TITLE: ESTIMATING AGGREGATE LOSS PROBABILITY AND INCREASED LIMIT FACTOR

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Aggregate loss probability is an effective tool in actuarial rate making, risk charging, and retention analysis for both primary and secondary insurance companies. A noticeable trend over recent years indicates that it also is becoming an indispensable element in the risk management operations of many manufacturing and commercial firms. Some major insurance brokerage houses in the U.S., in step with the trend, already employ this technique routinely in selecting a retention plan for their clients. In its broadest form, the application extends beyond the actuarial domain into the broader area of corporate financial planning.

Most existing procedures for estimating aggregate loss probability distributions have significant disadvantages. Most often, these disadvantages are associated with inadequate treatment of skewed data. The purpose of this paper is to present a recently developed technique which seems to handle the aggregate loss estimation problem more effectively.

The first section presents a brief review of the strength and weakness of most popular techniques currently in use. This is followed by a brief description of the newly developed technique. Next, the results of a comparative study of the cost and effectiveness of these alternative procedures are reported. Finally, we illustrate the impact of improved aggregate loss estimation on the pricing of reinsurance. An appendix contains the mathematical derivation for those who would like to verify our results.

Standard Aggregate Loss Estimation Procedures

In dealing with the estimation of aggregate loss probability, there are three fundamental approaches commonly in use. They are analytical, approximation, and simulation models. Each is distinguished from the others by its own characteristics, advantages and disadvantages. The pure analytical model[1] generally is the most accurate. The handicap is that it can be applied to only a few distribution types. A frequently used approximation model is the Normal Power approximation. This is easy to implement but yields disturbingly large approximation errors when applied to highly skewed data.[2] Another less well-known approximation technique is the Gamma approximation,[3] which seems more accurate than the NP approximation in most occasions.[4] The only weakness of the Gamma approximation is that, like the NP approximation, it does not respond to the sensitive choice of frequency distribution. Simulation modeling is perhaps the most widely used technique in the field of management sciences; however, like the other techniques, it has disadvantages, too. First, since the error brought on from simulation is statistical rather than mathematical, it can be reduced significantly only by increasing considerably the number of iterations.[5] This would be an unfavorable element should the consideration of computing time and cost become crucial. Secondly, simulation is a brute force technique and offers limited insight into how a system works. Thus, any sensitivity analysis or optimization drawn from a simulation model is virtually a trial and error process and can not be justified mathematically.

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1 See Appendix B for a summary.

2 Reports compiled from experiments decline to recommend the use of NP approximation on data of skewness exceeding 1 or 2, see [2] and [13] .

3 See Appendix C for background materials on this technique.

4 There is a controversy in the literature [16] and [18] concerning which approximation is superior. In our study, we found out that at least for the distributions listed in this article, the result for the gamma approximation is much better than that from the NP approximation.

5 See Table 7.1 given in [2] , p. 93 for relation between the degree of error and number of iterations.

The aim of this paper is to introduce a new model which is designed to meet the dual requirements of accuracy and simplicity in implementation. Our approach is a blend of the analytical and approximation models. It is approximate, because the answer is not the exact, and analytical primarily because the formula is derived from the fundamental characteristics of collective risk theory. To demonstrate the precision of our model, apart from the mathematical deduction attached as an appendix, we compare the results of the new model with those where the exact probability can be calculated directly using the analytical method.

A New Model (Modified Gamma Approximation)

Aggregate loss, occurring as a random process, is compiled from two variables: one is identified as the number of claims experienced in a given time span (normally one year) and denominated as the "frequency of loss." The other is the size of an individual claim and is termed the "severity of loss." Jointly, frequency and severity determine total or aggregate loss from all claims in the given time span. The most often used frequency distributions are poisson and negative binomial.[6] For severity distributions, experience[7] indicates that normal, gamma, inverse normal, pareto, log-normal and log-gamma[8] are appropriate for casualty and property insurance.

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[6] Some authors also recommend a third type, the generalized Waring distribution, for details please see [19] .

[7] See [3] , [8] , [10] , [11] , and [18] .

[8] *A summary of these distributions can be found in Appendix A.*

For convenience, we shall adopt the term generalized poisson model for the aggregate loss distribution which uses the poisson distribution as the frequency function and leaves the choice of the severity function open. Similarly, the generalized negative binomial model reflects the application of the negative binomial distribution as the frequency function.

To describe our formula, we need the following statistics which can be estimated[9] from the sample data:

λ	:	frequency mean
σ_p	:	frequency standard deviation
μ_s	:	severity mean
σ_s	:	severity standard deviation

and statistics which can be derived intrinsically:

μ	:	aggregate mean (e.g., the product of λ and μ_s)[10]
σ	:	aggregate standard deviation (e.g., $\lambda(\mu_s^2+\sigma_s^2)$ for the generalized poisson model and $\lambda\sigma_s^2+\mu_s^2\sigma_p^2$ for the generalized $n.b.$ model)[10]
γ_s	:	severity skewness

Our formula states that the probability $F(x)$ of annual aggregate loss less than or equal to x is given by:[11]

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty f(t)\,\frac{\sin\left(xt/\sigma + g(t)\right)}{t}\,dt \qquad (1)$$

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9 See [7] for the estimation of these statistics.

10 See [14] p. 179 for the derivation.

11 The derivation of formula (1) and the subsequent tables are given in Appendix D.

where functions $f(t)$ and $g(t)$ in the integrand are defined by:

Table 1

Model	$f(t)$	$g(t)$
Generalized Poisson	$\exp(\lambda \ell(t))$	$-\lambda k(t)$
Generalized NB	$\left\{\left(1-\left(\frac{\sigma_p^2-\lambda}{\lambda}\right)\ell(t)\right)^2+\left(\left(\frac{\sigma_p^2-\lambda}{\lambda}\right)k(t)\right)^2\right\}^{-\frac{\lambda^2}{2(\sigma_p^2-\lambda)}}$,	$\left(\frac{\lambda^2}{\sigma_p^2-\lambda}\right)\tan^{-1}\left(\frac{(\sigma_p^2-\lambda)k(t)}{\lambda-(\sigma_p^2-\lambda)\ell(t)}\right)$

and functions $\ell(t)$ and $k(t)$ in both models are given by:

Table 2

$\ell(t)$	$k(t)$	$\delta(t)$	$\theta(t)$
$\delta(t)\cos(\theta(t))-1$	$\delta(t)\sin(\theta(t))$	$\left(1+\left(\frac{t\gamma_p\sigma_p}{2\sigma}\right)^2\right)^{-\frac{2}{\gamma_p^2}}$	$t\left(\frac{\mu_p-2\sigma_p/\gamma_p}{\sigma}\right)+\frac{4}{\gamma_p^2}\tan^{-1}\left(\frac{t\gamma_p\sigma_p}{2\sigma}\right)$

The only quantity which has not been expressed explicitely in the formula is the severity skewness γ_p . Since each of the six severity distribution functions has exactly two parameters, each is defined and described completely by the severity sample mean and standard deviation. All the other quantities, including γ_p , depend ultimately on the type of the severity distribution chosen; that is on the sample mean and standard deviation. The corresponding severity skewness of the six alternative severity distributions are tabulated as follows:[12]

Table 3

Type	Skewness[13] γ_p
Normal	0
Gamma	$2\left(\frac{\sigma_p}{\mu_p}\right)$

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[12] The derivation of Table 3 is given in Appendix A.

