is the first row for each class, the medium θ is the second row for each class and the high θ is the third row for each class. In Section VII these parameters will be input to a Prudential Reinsurance Company computer package named RISKMODEL which will calculate the moments of the aggregate loss L for the layer \$750,000 excess of \$250,000 for the coverage year 1980/81 using the formulas from Section IV. The package also approximates selected percentiles of the distribution of L.

The form of the parameterized c.d.f.'s we shall use for the distribution of the number of loss occurrences N_i for class i is the negative binomial defined in Appendix D. Thus, we must specify two parameters for each c.d.f.; we will specify $E[N_i|\theta]$ and the ratio $Var[N_i|\theta] \div E[N_i|\theta]$ for each class i for each θ. The expected number of ground-up loss occurrences $E[N_i|\theta]$ is based upon the exposure and loss frequency estimates in Table 5A, columns (2) and (3). The estimates of exposure by class are based upon ISO exposure data

TABLE 5A

GROUP	EXPOSURE	FREQUENCY	VENJ-CENT	DST CODE	PAR 1	PAR 2	TRUNC. PT T	EXCESS PROP XP	WGT	INDEX
CLASS1 1	215.0000	.0050	1.0000	2	23640.0000	1.4040	1000	.0000	.2500	1
CLASS1 2	215.0000	.0062	1.5000	2	18450.0000	1.2980	1000	.8560	.5000	0
CLASS1 3	215.0000	.0075	2.0000	2	18155.0000	1.1910	1000	.8380	.2500	0
CLASS2 1	77.0000	.0072	1.0000	2	23640.0000	1.4040	1000	.0000	.2500	1
CLASS2 2	77.0000	.0090	1.5000	2	18450.0000	1.2980	1000	.8560	.5000	0
CLASS2 3	77.0000	.0108	2.0000	2	18155.0000	1.1910	1000	.8380	.2500	0
CLASS3 1	66.0000	.0085	1.0000	2	23923.0000	1.4650	1000	.8560	.2500	1
CLASS3 2	66.0000	.0106	1.5000	2	20106.0000	1.2700	1000	.8860	.5000	0
CLASS3 3	66.0000	.0127	2.0000	2	20597.0000	1.1890	1000	.8950	.2500	0
CLASS4 1	10.0000	.0104	1.0000	2	22253.0000	1.4650	1000	.8560	.2500	1
CLASS4 2	10.0000	.0130	1.5000	2	21724.0000	1.2780	1000	.8860	.5000	0
CLASS4 3	10.0000	.0156	2.0000	2	21762.0000	1.1890	1000	.8950	.2500	0
CLASS5 1	46.0000	.0130	1.0000	2	27911.0000	1.4650	1000	.8560	.2500	1
CLASS5 2	46.0000	.0163	1.5000	2	25450.0000	1.2700	1000	.8860	.5000	0
CLASS5 3	46.0000	.0195	2.0000	2	24031.0000	1.1890	1000	.8950	.2500	0
CLASS6 1	35.0000	.0169	1.0000	2	23923.0000	1.4650	1000	.8560	.2500	1
CLASS6 2	35.0000	.0212	1.5000	2	20106.0000	1.2700	1000	.8860	.5000	0
CLASS6 3	35.0000	.0254	2.0000	2	20597.0000	1.1890	1000	.8950	.2500	0
CLASS7 1	51.0000	.0156	1.0000	2	30569.0000	1.4650	1000	.8560	.2500	1
CLASS7 2	51.0000	.0195	1.5000	2	25692.0000	1.2700	1000	.8860	.5000	0
CLASS7 3	51.0000	.0234	2.0000	2	26320.0000	1.1890	1000	.8950	.2500	0

and the assumption that there will be 500 doctors. Possible variance of the actual exposures from these estimates will be simply accounted for when selecting the low and high frequency estimates. The medium frequency (ground-up) estimates are derived in Appendix A, p. A1. They are based upon projections of overall countrywide doctor loss frequency at the mid-point (January 1, 1981) of coverage year fiscal 1980/81, modified by various offsets: 1) class, 2) state, 3) year in claims-made program (in this case, first year) and 4) contagion (multiple doctors per incident). It is necessary to use a contagion factor to adjust the basic ISO data, which are number of occurrences per doctor, since the treaty will cover loss per occurrence for all covered doctors added together. All the offsets are selected on the basis of ISO data and NAIC (1977, 1978) information. The low and high loss frequencies are selected to be ± 20% of the medium loss frequencies; this is pure judgement to reflect the uncertainty regarding the actual exposure and the "true" expected frequency per class. The ratio $\mathrm{Var}[N_i|\theta] \div E[N_i|\theta]$ values 1.0 (low), 1.5 (medium) and 2.0 (high), Table 5A, column (4), are selected on the basis of research by the ISO Increased Limits Subcommittee.

The parameters for the loss amount c.d.f.'s are in Table 5A, columns (5) - (9). The number 2 in column (5) specifies to the computer package RISKMODEL that the form of the c.d.f. is the 4-parameter modified Pareto distribution defined in Appendix D; the other choices are 1 = lognormal and 3 = Weibull. Columns (6) - (9) are its four parameters for each class and each θ. We and the ISO Increased Limits Subcommittee have found this general Pareto c.d.f. to be very useful for describing loss amount distributions. The particular parameters derived on Appendix A, pp. A2 and A3 are based upon ISO countrywide loss amount data and modified by various offsets (class, state and contagion) selected on the basis of other ISO data and NAIC (1977, 1979) information. Note that the offsets apply to the β parameter (PAR1) only. We do not presently offset according to year in claims-made program, although we might if we ever see any claims-made loss data sufficient for this purpose. The low, medium and high parameters are selected from c.d.f.'s fitted to five policy years of ISO data via the maximum likelihood techniques described in Patrik (1980) and are indeed the low, medium (all five years combined) and high c.d.f.'s.

Column (10) of Table 5A displays the subjective weights assigned to the three parameters. In this case, they are purely judgemental, with the medium parameter assigned a likelihood of 50% and the low and high parameters assigned 25% likelihoods.

VI. PARAMETER ESTIMATION: EXAMPLE B

The parameter estimation for example B, the exeess proposal for P&C Insurance Company's general liability coverage, will illustrate the case where there are credible historical loss data directly related to the exposure. In this case, we will use as much of the data as we can to select the homogeneous coverage groups, to estimate the forms of the loss amount c.d.f.'s and to estimate some θ's and $U(\theta)$'s (the loss count c.d.f.'s are assumed to be adequately modeled by negative binomial distributions). Recall from Section II that the proposal is for $750,000 excess of $250,000 and that the P&C Insurance Company has provided a detailed history of large losses (greater than $25,000), gross earned premiums, an overall rate history and more.

The steps of the procedure we will follow are:

1. Select the homogeneous coverage groups.
2. Decide which historical exposure years are most indicative of (can be easily adjusted to) next year's exposure.
3. Estimate loss amount inflationary trend factors.
4. Select a primary retention to directly evaluate loss count and amount distributions for the next coverage year and restrict attention to those

large losses whose trended values are greater than this retention. This retention is not necessarily the proposed retention, but is instead the one which we believe will yield the most credible estimates of the potential loss.

5. Decide how to adjust the large loss data to an ultimate settlement basis.
6. Estimate ground-up loss amount c.d.f.'s for the next coverage year, both forms and parameters, from the large loss data and general information.
7. Estimate the number of excess IBNR losses (excess of the deflated values of the selected retention (4)).
8. Estimate excess loss frequencies for the next coverage year.
9. Estimate base (ground-up) loss count c.d.f.'s for the next coverage year based upon (6), (8) and the estimated exposure.
10. Select the parameter weights $U(\theta)$.

The procedure for example B will follow this outline very cleanly. In practice, however, any of the steps may be reversed and any of the decisions may be changed later during the procedure if the analysis so indicates.

We decided not to display the complete P&C Insurance Company data in an appendix for three reasons:

1. We would like to focus on the general procedure, not all the details. Most of the detailed steps could be done in many different ways.
2. The data are voluminous.
3. The data, used with the primary company's permission, should remain confidential.

Many summary exhibits are displayed in Appendix B.

Step 1

The groups are defined by the major policy limits based upon the policy limits listed on the large loss records and P&C Insurance Company's estimate of their policy limits distribution for 1980. However, the general liability coverage will be analyzed as a whole; thus, the parameters of the estimated ground-up loss amount c.d.f.'s and the loss frequencies will be the same for each group - only the policy limits and the underlying exposure will be different. The complete parameter matrix which will later be input to the RISKMODEL computer package is displayed in Table 6A. In this case, there are four policy limit groups: \$200,000, \$250,000, \$350,000 and \$500,000 or more; there are four parameters θ: the first is the combination of the first row for each group, and so on.

