TITLE: CREDIBILITY AND SOLVENCY

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I. Introduction

An important fact brought into sharp focus by the papers submitted to last year's program is that a healthy insurance enterprise not only must produce adequate earnings but must do so steadily and predictably. That is to say: the risks that confront the enterprise must be tightly controlled. Management must steer a course mindful not only of reasonable expectations but also of unforeseen deviations therefrom. This is true of the insurance enterprise in particular because of the urgent character of its obligations to policyholders who have suffered insured losses.

Last year's contributors also made clear that the risks inherent in the operation of the insurance enterprise cannot be isolated or sequestered: every source of financial uncertainty either in the environment or internal to the enterprise inevitably impinges on its ability to make that essential transaction - the loss payment. One of the most conspicuous areas where control of financial risk is essential is in the pricing of the business. It is misleading to think - let alone say - that the actuary can exert detailed control of the pricing process by prescribing and enforcing a static, cost-plus formula. The keyword here is "process". Pricing is a dialectical process engaging the potential insurer and insured within the market environment. Neither participant in the process can count on having complete control. Much that passes under the name of "pricing" is what engineers call "costing" and provides the insurer with the indispensable knowledge of the region in which to break off negotiation: between the economist's "break-even" and "shutdown" points. Successful pricing involves using the results of the costing formulas with art and judgment to anticipate the condition of the market. This is the first source of risk which the enterprise must face: that of misjudging the market and rushing in at a price at which too little (or too much) can be sold.

A second source of risk is that involved in estimating the aggregate cost of benefits which have been purchased in the market. Minimizing this risk is the chief endeavor of the pricing actuary. His chief ally is the law of large numbers; his chief foe is the passage of time: the time it takes for data to reach him and the time that elapses between his calculation and the losses it purports to forecast.

A third source of risk arises from a unique characteristic of insurance: the true ultimate meaning of the insurer's contractual obligation is as varied as the insureds themselves: no two alike. This is the joint domain of actuary and underwriter and this risk of inhomogeneity within the aggregate is attacked by such devices as classification ratemaking and experience rating, and, after all other bolts are shot, underwriting judgment. Since the underwriting decision is essentially yes/no, it is less risky to the extent that the actuary can propose a price that is graduated to be appropriate to the individual applicant. That is to say, as we all know, the underwriter is on the razor's edge and needs all the help he can get in deciding whether or not to expose the insurer's assets to risk. It is on this problem of resolving the inhomogeneity of the actuarial aggregate, in a way that minimizes the insurer's financial risk, that we shall concentrate in this paper: the problem of credibility. We have been careful in setting the stage because we shall follow the same course mathematically as we have conceptually: we shall define the problem in a way that leaves it distinct, as much as possible, from the other great problems of market strategy and aggregate valuation and forecasting. On the other hand we shall argue strenuously that credibility in classification ratemaking and credibility in individual risk rating are essentially the same problem on different scales, differing only in the relative importance of small sample corrections, and are not distinct fields of study as traditional actuarial practice would lead us to suppose.

This program will not lead us to present new work on credibility or even the most advanced. Our emphasis will be on simplicities rather than complexities, on the robust rather than the ethereal. For example it will be emphasized that it is more important for a credibility scheme to be well tuned - adjusted to minimize practical risk - than that it should embody a sophisticated model with the most general assumptions. And we shall spend significant space discussing the estimation problem for the parameters of the credibility model. Further we shall restrict ourselves to quadratic risk: that is, we shall consistently characterize risk in terms of the second moments of the financial variables. This is time-honored practice in risk theory, and we adopt it here on the premise that it is better to have a primitive definition that one can work with than to waste time and effort refining it to something unworkable.

We shall attempt to focus attention on the conceptual streams which feed into current practical work on credibility, particularly that going forward at I.S.O. These two converging streams are that initiated by Bülhmann and Straub in their landmark 1972 paper (1), which established the requirements for practical application of Bayesian Credibility, and that began by Charles Stein in his work on estimates with minimal quadratic loss (2).

In a concluding section we shall examine the benefits which flow from a credibility program tuned and managed for optimal risk control. In particular we shall see the implications for market strategy, social responsibility, and financial management.

Finally, in a series of technical appendices, we review a number of simple stochastic models for the loss ratios of a class of individual risks (or a group of classes). We show how these models all lead to a simple model for the covariance structure of the loss ratios over time and how this covariance model leads naturally to the familiar credibility formula as a predictor of future loss ratios. A final appendix shows an application of a simplified model to numerical data.

II. Decomposing the Chain of Risks

In the introduction we identified the sources of risk in the pricing process and declared that these can be dealt with separately. This is a declaration of hope because these risks are dealt with separately in practice. Indeed, the structure of the pricing process makes such separate treatment almost a practical necessity. Market risk is dealt with at the executive levels of underwriting, marketing, and financial management using as input knowledge of market conditions, aggregate loss costs, and administrative expenses. These deliberations can be knocked badly askew if aggregate loss cost projections are inaccurate.