[13] If the skewness is zero, replace it with any small number (e.g., 10^{-9}) in the computation, since dividing by zero is prohibited in our formula.

Inverse Normal	$3\left(\frac{\sigma_p}{\mu_p}\right)$
Pareto[14]	$2\left(\frac{\sigma_p}{\mu_p}\right)\left(\frac{3\sigma_p^2-\mu_p^2}{3\mu_p^2-\sigma_p^2}\right)$
Log-normal	$\left(\frac{\sigma_p}{\mu_p}\right)\left(\left(\frac{\sigma_p}{\mu_p}\right)^2+3\right)$
Log-gamma[14]	*[15]

Formula (1) and its consequent computations may seem complex in the form shown above. However, the implementation is quite simple. Any standard numerical integration technique would handle the computation effectively; for example, the extended Simpson's rule is adequate to calculate the integration in (1) and is easy to code in any scientific programming language. A practical discussion on the use of extended Simpson's rule and the truncated range of integration in formula (1) is given in Appendix D.

Effectiveness of the Modified Gamma Approach

From a conceptual point of view, the new model seems to satisfy the objective of increased accuracy at nominal cost. The ultimate test, however, lies in its effectiveness in handling actual loss data.

By combining the poisson or negative binomial (for frequency) with the normal, gamma, or inverse normal (for severity) it is possible to compute an exact aggregate distribution using the pure analytical method (A).

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14 In the cases of the pareto and log-gamma distributions, the skewness may not always exist; it depends on the relation between the sample mean and standard deviation. Thus, if the following conditions

$$3\mu_p^2 > \sigma_p^2 \qquad \text{for pareto ,}$$

$$\log(\sigma_p^2+\mu_p^2)/\log\mu_p \leq 2.7095 \qquad \text{for log-gamma}$$

are not met, the new model is not applicable. See Appendix A for details.

15 Since the skewness of log-gamma does not admit any closed form in terms of the sample mean and standard deviation, it is best expressed by its functional parameters, see Appendix A.

This procedure was used to provide a series of control distributions for a comparison of the relative accuracy of the normal power approximation (NP), standard gamma approximation (G), and the new, modified gamma approximation (MG).

In the analysis each of the four methods was used to generate aggregate probability distributions for several sets of hypothetical loss data. The primary variation in the data reflected differences in skewness (from a relatively modest .5 to a substantial skewness factor of 5). The points on the probability distribution were chosen in terms of standardized deviations from the mean rather than in absolute dollar amounts. Calculations were made utilizing both the generalized poisson and the generalized negative binomial models.[16]

Exhibit I, presents three sets of data for the generalized poisson model, as does exhibit II for generalized negative binomial model. In each set, the severity type, severity coefficient of variation and frequency mean are selected (in the case of negative binomial the frequency variance is also required), and the aggregate skewness is calculated by the aid of Table A6 given in appendix C2. Two auxiliary exhibits, labeled by Ia and IIa respectively, display the difference between results obtained from analytic method and the other three methods. At the bottom row, their variances, calculated by summing the squares of the difference dividing by the number of rows, are computed respectively.

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16 *The objective of the analysis was to uncover any systematic bias* or approximation errors inherent in the alternative approximation techniques. In normal practice the candidate distributions would be determined by a goodness of fit criterion.

As can be seen from both exhibits I and II, the new model clearly is superior to the other two approximation models in all scenarios. The discrepancy of NP approximation is particularly serious not only on highly skewed data but also on modestly skewed data (e.g., $\gamma_p =$ [illegible]). Also notice that in both the generalized poisson and negative binomial models, the results from the standard gamma and NP approximation are determined ultimately by the skewness, e.g., the differences in the control distributions reflecting the choice of frequency distribution are not captured by either traditional approximation methods. The new model does detect the difference between poisson and negative binomial frequency distributions.

Finally, we want to indicate the degree of sensitivity of the estimated aggregate loss probability to the selection of the type of severity function. Exhibit IV assumes that the frequency distribution is poisson (with mean = 60.383) and the estimated severity coefficient of variation is equal to 4. If the severity function is the inverse normal the aggregate skewness would be 1.5. The same parameter would be 9.02 for log-normal. Also a tail appears in the aggregate picture when the log-normal is selected for the severity. This phenomenon can be explained mathematically by the following observation: given a severity sample mean and variance, the magnitude of the severity skewness, according to Table 3, can be arranged in the following increasing order:

$$\text{normal, gamma, inverse normal,} \quad \left\{ \begin{array}{l} \text{pareto} \\ \text{log-normal} \end{array} \right\} .$$

Since the aggregate skewness varies along with the severity skewness, the selection of log-normal as severity function always yields a larger aggregate skewness than does the selection of inverse normal.

Increased Limits Factors for Stop-Loss Reinsurance

One of the practical applications of estimating aggregate loss probability is its use in excess of loss pricing, aggregate pricing and stop-loss reinsurance. The case of excess of loss pricing has been covered extensively in a recent article by Robert S. Miccolis.[17] We would like to concentrate on the latter two situations here.

Aggregate pricing and stop-loss reinsurance are fundamentally one concept. Stop-loss reinsurance is a process which transfers the risk above an aggregate limit to a reinsurer. Aggregate pricing structure can be envisaged as zero limit stop-loss reinsurance pricing structure, e.g., the reinsurer absorbs all the loss. Thus, as far as the pricing structure is concerned, we can treat aggregate pricing as a special case of stop-loss reinsurance pricing.

If F(x), as before, represents the aggregate loss probability distribution, without an aggregate limit, then let $F_L(x)$ be the truncated distribution where an aggregate limit L is introduced, μ_L and σ_L are respectively the mean and standard deviation of $F_L(x)$. The formula for premium, excluding loss expense, charged for stop-loss coverage of an aggregate limit L is given[18] as follows:

$$P_L = \mu_L + c\,\sigma_L^2 \qquad (2)$$

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17 See [15] .