TABLE 6A

GROUP	EXPOSURE	FREQUENCY	V[N]+E[N]	DST CODE	PAR 1	PAR 2	TRUNC. PT T	EXCESS PROP XP	WGT	INDEX
GL/200 1	1175.0000	.0108	1.5000	2	124016.0000	3.6795	0	1.0000	.1000	1
GL/200 2	1175.0000	.0135	2.0000	2	89251.0000	3.1290	0	1.0000	.4000	0
GL/200 3	1175.0000	.0096	1.5000	2	130747.0000	3.8769	0	1.0000	.1500	0
GL/200 4	1175.0000	.0104	2.0000	2	130693.0000	3.7558	0	1.0000	.3500	0
GL/250 1	1175.0000	.0108	1.5000	2	124016.0000	3.6795	0	1.0000	.1000	1
GL/250 2	1175.0000	.0135	2.0000	2	89251.0000	3.1290	0	1.0000	.4000	0
GL/250 3	1175.0000	.0096	1.5000	2	130747.0000	3.8769	0	1.0000	.1500	0
GL/250 4	1175.0000	.0104	2.0000	2	130693.0000	3.7558	0	1.0000	.3500	0
GL/350 1	2350.0000	.0108	1.5000	2	124016.0000	3.6795	0	1.0000	.1000	1
GL/350 2	2350.0000	.0135	2.0000	2	89251.0000	3.1290	0	1.0000	.4000	0
GL/350 3	2350.0000	.0096	1.5000	2	130747.0000	3.8769	0	1.0000	.1500	0
GL/350 4	2350.0000	.0104	2.0000	2	130693.0000	3.7558	0	1.0000	.3500	0
GL/500+ 1	18800.0000	.0108	1.5000	2	124016.0000	3.6795	0	1.0000	.1000	1
GL/500+ 2	18800.0000	.0135	2.0000	2	89251.0000	3.1290	0	1.0000	.4000	0
GL/500+ 3	18800.0000	.0096	1.5000	2	130747.0000	3.8769	0	1.0000	.1500	0
GL/500+ 4	18800.0000	.0104	2.0000	2	130693.0000	3.7558	0	1.0000	.3500	0

Step 2

We restrict our attention to the large loss data from accident years 1973 through 1978 since we believe that these data are more easily adjustable to 1980 level in a reasonable manner. Also, there does not appear to be any significant development of loss counts or amounts beyond the 78 month evaluations of the data presented in P&C's June 30, 1975, . . ., June 30, 1979 loss evaluations. With this decision, we still have quite enough data, over 200 large losses, to analyze.

Step 3

Many different loss amount inflationary trend models may be developed using many different economic and actuarial assumptions. We shall use two very simple models:

1. Exponential trend model: ISO general liability bodily injury average loss amounts of various kinds from the past several years may be fit by exponential curves in the usual manner. In this case, our model produces an annual trend estimate of 16.8%.

2. Econometric trend model: Slightly more sophisticated trend estimates are derived via a primitive but reasonable econometric model using the Bureau of Labor Statistics' Consumer Price Index and its Medical Care Services component as independent variables and some ISO loss amount

index as the dependent variable. The trend factors to adjust each accident year's data to 1980 level are displayed in Appendix B, p. B2, column (1).

Loss parameters will be derived separately from the two sets of data adjusted by these two trend models. In general, use as many reasonable trend models as possible and assign subjective weights to them.

Step 4

Our objective is to estimate 1980 ground-up loss amount and loss count c.d.f. models which produce accurate estimates of the losses in the layer $400,000 excess of $100,000. However, to estimate these models, it is not necessary to restrict our attention to only those historical losses whose 1980 level values are greater than $100,000. With the exponential and econometric trend models, a 1980 retention of $75,000 deflates to 1973 values of $25,291 and $25,299, respectively (see column (2) of Appendix B, pp. B1 and B2). Since these deflated values are larger than $25,000, the 1973 - 78 large loss data contain all known losses whose 1980 values are larger than $75,000. Furthermore, more credible excess frequency and loss amount estimates may be obtained from evaluating a lower retention of $75,000. That is, there are 171 (exponential) and 158 (econometric) known losses whose 1980 values are greater than $75,000 (see Appendix B, pp. B1 and B2), while only 109 (exponential) and 104 (econometric) have 1980 values greater than $100,000. Therefore, we restrict our attention to those large losses whose 1980 level values are greater than $75,000. The 1980 level average values and number of occurrences at each evaluation date are shown in Appendix B, pp. B1 and B2.

Step 5

For each historical coverage year, we want an estimate of the distribution of ultimate settlement values (1980 level) of losses greater than $75,000. The age-to-age development factors displayed in Appendix B, pp. B1 and B2, for the 1980 level average values indicate that the large loss distribution for the recent years will change as more losses pierce the retention and as the losses are settled. Thus, these data must be adjusted. In this case we observe that the loss amount distribution appears to develop little beyond the 42 month evaluation. Also, the two years for which we can expect the data to substantially develop, 1977 and 1978, have only 14 and 3 large losses respectively. Thus, in this case we choose to use multiplicative average size development factors applied to the large loss values. These factors are displayed in Appendix B, pp. B1 and B2. (For a more sophisticated approach, which simultaneously accounts for the development of loss counts and amounts, see Hachemeister (1976)).

Step 6

The 1980 loss amount c.d.f.'s are derived from four data sets by using the maximum likelihood estimation techniques and testing procedures described in Patrik (1980). The data sets are:

1. The large losses together with their policy limits adjusted to 1980 level via the exponential trend model and developed to ultimate settlement.
2. Same as (1) except that the losses and policy limits are censored at (limited to) $500,000.
3. The large losses together with their policy limits adjusted to 1980 level via the econometric trend model and developed to ultimate settlement.
4. Same as (3) except that the losses and policy limits are censored at $500,000.

Censorship at $500,000 is used in (2) and (4) for two reasons:

1. The proposed reinsurance layer stops at $500,000. Thus, we may focus upon the loss amount distribution below $500,000.
2. In general, we have found that censored (by policy limits) loss amount c.d.f.'s estimated via the method of maximum likelihood fit better when there are some losses at the censorship points: the parameter estimates appear to have smaller sample error.

However, the data in this case have no losses at
their policy limits.

The parameters for c.d.f.'s (1) - (4) are displayed in Table 6A, columns (5) - (9). Both the Kolmogorov-Smirnov Test and an "actuarial ad-hoc expected value test" (see Patrik (1980)) show the Pareto model fitting much better than either the lognormal or the Weibull models. Thus, each selected c.d.f. is Pareto (column (5) entry is 2). The column (8) and (9) entries are selected for convenience to be 0 and 1, respectively, because we are not concerned with the lower end of the loss amount distribution. See Appendix D and note that if XP = 1, then the four parameter model reduces to a two parameter model with the parameters PAR1 and PAR2 in Table 6A, columns (6) and (7). C.d.f.'s (2) and (4) fit well, while the fit of (1) and (3) is only fair. This information will be used later when selecting the subjective likelihoods (weights) of the parameters.

<u>Step 7</u>

The number of IBNR (incurred but not reported) 1980 level losses excess of $75,000 for each year 1973, . . ., 1978 are estimated using a method developed by James Stanard and described in Patrik (1978). The first step is to estimate a c.d.f. model for the distribution of report lags. In this case, the report lag is defined as the time in months between the date of occurrence of a loss and the date its 1980 level incurred value first exceeds \$75,000. Weissner (1978) showed how to estimate this c.d.f. using the method of maximum likelihood when the data include month of occurrence and month of report for every loss. However in this case, such detail is not available: the data have only year of occurrence (accident) and year of report. Thus, we select a report lag c.d.f. model by comparing the actual number of occurrence age-to-age factors in Appendix B, pp. B1 and B2, to tables of annual age-to-age factors generated by various theoretical report lag distributions, such as the exponential, lognormal or Weibull. In this case, a Weibull distribution with parameters $\beta = 34.0$ and $\delta = 2.75$ (see Appendix D) appears to describe both sets of actual age-to-age factors best; so we will use it to calculate IBNR. The annual age-to-age factors generated by this Weibull are the row underlined in the table in Appendix B, p. B3. The IBNR calculations are displayed in Appendix B, pp. B4 and B5.