These projections, in turn, are of diminished utility if the means of dealing with inhomogeneity in the aggregates are inadequate. This is particularly true in a competitive environment where the insurer's competitors are using accurate class rates and experience rating plans to reach pricing and underwriting decisions. In such a situation a company with a crude pricing apparatus will find that its aggregates are in fact unstable, with better business leaving and worse business coming in.

Thus we see that each level of the pricing process depends on the adequacy of risk control at the next, more detailed level. The question confronting us on the technical plane is whether or not these levels can be modeled independently:

Can the computation and trending of the aggregate be carried out without regard to its composition in terms of classes and in such a way that classification relativities can be defined free of secular trend?

Can the rate for a particular classification be further split by an individual experience model utilizing only information pertaining to that classification?

The first question can be answered in the affirmative if we give due consideration to two issues: first, the obvious problem encountered in tracking any aggregate through time, that of shifts in the makeup of the insured population by class; second, the rather more arcane problem that arises because the aggregate from which we split the classification relativities is not exactly known but is subject to sampling uncertainties.

The problem of shifting risk populations can be dealt with by a method familiar to demographics: define or project a standard risk population and restate historical aggregates in terms of it. Once this is done, there is an extensive repertoire of trending models that can be applied to the adjusted numbers - pure premiums are the most likely candidates for such treatment. Not to get too far from our main thrust, we should remark that the ubiquitous least squares trend line is neither the simplest nor - it is likely - the most robust of these models. The process has twin but complementary goals: to achieve an accurate projection of aggregate rate level and to allow classification relativities to be treated as a bundle of stationary time series, that is, to allow the problem of assigning relativities to be detached from that of projecting secular movements.

The more arcane problem; that the relativities will be based on aggregate numbers which are themselves uncertain, will be addressed in succeeding sections.

The second question, whether individual risk relativities can be split reasonably from the classification relativities, is somewhat more subtle and elusive. Classification rates are dependent on objective characteristics of the individual risks. The information used is limited to what can be recorded economically and more or less reliably. The most efficient use of this information is a matter beyond the scope of this paper. But there can be no dispute that, no matter how efficiently classification variables are utilized, a residuum of useful information will remain in the form of individual risk experience records. It has long been recognized that this information can be utilized to refine the pricing of individual contracts. How to do this most efficiently and, indeed, how to define efficiency in this situation is one of our principal topics. Some difficult problems make this task less than straightforward. One of the hardest is how to define and establish the identity of an individual risk whose characteristics may be changing over time. We shall have more to say about this later.

We have advanced about as far as we can using words only. In the next section we shall invoke a particular stochastic model to illustrate our previous remarks.

III. A Stochastic Model for Credibility

Without attempting to settle all the questions raised in the last section, we shall propose here a simple and plausible covariance model for the behavior of individual risk relativities within a given class. Assuming stationarity and a stable covariance structure, we shall arrive at a best linear unbiased estimator for the individual relativity, in terms of past experience, presuming a known aggregate value. The model has a classification structure also, and the best method for computing class rates and assigning credibility emerge from the model as well.

A. Variables and Notations

For the sake of concreteness we shall couch the model in terms of loss ratios. This allows closer comparison with the pioneering work of Buhlmann and Straub. The loss ratio is also a recognized index of equity, and the goal of most experience rating plans is to equalize loss ratios - properly defined - across all insureds, insofar as possible.

The variable of interest, then, we shall denote as

$$\tilde{x}_{Aa}(t),$$

the loss ratio for the a-th risk in class, A, during period, t. The superposed tilde denotes treatment as a random variable. For convenience, the time label, t, will be reckoned backwards, and we shall attempt to forecast experience in period, 0.

The only assumptions we shall need to characterize the variable for our purposes apply to the first two moments:

$$E[\,\tilde{x}_{Aa}(t)\,] = \xi, \qquad (1)$$

$$E\left[(\tilde{x}_{Aa}(t)-\xi)(\tilde{x}_{Bb}(u)-\xi)\right] = \delta_{AB}\,\sigma^3\left[\frac{1}{K} + \delta_{ab}\left(\frac{1}{K_A} + \frac{\delta_{tu}}{W_{Aa}(t)}\right)\right]. \qquad (2)$$

The first condition is the global expectation, without reference to class membership or individual experience records. The notation, δ_{ij}, is the Kronecker delta which is unity if the indices match, zero otherwise. The parameter, K, measures the homogeneity of the entire aggregate; K_A measures the homogeneity of the particular class, A. The quantity, $W_{Aa}(t)$, measures the relative statistical weight of individual risk experience, thus characterizing the size of the risk. For practical purposes, we may take it to be the individual risk premium computed at the aggregate rate before classification and experience rating. Statistical research with copious and accurate data might yield a better choice, but experience teaches not to hold one's breath but to proceed as best one can.

B. <u>Credibility Estimator</u>

The next step in our conceptual process is to propose an Ansatz for the form of the best estimator for predicting individual risk experience in time period, zero. This estimator should be:

1) linear in the aggregate and experience variables,
2) confined to experience within the class (since we have assumed a fixed, known aggregate),

3) unbiased, as our method of construction will ensure.