18 See [5] p. 85-87. An alternate suggestion for the safety loading in formula (2) is to use the standard deviation σ_L instead of variance σ_L^2 , see [1] .

where the loading coefficient C generally is chosen from experience. If L is zero (e.g., this is a full stop-loss coverage for the primary carrier) μ_L and σ_L become aggregate μ and σ as specified before. Suppose a is the loss expense ratio, then the total premium charged for a stop-loss coverage with limit L is $(1+a)P_L$. Then by definition, the increased limit factor, $I(L)$, of a stop-loss policy limit L imposed on a stop-loss basic limit L_0 is

$$I(L) = \frac{\text{Total premium of policy limit L}}{\text{Total premium of basic limit } L_0}$$

$$= \frac{(1+a)P_L}{(1+a)P_{L_0}} = \frac{\mu_L + c\,\sigma_L^2}{\mu_{L_0} + c\,\sigma_{L_0}^2} \qquad (3)$$

A formula is needed to calculate μ_L and σ_L^2. This can be worked out from the truncated distribution $F_L(x)$. Since the aggregate loss of the reinsurer under a stop-loss coverage with a policy limit L is reduced by an amount of L dollars, the probability $F_L(x)$ is given by:

$$F_L(x) = F(x+L) \qquad (4)$$

Hence, the jth moment, $\mu_{L,j}$ accordingly is defined by:

$$\mu_{L,j} = \int_0^\infty x^j \, dF_L(x)$$

$$= \int_0^\infty x^j \, dF(x+L)$$

(replaced variable $x+L$ by x)

$$= \int_L^\infty (x-L)^j \, dF(x)$$

particularly, when j = 1 and 2, we have

$$\mu_L = \int_L^\infty (x-L)\,dF(x) = \int_0^\infty x\,dF(x) - \int_0^L x\,dF(x) - L\int_L^\infty dF(x)$$

$$= \mu - \int_0^L x\,dF(x) - L\left(1-F(L)\right) \qquad (5)$$

$$\mu_{L,2} = \int_L^\infty (x-L)^2 \, dF(x)$$

$$= \int_0^\infty x^2 dF(x) - \int_0^L x^2 dF(x) - 2L\int_L^\infty (x-L)\,dF(x) + L^2\int_L^\infty dF(x)$$

$$= \sigma^2 + \mu^2 - \int_0^L x^2 dF(x) - 2L\mu_L + L^2\left(1-F(L)\right) \qquad (6)$$

Notice that $\sigma_L^2 = \mu_{L,2} - \mu_L^2$, thus two more values:

$$\int_0^L x^j \, dF(x), \qquad j = 1, 2 \tag{7}$$

have to be calculated before we compute formula (3). For this, the precise form of $F(x)$ would come into play. Since by (1),

$$\frac{dF(x)}{dx} = \frac{1}{\pi\sigma}\int_0^\infty f(t)\cos(xt/\sigma + g(t))\,dt \tag{8}$$

Substitute $dF(x)$ in (7) by (8). We have

$$\int_0^L x^j \, dF(x) = \frac{1}{\pi\sigma}\int_0^L x^j \, dx \int_0^\infty f(t)\cos(xt/\sigma + g(t))\,dt$$

(exchange the order of integration)

$$= \frac{1}{\pi\sigma}\int_0^\infty f(t)\,dt \int_0^L x^j \cos(xt/\sigma + g(t))\,dx \tag{9}$$

Now the first integrand $\int_0^L x^j \cos(xt/\sigma + g(t))\,dx$ (denoted by $w_j(t)$) in (9)

has a closed form, and the desired values are given as follows:

Table 4

$w_1(t)$	$w_2(t)$
$\frac{L\sigma \sin(Lt/\sigma + g(t))}{t} + \frac{\sigma^2\{\cos(Lt/\sigma + g(t)) - \cos(g(t))\}}{t^2}$,	$\frac{L^2\sigma \sin(Lt/\sigma + g(t))}{t} + \frac{2L\sigma^2\cos(Lt/\sigma + g(t))}{t^2} - \frac{2\sigma^3\{\sin(Lt/\sigma + g(t)) - \sin(g(t))\}}{t^3}$

where g(t) and f(t) are given as before in Tables 1 and 2.

In summary, the increased limits factor I(L) is calculated by formula (3), where $\sigma_L^2 = \mu_{L,2} - \mu_L^2$ with μ_L and $\mu_{L,2}$ given by (5) and (6). Whereas in formula (5) and (6), $\int_0^L x^j dF(x)$, $j=1,2$ is calculated by:

Table 5

$\int_0^L x\,dF(x)$	$\int_0^L x^2\,dF(x)$
$\frac{1}{\pi\sigma}\int_0^\infty f(t)\,w_1(t)\,dt$	$\frac{1}{\pi\sigma}\int_0^\infty f(t)\,w_2(t)\,dt$

The integrations in Table 5 can be handled by any numerical integration technique as discussed before, e.g., extended Simpson's rule, etc. Exhibit III illustrates an increased limits table derived by formula (3) and tables 4 and 5.

Conclusion

The effectiveness of the estimation of aggregate loss probability and the aggregate pricing model introduced in this article will, to a great extent, depend on how consistantly the loss-experience data is treated. In our model, we assume that all the losses have already been adjusted to the present or ultimate level. That is: losses have been developed to the ultimate; IBNR has been adjusted and inflation has been trended to the forecasting year, etc. The reason that we did not discuss those in here is because they are rather standard actuarial techniques practiced in most areas of rate-making and have been covered extensively elsewhere in the literature.[19]

The analysis shown above indicates that for many classes of distributions the new modified gamma approximation is superior in estimation accuracy and poses no significant increase in computation effort or expense. The new technique thus, is potentially valuable in more effective pricing of certain classes of reinsurance.

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[19] An alternate approach is to incorporate those effects into the parameters of distribution as suggested in [12] and [15].

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Exhibit I

Generalized Poisson Model

Aggregate Probability $F(x)$ (%)

$z\left(=\frac{x-\mu}{\sigma}\right)$	γ = .5, Sev = Gamma				γ = 1, Sev = Inv. Normal				γ = 5, Sev = Gamma			
	A	MG	G	NP	A	MG	G	NP	A	MG	G	NP
-1.5	4.87	4.87	4.87	5.04	2.49	2.60	1.90	2.28	*	*	*	*
-1	15.58	15.58	15.56	15.87	14.17	14.21	14.29	15.87	*	*	*	*
- .5	32.93	33.09	33.06	33.28	34.69	34.79	35.28	36.16	*	*	*	*
0	53.14	53.33	53.33	53.30	56.33	56.36	56.65	56.45	78.49	78.47	78.52	71.44
.5	71.28	71.32	71.33	71.14	73.52	73.51	73.50	72.76	87.18	87.20	87.20	78.81
1	84.33	84.33	84.35	84.13	85.05	84.99	84.88	84.13	91.44	91.44	91.46	84.13
1.5	92.30	92.30	92.31	92.16	91.99	91.85	91.82	91.29	94.00	94.00	94.01	88.05
2	96.56	96.56	96.56	96.49	95.86	95.80	95.76	95.45	95.68	95.68	95.68	90.97
3	99.44	99.44	99.46	99.45	98.98	98.98	98.97	98.90	97.63	97.63	97.63	94.81
4	99.93	99.93	99.93	99.94	99.77	99.77	99.77	99.78	98.64	98.64	98.64	97.01
5	99.99	99.99	99.99	99.99	99.95	99.95	99.95	99.96	99.20	99.20	99.20	98.27
Frequency Mean	100				77.84				100.5			
Severity Coefficient of Variation	2.5				3				25			

Note: (*) Points below zero dollar limit.

(i) All four models are calculated by a HP-19 programmable calculator.