Step 8

Appendix B, pp. B6 and B7, displays the estimated IBNR per year (column (4)) and the implied 1980 level frequency excess-of-\$75,000 per year (column (6)) with respect to gross direct earned premium at present (1980) rate level (column (2)). Columns (7) and (8) display our estimates of the 1980 level base frequency per year. We use the term "base frequency" to distinguish these numbers from the true ground-up loss frequency. The base frequencies are slightly fictitious numbers derived solely as convenient input for the RISKMODEL computer package (table 6A, column (3)). They are interpolated downward from the excess frequencies by use of the previously selected loss amount c.d.f. models. For example, the base frequency of .0108 for 1973 in column (7) of Appendix B, p. B6, is derived from the excess frequency of .0019 in column (6) via:

$$
\begin{aligned}
& (\text{excess frequency}) \div \operatorname{Prob}[X>\$75{,}000 \mid \text{c.d.f.}(1)] \\
(6.1) \qquad & = (.0019) \div \left(\frac{\beta}{\beta + 75{,}000}\right)^{\delta} \\
& = .0108
\end{aligned}
$$

where $\beta = 124{,}016$ and $\delta = 3.6795$.

The base frequencies with respect to all four loss amount c.d.f.'s are displayed in Appendix B, pp. B6 and B7, along with four selected values which are input in Table 6A, column (3).

Step 9

The negative binomial c.d.f. is selected as the general form for the distribution of N_i, the number of 1980 base losses for policy limit group i. The expected value for each particular c.d.f. is the base frequency times the estimate of the 1980 gross direct earned premium in Table 6A, column (2). The ratios $\operatorname{Var}[N_i|\theta] \div E[N_i|\theta]$ in column (4) are again selected on the basis of research by the ISO Increased Limits Subcommittee.

Step 10

The parameter weights $U(\theta)$ in Table 6A, column (10), are selected on the following basis:

1. Each trend model is given weight .50.
2. The weight selected for loss amount c.d.f. (2) together with its implied base frequency is .40 (out of .50 possible) since it fit best; the remaining .10 goes to c.d.f. (1). Likewise, loss amount c.d.f. (4) together with its implied base frequency is given a weight of .35 because of its good fit, with the remaining .15 going to c.d.f. (3).

As a final remark on the parameter estimation for example B, it should be apparent that if we believe that the P&C large loss data is not fully credible, then we can append more parameters based upon general industry information as in example A. The parameter weights would be adjusted accordingly, perhaps via some credibility procedure.

VII. MOMENTS AND PERCENTILES OF THE DISTRIBUTION OF AGGREGATE LOSS

This section describes a computer package named RISKMODEL which takes information such as in Tables 5A and 6A and transforms it into moments and percentiles of the distribution of aggregate loss for any selected mixture of loss layers. Tables 7A, 5A and 7B-7D document a RISKMODEL run for example A; the run for example B is contained in Appendix C and Tables 6A and 7E. In both cases the printout displays both the package interrogatories and the user's input. Almost complete runs are displayed so that the reader can see how easily the complicated model formulas translate into a working computer package; the only parts eliminated are the step-by-step data input process and some ending details regarding further displays and memory storage.

Table 7A displays the beginning of the RISKMODEL run for example A. The user enters the group names "class 1, class 2, . . ., class 7", specifies that there will be three parameters and indicates that he wants the limits matrix LIM in the package to be assigned the elements of a previously created matrix LIMA. Since the proposed coverage is $750,000 excess of $250,000, the loss layers we want to consider are 0 - $250,000 and $250,000 - $1,000,000; we observe the output for the lower layer to provide an extra check on the reasonableness of the output for the excess layer. For each group (class), there are two rows with lower and upper limit columns and a third column, INDEX, which indicates when there is a change in group.

The user next specifies that he wants the parameter matrix PAR in the package to be assigned the elements of a previously

TABLE 7A

```
      RICKMODEL
DO NOT PANIC IF YOU MAKE AN ERROR WHILE INPUTTING.
OPPORTUNITY TO CHANGE LATER.

ENTER MAJOR GROUP NAMES AS FOLLOWS:   /GRP1/GRP2..........
NOTE: MUST BE IN QUOTES.   FOR MORE THAN 1 LINE OF INPUT, USE ,D
D:
      '/CLASS1/CLASS2/CLASS3/CLASS4/CLASS5/CLASS6/CLASS7'

ENTER THE NUMBER OF PARAMETERS,  E.G.   5
D:
      3

DO YOU WISH TO  (1) INPUT VECTOR OF LIMITS,  OR
(2) USE MATRIX OF LIMITS PREVIOUSLY CREATED. 1 OR 2.
D:
      2
ENTER THE NAME OF THE MATRIX OF LIMITS PREVIOUSLY CREATED
NOTE: NAME SHOULD HAVE PREFIX LIM
LIMA

DO YOU WISH TO SEE THE LIM MATRIX.  Y OR N
Y
                        LIMITS
             LOWER              UPPER              INDEX
                 0             250000                  1
            250000            1000000                  0
                 0             250000                  1
            250000            1000000                  0
                 0             250000                  1
            250000            1000000                  0
                 0             250000                  1
            250000            1000000                  0
                 0             250000                  1
            250008            1000000                  0
                 0             250000                  1
            250000            1000000                  0
                 0             250000                  1
            250000            1000000                  0

DO YOU WISH TO MAKE ANY CHANGES IN THE LIM MATRIX.  Y OR N
N
DO YOU WISH TO
(1) INPUT VECTOR OF PARAMETERS FOR THE FIRST SUBGROUP  OR
(2) USE MATRIX OF PARAMETERS PREVIOUSLY CREATED.  1 OR 2
D:
      2
ENTER THE NAME OF THE MATRIX OF PARAMETERS PREVIOUSLY CREATED
NOTE: NAME SHOULD HAVE PREFIX PAR
PARA

DO YOU WISH TO SEE THE PAR MATRIX.  Y OR N
Y
```

created matrix PARA. The parameter matrix was displayed in Table 5A.

Table 7B continues the run after the display of the parameter matrix PAR. Next displayed is a matrix of intermediate calculations for layer 1: 0 - \$250,000. The notation here is:

(7.1) A = layer lower bound (here $A = 0$)

B = layer upper bound (here $B = 250{,}000$)

S = ground-up loss amount random variable

$$P[S>A] = 1-G_i(A|\theta) \qquad \text{for each group } i \text{ for each } \theta$$

$$P[S>B] = 1-G_i(B|\theta) \qquad \text{for each group } i \text{ for each } \theta$$

$$E[S*m] = \int_A^B x^m \, dG_i(x|\theta) \qquad \text{for each group } i \text{ for each } \theta$$

where $m = 1, 2, 3$

$$G_i(x|\theta) = \text{Prob}[S_i \leq x|\theta]$$

These values will be used to calculate the moments of the aggregate loss L given θ by using formula (4.7). They are displayed so that the user can check that the run is going alright.

Table 7C continues the run with a display of a matrix of intermediate calculations for layer 2: \$250,000 - \$1,000,000. These are similar to those for layers 1 except that here $A = 250{,}000$ and $B = 1{,}000{,}000$. Next input are the selected c's

TABLE 7B

DO YOU WISH TO MAKE ANY CHANGES IN THE PAR MATRIX. Y OR N
N

GROUPS AND PARAMETER INPUT COMPLETED
TO PROCESS INTERMEDIATE CALCULATIONS, HIT EXECUTE

DO YOU WISH TO PRINT THE INTERMEDIATE CALCULATIONS,
P[S>A],P[S>B],E[S],E[S*2],E[S*3]. Y OR N.
Y

INTERMEDIATE CALCULATIONS USED THROUGHOUT MOMENT CALCULATIONS

LAYER 1

GROUPS	P[S>A]	P[S>B]	E[S]	E[S*2]	E[S*3]
CLASS1 1	1.000	.021	2.216E04	1.813E09	2.415E14
CLASS1 2	1.000	.026	2.258E04	1.878E09	2.546E14
CLASS1 3	1.000	.034	2.363E04	2.065E09	2.868E14
CLASS2 1	1.000	.021	2.216E04	1.813E09	2.415E14
CLASS2 2	1.000	.026	2.258E04	1.878E09	2.546E14
CLASS2 3	1.000	.034	2.363E04	2.065E09	2.868E14
CLASS3 1	1.000	.024	2.392E04	1.985E09	2.660E14
CLASS3 2	1.000	.032	2.500E04	2.154E09	2.960E14
CLASS3 3	1.000	.042	2.717E04	2.459E09	3.456E14
CLASS4 1	1.000	.026	2.478E04	2.093E09	2.824E14
CLASS4 2	1.000	.034	2.585E04	2.261E09	3.124E14
CLASS4 3	1.000	.044	2.802E04	2.571E09	3.632E14
CLASS5 1	1.000	.030	2.642E04	2.304E09	3.146E14
CLASS5 2	1.000	.038	2.745E04	2.468E09	3.444E14
CLASS5 3	1.000	.050	2.962E04	2.787E09	3.973E14
CLASS6 1	1.000	.024	2.392E04	1.985E09	2.660E14
CLASS6 2	1.000	.032	2.500E04	2.154E09	2.960E14
CLASS6 3	1.000	.042	2.717E04	2.459E09	3.456E14
CLASS7 1	1.000	.033	2.795E04	2.507E09	3.463E14
CLASS7 2	1.000	.043	2.893E04	2.667E09	3.756E14
CLASS7 3	1.000	.055	3.110E04	2.994E09	4.303E14