Exhibit II

Generalized Negative Binomial Model

Aggregate Probability $F(x)$ (%)

$z\left(=\frac{x-\mu}{\sigma}\right)$	γ = 2, Sev = Normal				γ = 3, Sev = Normal				γ = 5, Sev = Gamma			
	A	MG	G	NP	A	MG	G	NP	A	MG	G	NP
-1.5	*	*	*	*	*	*	*	*	*	*	*	*
-1	*	*	*	*	*	*	*	*	*	*	*	*
-.5	39.86	39.32	39.35	42.97	40.43	40.63	41.11	50.00	*	*	*	*
0	63.51	63.39	63.21	61.90	69.33	69.35	69.25	66.06	78.61	78.59	78.52	71.44
.5	78.03	78.10	77.69	75.16	81.57	81.49	81.47	76.79	87.22	87.21	87.20	78.81
1	86.71	86.77	86.47	84.13	88.37	88.42	88.29	84.13	91.46	91.46	91.46	84.13
1.5	91.95	92.01	91.79	90.04	92.47	92.52	92.40	89.18	94.01	94.01	94.01	88.05
2	95.13	95.09	95.02	93.84	95.05	95.08	94.99	92.64	95.67	95.67	95.68	90.97
3	98.33	98.32	98.17	97.72	97.79	97.78	97.75	96.63	97.63	97.63	97.63	94.81
4	99.40	99.40	99.30	99.19	98.99	99.00	98.96	98.47	98.64	98.64	98.64	97.01
5	99.78	99.78	99.75	99.72	99.53	99.53	99.51	99.31	99.20	99.20	99.20	98.27
Frequency Mean	90.25				111				100			
Severity Coefficient of Variation	2				2				25			

Note: (*) Points below zero dollar limit

(1) All four models are calculated by a HP-19 programmable calculator.

Exhibit Ia

Generalized Poisson Model

Variances of Modified Gamma, Gamma and NP vs Analytical Model

$Z(=\frac{x-\mu}{\sigma})$	γ = .5 MG/A	Sev = G/A	Gamma NP/A	γ = 1 MG/A	Sev = Inv.Normal G/A	NP/A	γ = 5 MG/A	Sev = G/A	Gamma NP/A
-1.5	0	0	.17	.11	-.59	-.21	x	x	x
-1	0	-.02	-.01	.04	.12	1.70	x	x	x
-.5	.16	.13	-.65	.10	.59	1.47	x	x	x
0	.19	.19	.16	.03	.32	.12	-.02	.03	-7.05
.5	.04	.05	-.14	-.01	-.02	-.74	.02	.02	-8.37
1	0	.02	-.20	-.06	-.17	-.92	0	.02	-7.31
1.5	0	.01	-.14	-.14	-.17	-.70	0	.01	-5.95
2	0	0	-.07	-.06	-.1	-.41	0	0	-4.71
3	0	.02	.01	0	-.01	-.08	0	0	-2.82
4	0	0	.01	0	0	.01	0	0	-1.53
5	0	0	0	0	0	.01	0	0	-.93
Variance vs Analytical Model	.006	.005	.051	.005	.080	.652	0	0	30.243

Exhibit IIa

Generalized Negative Binomial Model

Variances of Modified Gamma, Gamma and NP vs Analytical Model

$Z\left(=\frac{x-\mu}{\sigma}\right)$	$\gamma = 2$ MG/A	Sev = G/A	Normal NP/A	$\gamma = 3$ MG/A	Sev = G/A	Normal NP/A	$\gamma = 5$ MG/A	Sev = G/A	Gamma NP/A
-1.5	x	x	x	x	x	x	x	x	x
-1	x	x	x	x	x	x	x	x	x
-.5	-.54	-.51	3.11	.20	.68	9.57	x	x	x
0	-.12	-.30	-1.61	.02	-.08	-3.27	-.02	-.09	-7.17
.5	.07	-.34	-2.87	-.08	-.10	-4.78	-.01	-.02	-8.41
1	.06	-.24	-2.58	.05	-.08	-4.24	0	0	-7.33
1.5	.06	-.16	-1.91	.05	-.07	-3.29	0	0	-5.96
2	-.04	-.11	-1.29	.03	-.06	-2.41	0	.01	-4.70
3	-.01	-.16	-.61	-.01	-.04	-1.16	0	0	-2.81
4	0	-.10	-.21	.01	-.03	-.52	0	0	-1.63
5	0	-.03	-.06	0	-.02	-.22	0	0	-.93
Variance vs Analytical Model	.036	.066	3.654	.006	.055	17.933	.000	.001	30.612

Sensitivity on the Selection of Severity Distribution

Aggregate Probability (%)

Frequency = Poisson

Frequency mean = 60.383

Severity coefficient of variation =4

$Z = \left(\frac{x-\mu}{\sigma}\right)$	Inv. Normal	log-normal
-1.5	0	.01
-1	10.71	.02
-.5	37.34	8.07
0	59.98	74.71
.5	75.66	92.52
1	85.64	94.92
1.5	91.70	96.23
2	95.28	97.09
3	98.52	98.15
4	99.55	98.77
5	99.87	99.15

Severity skewness	12	68
Aggregate skewness	1.5	9.02

Increased Limited Factors

Exhibit IV

(1) $\frac{L-\mu}{\sigma}$	(2) $F(L)$	(3) $\frac{1}{\sigma}\int_0^L x\,dF(x)$	(4) $\frac{1}{\sigma^2}\int_0^L x^2\,dF(x)$	(5) $\mu_L/\sigma = (\frac{\mu}{\sigma})-(3)-((1)+\frac{\mu}{\sigma})(1-(2))$	(6) $\sigma_L^2/\sigma^2 = 1+(\frac{\mu}{\sigma})^2-(4)-2(\frac{L}{\sigma})(5)-(\frac{L}{\sigma})^2(1-(2))-(5)^2$	(7)* $(5)+c(6)$	(8)* $I(L) = (7)/.430403$
-1.5	.048710	.284267	.186949	1.324071	11.745089	1.353434	3.14457
-1	.155801	.362889	.860435	1.061534	13.363123	1.094942	2.54399
-.5	.330885	.885834	2.425967	.680985	14.543811	.717345	1.66668
0	.533291	1.589031	4.873183	.397018	13.354180	.430403	1.00000
.5	.713208	2.302312	7.704674	.210377	10.460275	.236528	.54955
1	.843333	2.882251	10.292049	.101770	7.101659	.119524	.27770
1.5	.923029	3.276798	12.246940	.045233	4.248274	.055854	.12977
2	.965591	3.508574	13.509924	.018602	2.273409	.024286	.05643
3	.994601	3.685267	14.588247	.002567	.490539	.003793	.00881
4	.999351	3.718841	14.825890	.000284	.077647	.000478	.00111
5	.999937	3.723562	14.863940	.000026	.009605	.000050	.00012

(*) Parameters: Frequency mean = 100.551724, severity coefficient of variation =2.5, aggregate coefficient $\frac{\mu}{\sigma}$ = 3.724138,

laoding coefficient c= .0025*σ . Aggregate mean is selected as the stop - loss basic policy limit

Note: Columns (2), (3) and (4) are calculated by using extended simpson's rule with integration range [0,10] subdivided into 50 intervals.

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Appendix

A. Backgrounds on distributions listed in this paper.

A1. Function types

Table A1

Frequency Distributions

Type	Density Function $p(n)$	Range of parameters
Poisson	$e^{-\lambda}\lambda^{n}/n!$	$\lambda > 0$
Negative Binomial [20]	$(1-q)^{w}\binom{-w}{n}(-q)^{n}$	$w > 0,\ 0 < q < 1$

Table A2

Severity Distributions

Type	Notation	Cumulative Dist. Function	Range of parameters
Normal	$N(x;\mu,\sigma)$	$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^{2}}dy$	$\mu,\ \sigma > 0$
Gamma	$G(x;b,p)$	$\frac{b^{p}}{\Gamma(p)}\int_{0}^{x}y^{p-1}e^{-by}dy$	$b,\ p > 0$
Inverse Normal	$I(x;a,b)$	$\frac{b}{\sqrt{\pi}}e^{2ab}\int_{0}^{x}e^{-a^{2}y-b^{2}/y}y^{-\frac{3}{2}}dy$	$a,\ b > 0$
Pareto	$P(x;c,m)$	$1-(1+x/c)^{-m}$	$c > 0,\ m \geq 2$
Log-Normal	$LN(x;d,u)$	$\frac{1}{\sqrt{2\pi}u}\int_{0}^{x}e^{-\frac{1}{2}\left(\frac{\log y-d}{u}\right)^{2}}y^{-1}dy$	$u,\ d > 0$
Log-Gamma	$LG(x;a,v)$	$\frac{a^{v}}{\Gamma(v)}\int_{1}^{x}(\log y)^{v-1}y^{-(a+1)}dy$	$v > 0,\ a > 2$

A2. Characteristic Functions

A powerful feature in the study of distribution functions and their moments is the characteristc function, $\varphi_F(t)$, associated with a given distribution function F. which is defined as [21]

20. The parameters are given by: $q = (\sigma_p^2-\lambda)/\sigma_p^2,\ w = \lambda^2/(\sigma_p^2-\lambda)$
see [9] p.167

21. See [9] Chap. 4

$$\varphi_F(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$$

where $i = \sqrt{-1}$ is the imaginary number. The overwhelming advantages of using the characteristic function is evident from the following:

(1) A moment generating function is defined over real number; the characteristic function is its complex analogue. It retains all the desirable properties of the moment generating function and unlike the moment generating function, it always exists;

(2) a standard mathematical technique known as the Laplace transformation (or Fourier transformation) asserts that as long as the characteristic function is known, one can rediscover the associated distribution function. This invertible property (not valid for moment generating function) offers an algorithm to compute the aggregate loss probability directly.

Without using these two features of characteristic functions, the derivation of our formula for the new model would be virtually impossible.

Among the six severity distribution functions listed in Table A2, only the first three have an explicit form for their characteristic function. The last three do not admit any closed form for the characteristic functions. We will derive the characteristic function of the inverse normal distribution here, and leave that of the normal and gamma to the interested reader.[22]

Letting the variable $y = z^{-2}$ in the c.d. f of inverse normal (Table A2), we have

$$I(x; a, b) = \frac{2a}{\sqrt{\pi}} e^{2ab} \int_{\frac{1}{\sqrt{x}}}^{\infty} e^{-b^2 z^2 - a^2/z^2} dz \qquad (10)$$

22. See [9] pp. 147 and 152

Now, observe that $2b\,dz = d(bz+a/z) + d(bz-a/z)$, thus

$$I(x;a,b) = \frac{e^{2ab}}{\sqrt{\pi}}\left\{\int_{1/\sqrt{x}}^{\infty} e^{-b^2z^2-a^2/z^2}\,d(bz+a/z) + \int_{1/\sqrt{x}}^{\infty} e^{-b^2z^2-a^2/z^2}\,d(bz-a/z)\right\}$$

next setting variable $y = bz + a/z$ in the first integral and $y = bz - a/z$ in the second, it follows that

$$I(x;a,b) = \frac{1}{\sqrt{\pi}}\left\{e^{4ab}\int_{b\sqrt{x}+a/\sqrt{x}}^{\infty} e^{-y^2}dy + \int_{b\sqrt{x}-a/\sqrt{x}}^{\infty} e^{-y^2}dy\right\}$$

(change $y = -z/\sqrt{2}$)

$$= \frac{1}{\sqrt{\pi}}\left\{e^{4ab}\int_{-\infty}^{-\sqrt{2x}\,b-a\sqrt{2/x}} + \int_{-\infty}^{a\sqrt{2/x}-b\sqrt{2x}}\right\} e^{-z^2/2}\,dz$$

Thus, we have a practical form for the inverse normal distribution expressed in terms of the normal distribution:

$$I(x;a,b) = e^{4ab}N\left(-b\sqrt{2x}-a\sqrt{2/x};0,1\right) + N\left(a\sqrt{2/x}-b\sqrt{2x};0,1\right) \qquad (11)$$

The calculation of the characteristic function follows closely the approach which led to the derivation of formula (11). In fact, by definition,

$$\varphi_{IN}(t) = \frac{b}{\sqrt{\pi}}e^{2ab}\int_0^{\infty} e^{-(a^2-it)/y-b^2y}\,y^{-\frac{3}{2}}\,dy$$

(change $z = y^{-1/2}$)

$$= \frac{2b}{\sqrt{\pi}}e^{2ab}\int_0^{\infty} e^{-(a^2-it)/z^2-b^2z^2}\,dz$$

$$= e^{2b(a-\sqrt{a^2-it})}\left(\frac{2b}{\sqrt{\pi}}e^{2\sqrt{a^2-it}\,b}\int_0^{\infty} e^{-(a^2-it)/z^2-b^2z^2}\,dz\right)$$

Compare the form inside the parenthesis with the r.h.s of (10), it is identical to $I(\infty;\sqrt{a^2-it},b)$[23], taking the fact that cumulative probability is always equal to 1 when the argument tends to infinity, then

$$\varphi_{IN}(t) = \exp\left(2b(a-\sqrt{a^2-it})\right) \qquad (12)$$

The comparable results for the normal and gamma distributions are given by

$$\left\{\begin{array}{lll} \varphi_N(t) = \exp(it\mu-\frac{1}{2}\sigma^2t^2), & \text{for normal} \\ \varphi_G(t) = (1-it/b)^{-p}\ , & \text{for gamma} \end{array}\right. \qquad (13)$$

---

23. Since parameters of distribution have to be real numbers, while, here we have complex numbers involved, it is thus confusing to use same notation. However, in this particular case (and except this ambiguous notation) the property of distribution still holds in the extended situation, see M. Abramowitz and I.A. Stegan: "Handbook of Mathematical Functions" National Bureau of Standards, formula 7.4.3 p. 302

A3. Background information on the Inverse Normal Distribution

An immediate consequence of deriving the characteristic function is that one can readily determine the cumulants of a givwn distribution function. Since, by definition, log $\varphi_{..}(t)$ can be formally expanded as follows:[24]

$$\log \varphi_S(t) = \mu_s(it) + \frac{1}{2}\sigma_s^2(it)^2 + \frac{1}{6}\gamma_s\sigma_s^3(it)^3 + \cdots \qquad (14)$$

where μ_s, σ_s^2 and γ_s are the mean, variance and skewness of S. Now, apply (14) to inverse normal distribution, we have

$$\begin{aligned}\log \varphi_{IN}(t) &= 2ab\left(1-(1-it/a)^{\frac{1}{2}}\right)\\ &= 2ab\left(\frac{1}{2}\left(\frac{it}{a}\right)+\frac{1}{8}\left(\frac{it}{a}\right)^2+\frac{1}{16}\left(\frac{it}{a}\right)^3+\cdots\right)\end{aligned}$$