TABLE 7C

LAYER 2

GROUPS	P[S>A]	P[S>B]	E[S]	E[S*2]	E[S*3]
CLASS1 1	.021	.003	8.015E03	4.093E09	2.431E15
CLASS1 2	.026	.005	9.754E03	5.103E09	3.097E15
CLASS1 3	.034	.007	1.227E04	6.522E09	4.014E15
CLASS2 1	.021	.003	8.015E03	4.093E07	2.431E15
CLASS2 2	.026	.005	9.754E03	5.103E09	3.097E15
CLASS2 3	.034	.007	1.227E04	6.522E09	4.014E15
CLASS3 1	.024	.003	9.025E03	4.622E09	2.753E15
CLASS3 2	.032	.006	1.176E04	6.177E09	3.763E15
CLASS3 3	.042	.009	1.513E04	8.059E09	4.967E15
CLASS4 1	.026	.004	9.701E03	4.973E09	2.964E15
CLASS4 2	.034	.006	1.253E04	6.589E09	4.016E15
CLASS4 3	.044	.009	1.605E04	8.556E09	5.276E15
CLASS5 1	.030	.004	1.108E04	5.690E09	3.396E15
CLASS5 2	.030	.007	1.409E04	7.419E09	4.527E15
CLASS5 3	.050	.010	1.790E04	9.551E09	5.897E15
CLASS6 1	.024	.003	9.025E03	4.622E09	2.753E15
CLASS6 2	.032	.006	1.176E04	6.177E09	3.763E15
CLASS6 3	.042	.009	1.513E04	8.059E09	4.967E15
CLASS7 1	.033	.005	1.248E04	6.423E09	3.841E15
CLASS7 2	.043	.008	1.566E04	8.258E09	5.045E15
CLASS7 3	.055	.011	1.973E04	1.055E10	6.520E15

```
TO PROCESS MORE INTERMEDIATE CALCULATIONS, HIT EXECUTE

ENTER EPSILON(S) FOR WHICH PROB(LOSS>MAX. PROB. LOSS) = EPSILON. (0<ε≤.5)
⎕:
      .1 .05 .01
NOW FOR THE FINAL PRINTOUT

ENTER COMPANY NAME
EXAMPLE A:   A DOCTORS MUTUAL INSURANCE COMPANY

ENTER YOUR NAME (EG. J. SMITH)
HOWARD H. FRIEDMAN

ENTER TODAY'S DATE (EG. JAN. 1, 1979)
APRIL 1, 1980

ENTER IN PARENTHESIS AND QUOTES A SEVEN CHARACTER NAME FOR THE UNITS
(E.G. '(DOCTORS)' OR '__(BEDS)_')
⎕:                                        OF EXPOSURE CENTERED IN 9 SPACES
      '(DOCTORS)'

ADJUST PAPER TO TOP OF NEW PAGE & HIT EXECUTE
```

TABLE 7D

EXAMPLE A: A DOCTORS MUTUAL INSURANCE COMPANY

LAYER 1

GROUPS	LIMITS LOWER ($ 000)	LIMITS UPPER ($ 000)	EXPOSURE (DOCTORS)	EXPECTED NUMBER OF LOSSES	EXPECTED LOSS ($)	STANDARD DEVIATION ($)	COEFF. OF SKEWNESS	MAXIMUM PROBABLE LOSS ONE IN 10.0 YEARS ($)	MAXIMUM PROBABLE LOSS ONE IN 20.0 YEARS ($)	MAXIMUM PROBABLE LOSS ONE IN 100.0 YEARS ($)
CLASS1	0	250	215.000	1.34	39,072	75,227	2.993	160,419	227,692	380.569
CLASS2	0	250	77.000	.69	20,626	53,964	4.128	113.657	172.766	310.046
CLASS3	0	250	66,000	.70	23,634	59,345	3.917	124,600	187,304	332.704
CLASS4	0	250	10,000	.13	4,500	26,204	8.026	45,800	91,600	236.386
CLASS5	0	250	46.000	.75	20,357	66,755	3.606	139,709	206,652	360.754
CLASS6	0	250	35,000	.74	25,015	61.076	3.810	128,230	191,699	338.324
CLASS7	0	250	51.000	1.00	40,319	80,936	3.055	170,547	243,018	410.534
TOTALS			500.000	5.35	182,404	169.614	1.513	427,306	534.476	765.907

LAYER 2

GROUPS	LIMITS LOWER ($ 000)	LIMITS UPPER ($ 000)	EXPOSURE (DOCTORS)	EXPECTED NUMBER OF LOSSES	EXPECTED LOSS ($)	STANDARD DEVIATION ($)	COEFF. OF SKEWNESS	MAXIMUM PROBABLE LOSS ONE IN 10.0 YEARS ($)	MAXIMUM PROBABLE LOSS ONE IN 20.0 YEARS ($)	MAXIMUM PROBABLE LOSS ONE IN 100.0 YEARS ($)
CLASS1	250	1000	215.000	.04	11,003	78,129	8.493	110,030	221.660	600.947
CLASS2	250	1000	77.000	.02	5,733	54,155	11.790	57,334	114,668	573,338
CLASS3	250	1000	66.000	.02	7,061	62,591	10.637	70,611	141.222	643,414
CLASS4	250	1000	10.000	.00	1,403	27,904	23 894	14,032	28,064	140.322
CLASS5	250	1000	46.000	.03	9,103	71,249	9.414	91,030	102,059	668,333
CLASS6	250	1000	35.000	.02	7,475	64,405	10.359	74,752	149,503	648.163
CLASS7	250	1000	51.000	.04	13,508	86,970	7.750	135,081	270.163	711,734
TOTALS			500.000	.18	55,367	176,305	3.862	354,204	539.083	966.439

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(.10, .05, .01) for the aggregate loss distribution percentiles. In the package, the 1 - ε percentile, L_ε, the point which L has subjective probability ε of exceeding, is called "the maximum probable loss for one in 'ε^{-1}' years". This wording was chosen to be more meaningful to the underwriters who see the main output.

The main output is displayed in Table 7D. Various information about the distribution of aggregate loss for each layer is shown. The display should be self-explanatory to actuaries. Note for example, the amount of "risk" being assumed by the reinsurer as evidenced by the coefficient of skewness: 1.513 for the primary layer versus 3.862 for the excess layer. Or, notice the coefficients of variation: .930 (169,614 ÷ 182,404) for the primary layer versus 3.184 (176,305 ÷ 55,367) for the excess layer. Approximations of the aggregate loss percentiles are in the last three columns.

There are many methods for approximating the percentiles of a distribution. The method used by RISKMODEL is the NP-approximation described by Beard, Pentikäinen and Pesonen (1969 - 2nd ed., 1977). This approximation is given by:

$$(7.2) \qquad L_\varepsilon \doteq E[L] + (Var[L])^{\frac{1}{2}} \cdot \{z_\varepsilon + \frac{\gamma_1}{6} \cdot (z_\varepsilon^2 - 1)\}$$

where L_ε is minimal such that $\text{Prob}[L>L_\varepsilon] \leq \varepsilon$

$z_\varepsilon = \Phi^{-1}(1-\varepsilon)$ for Φ the standard normal (0,1) c.d.f.

$\gamma_1 = \mu_3(L) \div (\text{Var}[L])^{3/2}$, the coefficient of skewness.