From a comparison with the right hand side of (14), it can be deduced that

$$\mu_s = b, \quad \sigma_s^2 = \frac{1}{2}\left(\frac{b}{a}\right), \quad \gamma_s\sigma_s^3 = \frac{3}{4}\left(\frac{b}{a^2}\right)$$

Next, solving the first two equations for a and b, and placing the results in the last equation, it can be seen that:

$$a = \frac{1}{2}(\mu_s/\sigma_s^2), \quad b = \mu_s, \quad \gamma_s = 3(\sigma_s/\mu_s) \qquad (15)$$

A4. The skewness of severity distribution

The last equation of (11) proved the case of inverse normal distribution stated in Table 3. The case of the normal is quite straight forward, since:

$$\log \varphi_N(t) = it\mu - \frac{1}{2}\sigma^2 t^2$$

24. See [9] formula (4.3.3) p. 111, note that in [9], the term semi-invariants, instead of cumulants, is used.

thus $\gamma_A \sigma_A^3 = 0$ i.e., $\gamma_A = 0$. As for the gamma distribution:

$$\log \varphi_G(t) = -p \log(1 - it/b)$$
$$= \frac{p}{b}(it) + \frac{p}{2b^2}(it)^2 + \frac{p}{3b^3}(it)^3 + \cdots$$

or $\mu_A = p/b,\ \sigma_A^2 = p/b^2$ and $\gamma_A \sigma_A^3 = 2p/b^3$. It follows that

$$\gamma_A = 2\left(\frac{\sigma_A}{\mu_A}\right)$$

which completes the case for the gamma distribution.

For the other three types, it is necessary to use an alternative definition of skewness, which (if it exists) is [25]:

$$\gamma_A = E\left[\frac{X-\mu_A}{\sigma_A}\right]^3$$
$$= \sigma_A^{-3}\left(E[X^3] - 3\mu_A E[X^2] + 2\mu_A^3\right) \qquad (16)$$

In carrying out the calculation of the first three moments of the pareto, log-normal and log-gamma distribution, we have the following table:

Table A3

Type	E(X)	$E(X^2)$	$E(X^3)$
Pareto	$c/(m-1)$	$2c^2/((m-1)(m-2))$	$6c^3/((m-1)(m-2)(m-3))$
log-normal	$\exp(d+\frac{u^2}{2})$	$\exp(2d+2u^2)$	$\exp(3d+\frac{9}{2}u^2)$
log-gamma	$(1-1/a)^{-v}$	$(1-2/a)^{-v}$	$(1-3/a)^{-v}$

where in order to ensure the existence of the integration in the derivation of the 3rd moment, the following conditions have to be satisfied:

$$\begin{cases} m > 3, & \text{in pareto case} \\ a > 3, & \text{in log-gamma case} \end{cases} \qquad (17)$$

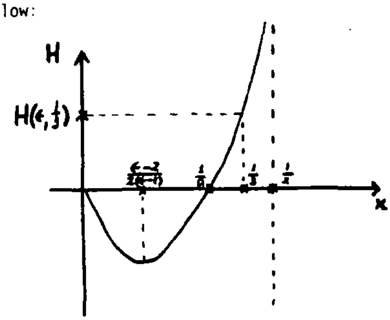
25. See [9] p. 73

Bysolving the first two columns of Table A3 for parameters in each case and substituting the results in the last column and in formula (16), we have the following table:

Table A4

Type	Parameters	Skewness
Pareto	$c=\mu_\Lambda(\mu_\Lambda^2+\sigma_\Lambda^2)/(\sigma_\Lambda^2-\mu_\Lambda^2)$, $m=2\sigma_\Lambda^2/(\sigma_\Lambda^2-\mu_\Lambda^2)$	$2\left(\frac{\sigma_\Lambda}{\mu_\Lambda}\right)\left(\frac{3\sigma_\Lambda^2-\mu_\Lambda^2}{3\mu_\Lambda^2-\sigma_\Lambda^2}\right)$
log-normal	$u=\sqrt{\log((\sigma_\Lambda^2+\mu_\Lambda^2)/\mu_\Lambda^2)}$, $d=\log(\mu_\Lambda^2/\sqrt{\mu_\Lambda^2+\sigma_\Lambda^2})$	$\left(\frac{\sigma_\Lambda}{\mu_\Lambda}\right)\left(\left(\frac{\sigma_\Lambda}{\mu_\Lambda}\right)^2+3\right)$
log-gamma[26]	a, $v=-\log\mu_\Lambda/\log(1-1/a)$	$\sigma_\Lambda^{-3}\left((1-3/a)^{-v}-3\mu_\Lambda\sigma_\Lambda^2-\mu_\Lambda^3\right)$

By using an explicit value for m in (17), the assertion of the footnote 14 for the pareto case is established. For the log-gamma case, since $(1/a)$ is the root of $H(\epsilon,x)$ (see footnote 26) and the graph of $H(\epsilon,x)$ can be portrayed as follow:



where $(\epsilon-2)/2(\epsilon-1)$ is a local minimum point of $H(\epsilon,x)$. Thus the requirement of (17) asserts that $(1/a)<\frac{1}{3}$, which is equivalent to $H(\epsilon,1/a)>0$, or

$$\epsilon \leq \frac{\log 3}{\log 3-\log 2} \doteq 2.7095$$

We thus prove the last assertion of footnote 14.

26. Since from the first two columns in Table A3, we have $\log\mu_\Lambda=-v\log(1-1/a)$ $\log(\sigma_\Lambda^2+\mu_\Lambda^2)=-v\log(1-2/a)$, hence $(1/a)$ is the solution of the following equation

$$H(\epsilon,x)=\log(1-2x)-\epsilon\log(1-x)=0, \quad \text{where } \epsilon=\log(\sigma_\Lambda^2+\mu_\Lambda^2)/\log\mu_\Lambda$$

B. Analytical Model

A fundamental equation in collective risk theory demonstrates that the aggregate cumulative distribution function F(z) of annual aggregate loss less than or equal to Z, is given by [27]

$$F(z) = \sum_{n=0}^{\infty} p(n)\, S^{n*}(z) \qquad (18)$$

where $S^{n*}(z)$ is the nth convolution of S or, equivalently, the cumulative distribution of exactly n claims with total loss less than or equal to Z and $p(n)$ is the frequency density function as listed in Table A1.

Formula (18) has practical value only when the characteristic function of S has a closed form, so that the precise form of $S^{n*}(z)$ can be derived. Among the severity functions in Table A2, only the first three meet this condition. For the rest three which do not admit a closed form for the characteristic function an alternative numerical technique has to be devised to calculate their characteristic functions, this would cause the whole computation not only time consuming but also, sometimes, very messy.

In the case of normal, gamma and inverse normal, where their characteristic functions are known, it is possible to use the following two fundamental properties of characteristic function:

(i) the characteristic function of the convolution of two functions is the product of their respective characteristic functions;

(ii) if two distribution functions possess identical characteristic functions, then the distribution functions are equal,

we can derive the explicit form for the $S^{n*}(z)$ as shown in the following table:

27. For an expository treatment of collective risk theory, please see [2] and [17].

Type	(1) $S(Z)$	(2) $\varphi_S(t)$	(3) $\varphi_S^n(t)\,(=\varphi_{S^{n*}}(t))$	(4) $S^{n*}(Z)$
Normal	$N(z;\mu,\sigma)$	$\exp(it\mu-\frac{1}{2}\sigma^2t^2)$	$\exp(itn\mu-\frac{1}{2}n\sigma^2t^2)$	$N(z;n\mu,\sqrt{n}\sigma)$
Gamma	$G(z;b,p)$	$(1-it/b)^{-p}$	$(1-it/b)^{-np}$	$G(z;b,\sqrt{n}p)$
Inverse Normal	$I(z;a,b)$	$\exp(2b(a-\sqrt{a^2-it}))$	$\exp(2nb(a-\sqrt{a^2-it}))$	$I(z;a,nb)$

In column (3), property (i) is used and in column (4), property (ii) is used.

C. Gamma Approximation

C1. The derivation of the gamma approximation

The gamma distribution $G(x;b,p)$ has only two parameters which are determined by the first two sample moments. If we add one more parameter α to the function, $G(x+\alpha;b,p)$, then the third moment is required to estimate the parameters. This procedure is called the gamma approximation.

To specify parameters x, b and p, one sets up three equations for the first two moment and skewness, then solves them for x, b and p. To do this let us first calculate the characteristic function of $G(x+\alpha;b,p)$. Since the density function of $G(x+\alpha;b,p)$ is

$$\frac{d}{dx}G(x+\alpha;b,p) = \frac{b^p}{\Gamma(p)}(x+\alpha)^{p-1}e^{-b(x+\alpha)}, \qquad -\alpha \le x < \infty$$

then the characteristic function is defined by

$$\varphi_S(t) = \frac{b^p}{\Gamma(p)}\int_{-\alpha}^{\infty} e^{itx}(x+\alpha)^{p-1}e^{-b(x+\alpha)}dx$$

(letting $y = x + \alpha$)

$$= \frac{b^p}{\Gamma(p)}\int_0^{\infty} e^{it(y-\alpha)}y^{p-1}e^{-by}\,dy$$

$$= e^{-it\alpha}(1-it/b)^{-p} \qquad (19)$$

take logrithm both sides and expand the left hand side into power series of (it), we have

$$\log \varphi_S(t) = -\alpha(it) + p\left(\left(\frac{it}{b}\right) + \frac{1}{2}\left(\frac{it}{b}\right)^2 + \frac{1}{3}\left(\frac{it}{b}\right)^3 + \cdots\right),$$

compare the coefficients of the three lowest terms of (it), we set up three equations:

$$\begin{cases} \mu_p = p/b - \alpha \\ \sigma_p^2 = p/b^2 \\ \gamma_p \sigma_p^3 = 2p/b^3 \end{cases}$$

solve them for x, b and p, we have

$$b = 2/(\sigma_p \gamma_p), \quad p = (2/\gamma_p)^2, \quad \alpha = 2/(\sigma_p \gamma_p) - \mu_p \qquad (20)$$

Therefore the gamma approximation is expressed as

$$G(x+\alpha; b, p) = \frac{b^p}{\Gamma(p)} \int_0^{x-\mu_p+2/(\sigma_p\gamma_p)} y^{\left(\frac{2}{\gamma_p}\right)^2 - 1} e^{-\left(\frac{2y}{\sigma_p \gamma_p}\right)} dy$$

(letting $z = 2y/(\gamma_p \sigma_p)$)

$$= G\left(\left(\frac{2}{\gamma_p}\right)^2 + \left(\frac{2}{\gamma_p}\right)\left(\frac{x-\mu_p}{\sigma_p}\right); 1, \left(\frac{2}{\gamma_p}\right)^2\right) \qquad (21)$$

C2. Aggregate skewness

In applying NP or gamma approximation, one needs the input of aggregate mean, standard deviation and skewness. In this section we are going to derive the aggregate skewness for both generalized poisson and negative binomial models.

As usual we first calculate the characteristic function of either model. Taking the fact that characteristic function of S^{n*} is the product of n characteristic functions of S, together with formula (18) and the explicit form of p(n) in Table A1, it is not difficult to see that the characteristic function of the aggregate distribution function F is given by.

$$\varphi_F(t) = \begin{cases} \exp(\lambda(\varphi_S(t) - 1)), & \text{for poisson} \\ \left(\frac{1 - q\varphi_S(t)}{1-q}\right)^{-w}, & \text{for negative binomial} \end{cases} \qquad (22)$$

Where $\varphi_S(t) = 1 + \mu_\Delta (it) + \mu_{\Delta,2}/2\,(it)^2 + \mu_{\Delta,3}/6\,(it)^3 + \text{----}$ is the characteristic function of the severity distribution S. Taking the logrithm both sides, it becomes

$$\log \varphi_F(t) = \begin{cases} \lambda(\varphi_S(t) - 1), & \text{for poisson} \\ -n \log\left(1 - \frac{q}{1-q}(\varphi_S(t) - 1)\right), & \text{for negative binomial} \end{cases} \qquad (23)$$

Identifying the coefficients of $(it)^3$ both sides, we can evaluate [28] the aggregate skewness. They are given as follows:

Table A5

Aggregate Skewness

Generalized poisson model	Generalized negative binomial model
$\dfrac{\mu_{\Delta,3}}{\sqrt{\lambda}\,(\mu_{\Delta,2})^{3/2}}$	$\dfrac{2(\sigma_p^2-\lambda)^2\mu_\Delta^3/\lambda + 3(\sigma_p^2-\lambda)\mu_{\Delta,2}\mu_\Delta + \lambda\mu_{\Delta,3}}{(\sigma_\Delta^2\mu_\Delta^2 + \lambda\sigma_\Delta^2)^{3/2}}$

Table A5 is a general formula for aggregate skewness and does not use the precise form of severity statistics. If the individual severity type is incorporated the formula for skewness would bear the following form.