A problem with the NP-approximation is that if γ_1 is very large (say $\gamma_1 > 8$), then for certain values of ε, the approximation is much too large. However, there is a natural bound on L_ε which RISKMODEL uses to bound the NP-approximation. This bound is:

$$L_\varepsilon \leq \varepsilon^{-1} \cdot E[L] \tag{7.3}$$

The necessity of this Chebyshev-like bound is seen immediately from:

$$\begin{aligned} E[L] &= \int_0^\infty x \cdot dF(x) && \text{since } F(x) = 0 \text{ for } x < 0 \\ &\geq \int_{L_\varepsilon}^\infty x \cdot dF(x) && \text{since } L_\varepsilon \geq 0 \\ &\geq \int_{L_\varepsilon}^\infty L_\varepsilon \cdot dF(x) \\ &= \varepsilon \cdot L_\varepsilon \end{aligned} \tag{7.4}$$

The extreme values of γ_1, which trigger this bound on the NP-approximation seem to occur only when the expected number of loss occurrences is very small. For example, the bound occurs in the example A main output, Table 7D, for the excess layer for each individual class when ε = .10, .05 and sometimes .01; in each case, the expected number of excess losses is less than .05. It does not happen for the overall excess layer where the expected number of losses is .18.

Thus, in certain extreme situations, the NP-approximation may not be very accurate. In fact, there has been quite a discussion in the recent literature regarding the accuracy of the NP-approximation versus its various alternatives. The reasonable alternatives presently include: 1) approximation via simulation, 2) an NP3-approximation which uses the fourth moment of L in addition to the first three and 3) approximation via the 3-parameter gamma distribution. See the argument carried on in Kauppi and Ojantakanen (1969), Seal (1977), Pentikäinen (1977) and Seal (1979) and also the discussion in Cummins and Freifelder (1978).

The reasons to use the NP-approximation are:

1. it is easier to compute than any of its reasonable alternatives.
2. in most situations, it is just as good.
3. it is slightly conservative; that is, L_ε is less than the NP-approximation.

In particular, it is as good as the alternatives for the usual excess-of-loss casualty working cover situation. Beard, Pentikäinen and Pesonen (1977), p. 5, said it well: "Thus it is important not to develop mathematical tools of disproportionate accuracy (and complication) without regard to the context in the problem being solved".

The example B run, Appendix C, has four policy limit groups and four parameters (see p. C1). The reason for grouping by policy limit should be obvious. Again, the limits and parameter matrices have been previously input. Since the proposed coverage is \$400,000 excess of \$100,000, the loss layers of interest are 0 - \$100,000 and \$100,000 - min {\$500,000, policy limit}. The parameters, Table 6A, were discussed in detail in Section VI. The intermediate calculations and the ϵ selection (pp. C2 and C3) are analogous to example A.

The main output is displayed in Table 7E. Again note the "risk" being assumed by the reinsurer as evidenced by the coefficient of skewness: .216 for the primary layer versus .437 for the excess layer. Or again notice the coefficients of variation: .129 (1,247,991 ÷ 9,678,618) for the primary layer versus .287 (641,998 ÷ 2,238,766) for the excess layer. Note that there is much less uncertainty in example B than there was for example A. Since we are using "base frequencies" as explained in Section VI, the expected number of losses in layer 1 are probably understated; the expected loss in layer 1 may also be understated. The estimates for layer 2 have no known systematic bias.

TABLE 7E

EXAMPLE D: P&C INSURANCE COMPANY-GENERAL LIABILITY

LAYER 1

GROUPS	LIMITS LOWER ($ 000)	LIMITS UPPER ($ 000)	EXPOSURE (000)	EXPECTED NUMBER OF LOSSES	EXPECTED LOSS ($)	STANDARD DEVIATION ($)	COEFF. OF SKEWNESS	MAXIMUM PROBABLE LOSS ONE IN 10.0 YEARS ($)	MAXIMUM PROBABLE LOSS ONE IN 20.0 YEARS ($)	MAXIMUM PROBABLE LOSS ONE IN 100.0 YEARS ($)
GL/200	0	100	1175.000	13.58	483,931	219,139	.662	780,360	885,753	1,100,600
GL/250	0	100	1175.000	13.59	483,931	219,139	.662	780,360	885,753	1,100,600
GL/350	0	100	2350.000	27.17	967,862	314,939	.481	1,207,750	1,529,073	1,812,045
GL/500+	0	100	18800.000	217.33	7,742,894	1,070,248	.227	9,140,652	9,572,683	10,411,589
TOTALS			23500.000	271.66	9,678,618	1,247,991	.216	11,307,066	11,808,457	12,780,606

LAYER 2

GROUPS	LIMITS LOWER ($ 000)	LIMITS UPPER ($ 000)	EXPOSURE (000)	EXPECTED NUMBER OF LOSSES	EXPECTED LOSS ($)	STANDARD DEVIATION ($)	COEFF. OF SKEWNESS	MAXIMUM PROBABLE LOSS ONE IN 10.0 YEARS ($)	MAXIMUM PROBABLE LOSS ONE IN 20.0 YEARS ($)	MAXIMUM PROBABLE LOSS ONE IN 100.0 YEARS ($)
GL/200	100	200	1175.000	1.46	77,023	79,995	1.224	190,046	236,486	335,192
GL/250	100	250	1175.000	1.46	91,014	100,994	1.352	235,895	296,823	427,286
GL/350	100	350	2350.000	2.92	213,774	180,223	1.123	466,456	567,854	782,016
GL/500+	100	500	18800.000	23.37	1,856,156	600,305	.486	2,656,854	2,926,809	3,467,655
TOTALS			23500.000	29.21	2,238,766	641,998	.437	3,091,686	3,374,779	3,938,912

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VIII. CONCLUSION

We have described a procedure for estimating the distribution of the aggregate loss for the next coverage year of an excess-of-loss casualty working cover reinsurance treaty. Recall that for both treaty proposals, for each individual loss the reinsurer shares the allocated loss adjustment expense (ALAE) pro rata according to his share of the loss (the reinsurer's unallocated loss adjustment expense is included in his general overhead expense). The ALAE share increases the reinsurer's aggregate loss by 3% to 6% depending upon the line of business and the excess layer. For both examples, we will increase all aggregate loss figures by 5%.

According to the list in Section III, there are four more general items to consider before deciding about the adequacy of the rate offered on example A or before proposing a rate for example B. Without offering complete, elegant solutions, let us briefly consider those items (2) - (4).

Item (2) is the potential distribution of cash flow. Both proposals are fairly typical excess-of-loss casualty working covers which we may assume will have standard monthly or quarterly premium payment patterns and typical long tail casualty loss payout patterns. That simple general cash flow models can be constructed should surprise no one who has read the CAS exam materials. In the long run, such general models should be constructed so that any two treaty proposals can be compared to each other. However, even without such models explicitly set up, we can say something about these two treaty proposals. For instance, based upon typical medical malpractice claims-made loss payment patterns, the one year aggregate loss expected values or higher percentiles for example A could be discounted from 10% to 15% on a present value basis with respect to rates-of-return on investments of 5% or greater. Based upon typical casualty loss payment patterns, the discount for example B would be 10% to 20%. The present values of the premium payments for both examples would be discounted around 5%. How this is viewed by the reinsurer depends upon items (3) - (5).

Item (3) is the collection of the reinsurer's various corporate financial parameters and decision-making criteria. Assuming that the reinsurer is at least moderate sized and is in good financial condition, then neither proposal in isolation leads to overwhelmingly complex decision problems; there is nothing unusual or very exciting here. It is highly unlikely that either treaty by itself could hurt such a reinsurer very much. However, the loss results of a whole portfolio of typical medical malpractice treaties, for example, would be correlated and could hurt a lot if priced badly.

Item (4) is the surplus necessary to "support" a treaty from the reinsurer's point-of-view and item (5) is the potential distribution of the rate-of-return on this "supporting surplus".

These are very ambiguous but we believe useful concepts. Strictly for illustration, let us define an ad-hoc measure of supporting surplus for our two treaty examples. In each case, we will consider the supporting surplus to be the difference of the 90th percentile of the distribution of aggregate loss and ALAE minus the pure premium (that part of the premium available to pay losses).

The A Doctors' Mutual Insurance Company proposal, example A, is expected to be profitable to the reinsurer based upon the 1980/81 expected aggregate loss of \$55,367 in the layer \$750,000 excess of \$250,000 (Table 7D) and an expected net reinsurance premium of \$115,248 (Appendix A, p. A3). But the 90th percentile of the reinsurer's subjective distribution of aggregate loss is \$354,284 (Table 7D), over three times the net premium. This is very risky, and our ad-hoc supporting surplus is (1.05 x \$354,284) - (.97 x \$115,248) = \$260,208 (take 3% out of the net premium for overhead expenses). The expected rate-of-return on this supporting surplus is 21% (((.97 x \$115,248) - (1.05 x \$55,367)) ÷ \$260,208). The reinsurer's decision to accept or reject the proposal would be based upon his attitude toward risk and upon the extra premium he wants for assuming such risk.

Example B could be profitable to the reinsurer if he can negotiate a reasonable net rate with the P&C Insurance Company. Exactly what the final rate will be depends upon the two companies' attitudes toward risk, their separate evaluations of the loss potential, the rates that are available for such coverage in the reinsurance marketplace and finally the amount of premium that P&C is collecting from his insureds for the layer $400,000 excess of $100,000. A quick check of the ISO increased limits factors for state B for this coverage, i.e., the premises/operations bodily injury table B (ISO Subline Code 314), indicates that about 15% of P&C's gross general liability premium is collected for this layer. Since the expected excess aggregate loss is $2,238,766 (Table 7E) and the expected gross direct earned premium is $23,500,000 (Table 7E total exposure), there is room to negotiate.

Purely for illustration, suppose that a flat net rate of 12% is negotiated for example B. Then the reinsurer's premium is .12 x $23,500,000 = $2,820,000 and his pure premium is .97 x $2,820,000 = $2,735,400. The 90th percentile of the reinsurer's subjective distribution of aggregate loss is $3,091,686, so our ad hoc supporting surplus is (1.05 x $3,091,866) - $2,735,400 = $511,059. The expected rate-of-return on this supporting surplus is 75% (($2,735,400 - 1.05 x $2,238,766) ÷ $511,059).

If the insurer and the reinsurer disagree strongly on the loss potential, the rate could be negotiated to include a profit commission arrangement by which they would share good years and bad years fairly. Reinsurance contract wording is often very inventive; treaties are custom-made for the particular situation;

the terms are adjusted to suit both parties. This is an example of a fundamental principle of reinsurance: reinsurance works best when it is a long term beneficial partnership between the parties.

We hope you noticed that the models, estimation techniques and decision procedures presented in this paper are not really specific to excess-of-loss reinsurance. They may be useful for pricing any large casualty contracts; with suitable modifications, they are useful for property insurance also. You may have noticed that we have presented no cookbook formulas for pricing reinsurance; the area is too rich in diversity and too interesting for such simplistic nonsense. We consider the work described here as only the beginning of a truly satisfying pricing procedure.

We close by noting that the Bibliography contains some papers on excess reinsurance pricing in addition to those previously mentioned. You will find most of these to be informative and interesting.

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APPENDIX A

EXAMPLE A: A DOCTORS' MUTUAL INSURANCE COMPANY

Parameter Selection

(1) Doctor Class	(2) Frequency Offset	(3) Medium Frequency	(4) Severity Offset	(5) Low β	(6) Medium β	(7) High β
1	.90	.0062	1.00	23,640	18,450	18,155
2	1.30	.0090	1.00	23,640	18,450	13,155
3	.65	.0106	.90	23,923	20,106	20,597
4	.80	.0130	.95	25,253	21,224	21,742
5	1.00	.0163	1.05	27,911	23,458	24,031
6	1.30	.0212	.90	23,923	20,106	20,597
7	1.20	.0195	1.15	30,569	25,692	26,320

(1) ISO old class plan.

(2) Selected on the basis of ISO data; the class 1, 2 countrywide mean frequency is selected to be .0385 and the class 3 - 7 countrywide mean frequency is selected to be .0904 for 1/1/81.

(3) The state A frequency offset is selected to be .90; the first year claims-made offset is selected to be .25; the contagion (multiple doctors per incident) is selected to be .80. Together with col. (2), these offset the countrywide mean frequencies in note (2). For example A, the low and high frequencies are selected to be $\pm$ 20% of the medium frequencies.

(4) Selected on the basis of ISO data.

(5)- (7) The state A severity offset is selected to be .70; the contagion offset is selected to be 1.25. Together with col. (4), these offset the countrywide β parameters on p.A2.

APPENDIX A

EXAMPLE A: A DOCTORS' MUTUAL INSURANCE COMPANY

General Loss Amount Distribution Model

		Countrywide Loss Amount Parameters: 1/1/81			
		β	δ	t	XP
Physicians	- low	27,017	1.484	1000	.808
(1, 2)	- medium	21,086	1.293	1000	.856
	- high	20,749	1.191	1000	.838
Surgeons	- low	30,378	1.465	1000	.856
(3 - 7)	- medium	25,531	1.278	1000	.886
	- high	26,155	1.189	1000	.895

The parameters are selected based upon ISO medical malpractice data via maximum likelihood estimation - See Patrik (1980). The general loss amount c.d.f. is the 4-parameter Pareto described in Appendix D.

APPENDIX A

EXAMPLE A: **A DOCTORS'** MUTUAL INSURANCE COMPANY

Estimated Premium: 7/1/80 - 6/30/81

(1) Doctor Class	(2) # in Class	(3) 1980 1M/3M Rate	(4) 1980 1M/3M Premium
1	215	$ 400	$ 86,000
2	77	720	55,440
3	65	1,200	78,000
4	11	1,600	17,600
5	46	2,000	92,000
6	35	2,400	84,000
7	51	3,200	163,200
	500		$576,240

(1) These are older ISO doctor class plan.

(2) Based upon ISO doctor distribution and the estimate of 500 doctors.

(3) First year claims-made rates to be used by A Doctors' Mutual Insurance Company.

(4) The reinsurance net premium is .20 x $576,240 = $115,248.

EXAMPLE B: P&C INSURANCE COMPANY

Average Incurred (Ground-Up) and Occurrence Loss Development
Excess of $75,000 at 1980 Level as of 6/30/79

Exponential Trend Model

Trend Factor	Deflated Retention	Accident Year		Age in Months 18	30	42	54	66	78
2.966	$25,291	1973	Avg. $	NA	141,778	170,039	162,867	159,706	159,117
			#		19	30	45	44	46
2.539	29,540	1974	Avg. $	117,249	134,211	165,289	173,331	183,696	
			#	4	17	39	44	43	
2.174	34,502	1975	Avg. $	92,268	103,421	113,232	127,553		
			#	2	21	28	41		
1.861	40,299	1976	Avg. $	112,482	109,284	109,583			
			#	5	10	24			
1.593	47,069	1977	Avg. $	0	100,650				
			#	0	14				
1.364	54,976	1978	Avg. $	103,172					
			#	3					

Average Incurred Age-to-Ultimate Factors

	18 - Ult.	30 - Ult.	42 - Ult.	54 - Ult.	66 - Ult.
Actual*	1.73	1.22	1.04	1.02	1.00
Selected	1.20	1.12	1.05	1.02	1.00

Occurrence (count) Age-to-Age Factors

	18 - 30	30 - 42	42 - 54	54 - 66	66 - 78
Actual**	5.64	1.81	1.34	.98	1.05
Selected	5.22	1.94	1.36	.98	1.05

* based on weighted average incurred
** based on average number of occurrences

EXAMPLE B: P&C INSURANCE COMPANY

Average Incurred (Ground-Up) and Occurrence Loss Development
Excess of $75,000 at 1980 Level as of 6/30/79

Econometric Trend Model

Trend Factor	Deflated Retention	Accident Year		Age in Months: 18	30	42	54	66	78
2.964	$25,299	1973	Avg. $	NA	141,960	170,257	163,076	159,911	159,321
			#		19	30	45	44	46
2.420	30,992	1974	Avg. $	111,925	140,014	160,026	172,256	183,063	
			#	4	14	38	41	40	
2.012	37,283	1975	Avg. $	100,600	118,962	124,117	131,588		
			#	1	11	18	32		
1.721	43,584	1976	Avg. $	121,726	121,905	102,667			
			#	5	9	23			
1.506	49,817	1977	Avg. $	0	95,201				
			#	0	14				
1.350	55,546	1978	Avg. $	102,150					
			#	3					

Average Incurred Age-to-Ultimate Factors

	18 - Ult.	30 - Ult.	42 - Ult.	54 - Ult.	66 - Ult.
Actual*	1.16	1.12	1.03	1.02	1.00
Selected	1.20	1.12	1.05	1.02	1.00

Occurrence (count) Age-to-Age Factors

	18 - 30	30 - 42	42 - 54	54 - 66	66 - 78
Actual**	4.80	2.06	1.37	.98	1.05
Selected	5.22	1.94	1.36	.98	1.05