28. This is done by expanding $\varphi_S(t)$ into series of (it), and using footnote 20 for the negative binomial model.

Table A6
Aggregate Skewness

Severity/Frequency	Poisson	Negative binomial
normal	$(1+3c_{\Lambda}^2)/\sqrt{\lambda(1+c_{\Lambda}^2)^3}$	$\{\lambda(1+3c_{\Lambda}^2)+A\}/B$
gamma	$(1+2c_{\Lambda})^2/\sqrt{\lambda(1+c_{\Lambda}^2)}$	$\{\lambda(1+c_{\Lambda}^2)(1+2c_{\Lambda})^2+A\}/B$
inverse normal	$(3c_{\Lambda}^4+3c_{\Lambda}^2+1)/\sqrt{\lambda(1+c_{\Lambda}^2)^3}$	$\{\lambda(3c_{\Lambda}^4+3c_{\Lambda}^2+1)+A\}/B$
pareto	$3\sqrt{1+c_{\Lambda}^2}/((3-c_{\Lambda}^2)\sqrt{\lambda})$	$\{3\lambda(1+c_{\Lambda}^2)/(3-c_{\Lambda}^2)+A\}/B$
log-normal	$(1+c_{\Lambda}^2)^{1.5}/\sqrt{\lambda}$	$\{\lambda(1+c_{\Lambda}^2)^3+A\}/B$
log-gamma[29]	$((1-3/a)/(1-2/a))^{-v}/\sqrt{\lambda}$	$\dfrac{\lambda(1-3/a)^{-v}+3(\sigma_p^2-\lambda)\mu_{\Lambda}\mu_{\Lambda 2}+2(\sigma_p^2-\lambda)^2\mu_{\Lambda}^3/\lambda}{(\sigma_p^2\mu_{\Lambda}^2+\lambda\sigma_{\Lambda}^2)^{3/2}}$

where $c_{\Lambda}=\sigma_{\Lambda}/\mu_{\Lambda}$ is the severity coefficient of vatiation,
A= $3(\sigma_p^2-\lambda)(1+c_{\Lambda}^2)+2(\sigma_p^2-\lambda)^2/\lambda$ and B= $(\sigma_p^2+\lambda c_{\Lambda}^2)^{3/2}$

D. Modified Gamma Approximation

D1. Derivation fo the new model

The approach we adopt for the new bodel is that, first we use the gamma approximation technique to match the selected severity distribution (one of the six types in Table A2) by identifying the statistics $\mu_{\Lambda}, \sigma_{\Lambda}$ and γ_{Λ} with those of the slected type, then utilize the following well known formula[30]

$$F(x) = \frac{1}{2} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \varphi_F(t)\, e^{-itx}\, t^{-1}\, dt \qquad (24)$$

to invert the aggregate distribution function from its characteristic function. Thus what we have to do here is to substitute $\varphi_F(t)$ in (24) by (22), then simplify it to the form given in (1), Table 1 and Table 2.

29. See footnote 26 for a and v.

30. This formula uses property (ii) discussed in section A. For detailed information on Laplace transformation please see [17].

Now, let $f(t)$ be the modules (or absolute value) of $\varphi_F(t/\sigma)$ and $-g(t)$ be the arguments of $\varphi_F(\frac{t}{\sigma})$, then [31]

$$\varphi_F(t/\sigma) = f(t)\, e^{-ig(t)} \tag{25}$$

where both f(t) and g(t) are real numbers. Replace $s = t\sigma$ in(24) and replace $\varphi_F(\frac{s}{\sigma})$ by (25), it turns out that

$$F(x) = \tfrac{1}{2} - \frac{1}{2\pi i}\left(\int_0^{\infty} + \int_{-\infty}^{0}\right) f(s)\, e^{-i(g(s) + sx/\sigma)} s^{-1}\, ds$$

(Change the 2nd integral by s =-t)

$$= \frac{1}{2} - \frac{1}{2\pi i}\left(\int_0^{\infty} f(t)\left(e^{-i(g(t)+tx/\sigma)} - e^{i(g(t)+tx/\sigma)}\right)\frac{dt}{t}\right)^{32}$$

(taking the fact that $e^{iy} - e^{-iy} = 2i\,\mathrm{Sin}(y)$)

$$= \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} f(t)\, \frac{\mathrm{Sin}(tx/\sigma + g(t))}{t}\, dt$$

which is the form given in (1). The next step is to find out f(t) and g(t) in either models. To continue our derivation, let us decompose $\varphi_S(\frac{t}{\sigma})-1$ into two parts:

$$\varphi_S(t/\sigma) - 1 = h(t) + i\,k(t) \tag{26}$$

then identify (25) with (22) via (26), we have

$$f(t)e^{-ig(t)} = \begin{cases} e^{\lambda h(t)} \cdot e^{i\lambda k(t)}, & \text{for poisson} \\ \left(1 - \left(\frac{q}{1-q}\right)h(t) - i\left(\frac{q}{1-q}\right)k(t)\right)^{-\alpha}, & \text{for negative binomial} \end{cases} \tag{27}$$

The case of the generalized poisson model in Table 1 is obvious from (27).

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31. An complex number can be expressed by its modulus and argument, see [6] p.6

32. Here we take the following fact that

$$\varphi_F(-t) = \int_{-\infty}^{\infty} \cos tx\, dF(x) - i\int_{-\infty}^{\infty} \sin tx\, dF(x) = \overline{\varphi_F(t)}$$ (the conjugate of $\varphi_F(t)$)

to demonstrate that $f(-t) = f(t)$ and $g(-t) = -g(t)$, and ultilize in our derivation, for conjugate number see [6].

For the generalized negative binomial, by the definition of modulus and argument[33], we find that

$$\begin{cases} f(t) = \left\{\left(1-\left(\frac{q}{1-q}\right)k(t)\right)^2 + \left(\frac{q}{1-q}\right)^2 k(t)^2\right\}^{-\frac{w}{2}} \\ g(t) = w \tan^{-1}\left(\frac{\lambda q k(t)/(1-q)}{\lambda - \lambda q k(t)/(1-q)}\right), \end{cases}$$

last item that has to be verified is Table 2. This is straight-forward, since by (19) and (20), we have

$$\varphi_S(t/\sigma) = \delta(t)e^{i\theta(t)} = \delta(t)\left(\cos\theta(t) + i\sin\theta(t)\right) \qquad (28)$$

where

$$\begin{cases} \delta(t) = \text{absolute values of } \left(1 - it\gamma_0\sigma_0/2\right)^{-\left(\frac{2}{\gamma_0}\right)^2} \\ \theta(t) = t\left(\frac{\mu_0 - 2\sigma_0/\gamma_0}{\sigma}\right) + \text{argument of } -\left(\frac{2}{\gamma_0}\right)^2 \log\left(1 - it\gamma_0\sigma_0/2\right) \end{cases} \qquad (29)$$

Comparing both sides of (26) and (28), and exploring the left hand side of (29), we come to the results of Table 2

D2. Formula (1) via Extended Simpson's Rule

The extended Simpson's rule is adequate to handle the numerical integration of formula (1) and Table 5. Since integration over an infinite interval is practically impossible, it is necessary to integrate over a truncated interval. The limit, R, of the range of integration has to be determined. Also the size, h, of the equally divided subinterval has to be chosen in ultilizing Simpson's rule.

33. See [6] p.5-7

34. For a detailed treatment of this, please see Stephen G. Kellison: 'Fundamentals of Numerical Analysis' Richard D. Irwin. Inc. 1975, Chap. 8

If a precision up to the sixth decimal point is required, h=.2 would be satisfactory. The selection for R would be more complicate. A quick and practical way to select the appropriate R is to input t until the value of

$$f(t)\, Sin\left(tx/\sigma + g(t)\right)/(\pi t)$$

is less than, say, 10^{-6}. Choose that t for R. Generally, the appropriate value of R would fall in to the range from 10 to 100, depend on γ_p and λ .