* based on weighted average incurred
** based on average number of occurrences

EXAMPLE B: P&C INSURANCE COMPANY

WEIBULL DISTRIBUTION
ACCIDENT YEAR AGE-TO-AGE FACTORS*

MEAN MON.	PARAMETERS SCALE	SHAPE	18 TO 30	30 TO 42	42 TO 54	54 TO 66	66 TO 78	78 TO 90	90 TO 102	102 TO 114	114 TO ULT
27	30.000	2.500	4.050	1.790	1.218	1.042	1.004	1.000	1.000	1.000	1.000
28	31.000	2.500	4.117	1.843	1.245	1.053	1.006	1.000	1.000	1.000	1.000
28	32.000	2.500	4.179	1.886	1.272	1.065	1.009	1.001	1.000	1.000	1.000
29	33.000	2.500	4.235	1.927	1.299	1.077	1.013	1.001	1.000	1.000	1.000
30	34.000	2.500	4.288	1.966	1.326	1.091	1.017	1.002	1.000	1.000	1.000
31	35.000	2.500	4.336	2.003	1.352	1.105	1.022	1.003	1.000	1.000	1.000
32	36.000	2.500	4.381	2.030	1.378	1.120	1.027	1.004	1.000	1.000	1.000
33	37.000	2.500	4.422	2.071	1.403	1.135	1.034	1.005	1.001	1.000	1.000
34	38.000	2.500	4.460	2.102	1.428	1.151	1.041	1.007	1.001	1.000	1.000
35	39.000	2.500	4.495	2.131	1.452	1.167	1.047	1.010	1.001	1.000	1.000
27	30.000	2.750	4.679	1.897	1.219	1.031	1.002	1.000	1.000	1.000	1.000
28	31.000	2.750	4.760	1.954	1.256	1.041	1.003	1.000	1.000	1.000	1.000
28	32.000	2.750	4.839	2.007	1.278	1.052	1.005	1.000	1.000	1.000	1.000
29	33.000	2.750	4.912	2.061	1.310	1.065	1.007	1.000	1.000	1.000	1.000
** 30	34.000	2.750	4.978	2.110	1.342	1.079	1.010	1.001	1.000	1.000	1.000
31	35.000	2.750	5.038	2.157	1.374	1.093	1.014	1.001	1.000	1.000	1.000
32	36.000	2.750	5.094	2.201	1.406	1.110	1.019	1.002	1.000	1.000	1.000
33	37.000	2.750	5.145	2.242	1.437	1.128	1.024	1.003	1.000	1.000	1.000
34	38.000	2.750	5.193	2.281	1.468	1.145	1.031	1.004	1.000	1.000	1.000
35	39.000	2.750	5.236	2.310	1.497	1.164	1.038	1.005	1.000	1.000	1.000

* Expected value of annual age-to-age factors that would be generated if the report lags of losses occurring in each month are distributed according to the Weibull distribution with specified parameters.

** Report lag c.d.f. selected with respect to both trend models.

APPENDIX B

EXAMPLE B: P&C INSURANCE COMPANY

Number of IBNR Occurrences Excess of <u>$75,000</u>
at 1980 Level as of <u>6/30/79</u>

Total number of IBNR occurrences excess of $75,000 for accident years 1973 - 78 as of 6/30/79 are estimated using the method described in Patrik (1978).

$$\text{Total IBNR} = \frac{(\text{Known})\cdot\omega}{1-\omega}$$

= 87.2 and 80.4 with respect to the exponential and econometric trend models, respectively.

where

Known = total number of known occurrences excess of $75,000 for accident years 1973 - 78 as of 6/30/79.

= 171 and 158 with respect to the exponential and econometric trend models, respectively.

$$\omega = \frac{\sum_m EP_m\cdot[1 - W(x_m)]}{EP}$$

= .3375 for months m such that $1/73 \leq m \leq 12/78$

EP_m = monthly exposure base, in this case GL gross direct earned premium at present rates, for $1/73 \leq m \leq 12/78$.

$EP = \sum_m EP_m$ for $1/73 \leq m \leq 12/78$

APPENDIX B

$W(\cdot)$ = selected report lag c.d.f. (see p.B3).

x_m = maximum observable report lag; that is, for accident month m the difference between 6/30/79 and the mid-point of m.

Letting IBNR(x;6/30/79) denote the number of IBNR occurrences for accident year x as of 6/30/79, the total IBNR is allocated to accident year x using the formula:

$$IBNR(x;6/30/79) = R \cdot \sum_m EP_m \cdot [1 - W(x_m)]$$

where

$$R = \frac{\text{Known + Total IBNR}}{EP},$$

$1/x \leq m \leq 12/x$, and $x = 73,\ldots,78$.

The assumptions underlying this IBNR method are:

1. homogeneous coverage groups
2. the ratio of ultimate number of occurrences to earned exposure is constant and independent of time
3. the report lag distribution does not vary with occurrence date.

EXAMPLE B: P&C INSURANCE COMPANY

Excess and Base Frequencies and Excess IBNR
by Accident Year at 1980 Level

Exponential Trend Model

	Present Level	Occurrences Excess of $75,000				Base Frequency*	
Accident Year	Gross Direct Earned Premium (000)	Known (6/30/79)	IBNR** (6/30/79)	Ultimate	Frequency Excess Of $75,000	c.d.f.(1)	c.d.f.(2)
1973	$24,524	46	0	46.0	.0019	.0108	.0128
1974	21,860	43	.5	43.5	.0020	.0114	.0135
1975	19,435	41	3.2	44.2	.0023	.0131	.0155
1976	19,685	24	12.5	36.5	.0019	.0108	.0128
1977	21,137	14	28.6	42.6	.0020	.0114	.0135
1978	22,701	3	42.4	45.4	.0020	.0114	.0135
Selected	-	-	-	-	-	.0108	.0135

* Base frequency = excess frequency divided by the probability of an occurrence exceeding $75,000 for loss amount c.d.f.(1) and c.d.f.(2).

** Based on the IBNR method described in Appendix B, pp. B4 and B5.

EXAMPLE B: P&C INSURANCE COMPANY

Excess and Base Frequencies and Excess IBNR
by Accident Year at 1980 Level

Econometric Trend Model

Accident Year	Present Level Gross Direct Earned Premium (000)	Occurrences Excess of $75,000 Known (6/30/79)	IBNR** (6/30/79)	Ultimate	Frequency Excess Of $75,000	Base Frequency* c.d.f.(3)	c.d.f.(4)
1973	$24,524	46	0	46.0	.0019	.0101	.0104
1974	21,860	40	.4	40.4	.0018	.0096	.0099
1975	19,435	32	2.9	34.9	.0018	.0096	.0099
1976	19,685	23	11.5	34.5	.0018	.0096	.0099
1977	21,137	14	26.4	40.4	.0019	.0101	.0104
1978	22,701	3	39.2	42.2	.0019	.0101	.0104
Selected	-	-	-	-	-	.0096	.0104

* Base frequency = excess frequency divided by the probability of an occurrence exceeding $75,000 for loss amount c.d.f.(3) and c.d.f.(4).

** Based on the IBNR method described in Appendix B, pp. B4 and B5.

APPENDIX C

EXAMPLE B: P&C INSURANCE COMPANY RISKMODEL RUN

```
      RISKMODEL
DO NOT PANIC IF YOU MAKE AN ERROR WHILE INPUTTING,
OPPORTUNITY TO CHANGE LATER.

ENTER MAJOR GROUP NAMES AS FOLLOWS.   /GRP1/GRP2..........
NOTE: MUST BE IN QUOTES.   FOR MORE THAN 1 LINE OF INPUT, USE .0
O:
      '&GL/200&GL/250&GL/350&GL/500+'

ENTER THE NUMBER OF PARAMETERS, E.G.   5
O:
      4

DO YOU WISH TO  (1) INPUT VECTOR OF LIMITS,  OR
(2) USE MATRIX OF LIMITS PREVIOUSLY CREATED. 1 OR 2.
O:
      2
ENTER THE NAME OF THE MATRIX OF LIMITS PREVIOUSLY CREATED
NOTE: NAME SHOULD HAVE PREFIX LIM
LIMP&C

DO YOU WISH TO SEE THE LIM MATRIX.  Y OR N
Y
                        LIMITS
             LOWER                UPPER              INDEX
                 0               100000                1
            100000               200000                0
                 0               100000                1
            100000               250000                0
                 0               100000                1
            100000               350000                0
                 0               100000                1
            100000               500000                0

DO YOU WISH TO MAKE ANY CHANGES IN THE LIM MATRIX.  Y OR N
N


DO YOU WISH TO
(1) INPUT VECTOR OF PARAMETERS FOR THE FIRST SUBGROUP  OR
(2) USE MATRIX OF PARAMETERS PREVIOUSLY CREATED.  1 OR 2
O:
      2
ENTER THE NAME OF THE MATRIX OF PARAMETERS PREVIOUSLY CREATED
NOTE: NAME SHOULD HAVE PREFIX PAR
PARP&C

DO YOU WISH TO SEE THE PAR MATRIX.  Y OR N
Y
```

(The PAR matrix is displayed in Table 6A)

C2

APPENDIX C

EXAMPLE B: P&C INSURANCE COMPANY RISKMODEL RUN

DO YOU WISH TO MAKE ANY CHANGES IN THE PAR MATRIX. Y OR N
N

GROUPS AND PARAMETER INPUT COMPLETED
TO PROCESS INTERMEDIATE CALCULATIONS, HIT EXECUTE

DO YOU WISH TO PRINT THE INTERMEDIATE CALCULATIONS,
P[S>A],P[S>B],E[S],E[S*2],E[S*3]. Y OR N.
Y

<u>INTERMEDIATE CALCULATIONS USED THROUGHOUT MOMENT CALCULATIONS</u>

LAYER 1

GROUPS		P[S>A]	P[S>B]	E[S]	E[S*2]	E[S*3]
GL/200	1	1.000	.114	2.544E04	1.270E09	8.047E13
GL/200	2	1.000	.095	2.394E04	1.147E09	7.112E13
GL/200	3	1.000	.122	2.591E04	1.312E09	8.384E13
GL/200	4	1.000	.118	2.568E04	1.292E09	8.225E13
GL/250	1	1.000	.114	2.544E04	1.270E09	8.047E13
GL/250	2	1.000	.095	2.394E04	1.147E09	7.112E13
GL/250	3	1.000	.122	2.591E04	1.312E09	8.384E13
GL/250	4	1.000	.118	2.568E04	1.292E09	8.225E13
GL/350	1	1.000	.114	2.544E04	1.270E09	8.047E13
GL/350	2	1.000	.095	2.394E04	1.147E09	7.112E13
GL/350	3	1.000	.122	2.591E04	1.312E09	8.304E13
GL/350	4	1.000	.118	2.568E04	1.292E09	8.225E13
GL/500+	1	1.000	.114	2.544E04	1.270E09	8.047E13
GL/500+	2	1.000	.095	2.394E04	1.147E09	7.112E13
GL/500+	3	1.000	.122	2.591E04	1.312E09	8.384E13
GL/500+	4	1.000	.118	2.568E04	1.292E09	8.225E13

APPENDIX C

EXAMPLE B: **P&C** *INSURANCE COMPANY RISKMODEL RUN*

LAYER 2

GROUPS		P[S>A]	P[S>B]	E[S]	E[S*2]	E[S*3]
GL/200	1	.114	.029	1.147E04	1.623E09	2.387E14
GL/200	2	.095	.025	9.503E03	1.342E09	1.972E14
GL/200	3	.122	.031	1.233E04	1.747E09	2.572E14
GL/200	4	.118	.031	1.195E04	1.692E09	2.490E14
GL/250	1	.114	.017	1.414E04	2.217E09	3.718E14
GL/250	2	.095	.015	1.171E04	1.834E09	3.074E14
GL/250	3	.122	.018	1.522E04	2.391E09	4.015E14
GL/250	4	.118	.018	1.474E04	2.315E09	3.886E14
GL/350	1	.114	.007	1.705E04	3.073E09	6.254E14
GL/350	2	.095	.007	1.418E04	2.562E09	5.233E14
GL/350	3	.122	.008	1.837E04	3.316E09	6.757E14
GL/350	4	.118	.008	1.780E04	3.214E09	6.550E14
GL/500+	1	.114	.003	1.893E04	3.848E09	9.493E14
GL/500+	2	.095	.003	1.507E04	3.259E09	8.154E14
GL/500+	3	.122	.003	2.037E04	4.144E09	1.021E15
GL/500+	4	.118	.003	1.977E04	4.026E09	9.944E14

TO PROCESS MORE INTERMEDIATE CALCULATIONS, HIT EXECUTE

ENTER EPSILON(S) FOR WHICH PROB(LOSS>MAX. PROB. LOSS) = EPSILON. (0<ε≤.5)
□:
    .1 .05 .01
NOW FOR THE FINAL PRINTOUT

ENTER COMPANY NAME
EXAMPLE B: P&C INSURANCE COMPANY-GENERAL LIABILITY

ENTER YOUR NAME (EG. J. SMITH)
RALPH M. CELLARS

ENTER TODAY'S DATE (EG. JAN. 1, 1979)
OCTOBER 31, 1979

ENTER IN PARENTHESIS AND QUOTES A SEVEN CHARACTER NAME FOR THE UNITS
(E.G. '(DOCTORS)' OR '__(BEDS)_')
□: OF EXPOSURE CENTERED IN 9 SPACES .
    ' (000) '

ADJUST PAPER TO TOP OF NEW PAGE & HIT EXECUTE

(The main output is displayed in Table 7E)

APPENDIX D

Probability Distribution Definitions

Negative Binomial

density: $f(x|p,\alpha) = \binom{\alpha+x-1}{\alpha} p^{\alpha}(1-p)^{\alpha}$ for $x = 0,1,2,\ldots.$

where $p,\alpha > 0$.

This is our basic model of the loss occurrence (count) process. Note, if $Var[N] \div E[N] = 1$, then RISKMODEL assumes that the occurrence process is Poisson with $\lambda = E[N]$.

Four Parameter Loss Amount Distributions

$$\text{c.d.f:}\quad G_S(x|\alpha,\beta,t,XP) = \begin{cases} \dfrac{XQ}{H(t|\alpha,\beta)} \cdot H(x|\alpha,\beta) & \text{for } 0<x\leq t \\ XQ + XP\cdot\{H(x|\alpha,\beta)-H(t|\alpha,\beta)\} & \text{for } x>t \end{cases}$$

where $t \geq 0$, $0 < XP \leq 1$

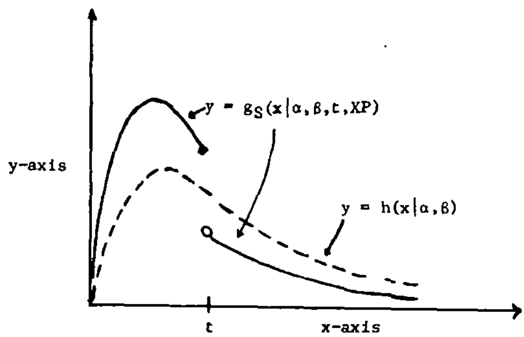
$$XQ = 1 - XP\cdot\{1 - H(t|\alpha,\beta)\}$$

$H(x|\alpha,\beta)$ is some c.d.f. for $x > 0$ with parameters (α,β). RISKMODEL's present library of choices for $H(\cdot|\alpha,\beta)$ are (1) = lognormal, (2) = Pareto and (3) = Weibull. Definitions of each of these distributions are given below.

$$\text{density:}\quad g_S(x|\alpha,\beta,t,XP) = \begin{cases} \dfrac{XQ}{H(t|\alpha,\beta)}\, h(x|\alpha,\beta) & \text{for } 0<x\leq t \\ XP\cdot h(x|\alpha,\beta) & \text{for } x>t \end{cases}$$

APPENDIX D

A graph of the density $g(\cdot|\alpha,\beta,t,XP)$ in general looks like:



(1) Lognormal

c.d.f: $H(x|\mu,\sigma^2) = \Phi\left(\frac{\log x - \mu}{\sigma}\right)$ for $0 < x < \infty$

where $\Phi(\cdot)$ is the standard normal (0,1) c.d.f. and $\sigma > 0$, $-\infty < \mu < \infty$.

density: $h(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\{-\frac{(\log x - \mu)^2}{2\sigma^2}\}$

(2) Pareto

c.d.f: $H(x|\beta,\delta) = 1 - \left(\frac{\beta}{x + \beta}\right)^{\delta}$ for $x \geq 0$

where $\beta,\delta > 0$.

density: $h(x|\beta,\delta) = \delta\beta^{\delta}(x + \beta)^{-\delta-1}$

APPENDIX D

(3) Weibull

c.d.f: $H(x|\beta,\delta) = 1 - \exp\{-\left(\frac{x}{\beta}\right)^{\delta}\}$ for $x \geq 0$

where $\beta,\delta > 0$

density: $h(x|\beta,\delta) = \delta\beta^{-\delta}x^{\delta-1}\exp\{-\left(\frac{x}{\beta}\right)^{\delta}\}$

For more details on probability distributions, see Hastings and Peacock (1975) or Johnson and Kotz (1969,1970).