These requirements lead to the Ansatz,

$$\hat{x}_{Aa}(\theta)-\xi = \sum_{b,t} A_b^{Aa}(t)\left(\hat{x}_{Ab}(t)-\xi\right) + \tilde{\varepsilon}_{Aa}(o), \quad (3)$$

where $\tilde{\varepsilon}_{Aa}(0)$ is a random variable expressing the residual variation after our attempt at prediction. It is this variation that we wish to minimize in some sense by seeking an optimal form of the coefficients, A.

How best to go about this requires some thought. We might follow Mayerson's approach in his famous paper on Bayesian credibility (3) and minimize $E[\tilde{\varepsilon}_{Aa}(0)^2]$. This optimizes individual equity. More important than individual equity, however, is the insurer's aggregate risk in this class of business since the security of all policyholders depends in part on the control of this risk. This would lead us to seek a solution minimizing $E\left[\left(\sum_a W_{Aa}(0)\tilde{\varepsilon}_{Aa}(0)\right)^2\right]$. If there were indeed a conflict, we should have to choose the latter. The marvelous fact is that no conflict exists: the individual equity solution also minimizes the aggregate risk, as we shall demonstrate in Appendix B.

Let us now compute the expectation of the squared individual risk residual. In our abbreviated notation an asterisk denotes summation on the indicated index.

$$R_{Aa} \equiv \frac{1}{\sigma^2} E\left[\, \tilde{\varepsilon}_{Aa}(0)^2 \,\right] = \frac{1}{K} + \frac{1}{K_A} + \frac{1}{W_{Aa}(0)} \tag{4}$$
$$- \frac{2}{K} A_*^{Aa}(*) - \frac{2}{K_A} A_a^{Aa}(*) + \frac{1}{K}\left(A_*^{Aa}(*)\right)^2 + \frac{1}{K_A}\sum_b \left(A_b^{Aa}(*)\right)^2$$
$$+ \sum_{bt} \frac{1}{W_{Ab}(t)}\left(A_b^{Aa}(t)\right)^2 + 2\lambda\left(A_*^{Aa}(*) - \sum_b A_b^{Aa}(*)\right) + 2\sum_b \mu_b \left(A_b^{Aa}(*) - \sum_t A_b^{Aa}(t)\right).$$

The extra terms with Lagrangian coefficients, λ and μ_b are added to enforce the sum constraints while treating the starred objects as independent variables. This model is solved in Appendix B. Here we merely state the solution:

$$A_b^{Aa}(t) = \delta_{ab}\, Z_{Aa}(t) + \left(1 - Z_{Aa}(*)\right) \frac{K_A Z_{Ab}(t)}{K + K_A Z_{A*}(*)},$$

$$\text{where} \quad Z_{Aa}(t) = \frac{W_{Aa}(t)}{K_A + W_{Aa}(*)}. \tag{5}$$

This leads to the prediction,

$$\tilde{x}_{Aa}(0) = \left(1 - Z_{Aa}(*)\right)\left\{\left(1 - \frac{K_A Z_{A*}(*)}{K + K_A Z_{A*}(*)}\right)\xi + \frac{K_A \sum_{bt} Z_{Ab}(t)\tilde{x}_{Ab}(t)}{K + K_A Z_{A*}(*)}\right\}$$
$$+ \sum_t Z_{Aa}(t)\,\tilde{x}_{Aa}(t) + \tilde{\varepsilon}_{Aa}(0). \tag{6}$$

The above is a beautiful result which should gladden the hearts of all actuaries. What it tells us is that, in the context of our covariance model, credibilities can be nested at different levels of aggregation, providing only that the aggregation is carried out properly.

Within this bristling expression, we may identify:

the partial credibilities, $Z_{Aa}(t)$, applying to the experience of risk, Aa, in period, t,

their complement, $\left(1 - Z_{Aa}(*)\right)$, applying to the class aggregate,

the experience estimate of the class aggregate,

$$\frac{1}{Z_{A*}(*)} \sum_{b,t} Z_b^{Aa}(t) \; \tilde{x}_{Ab}(t)$$

and its credibility, $\dfrac{K_A Z_{A*}(*)}{K + K_A Z_{A*}(*)}$, relative to the global aggregate, ξ.

All this falls out of the model and a simple Ansatz with no need for approximation.

This nested structure, with its easy identification and separation of individual and aggregate elements, is the structure that has always been assumed in actuarial practice. It is gratifying to know that it also has an appealing axiomatic justification in the context of risk and estimation theory. One loose end remains, however: the parameters K and K_A, one for the aggregate, one for each classification in the scheme. These must be estimated from the data available and this considerable problem will occupy us in the next section. Before embarking on this project, however, let us consider some straightforward extensions of our model.

C. Extensions of the basic Model

It is fairly easy to compute the consequences of relaxing the assumption that the aggregate loss ratio, ξ , is known and fixed. Doing so, we are led to the predictive relationship,

$$\tilde{x}_{Aa}(0) = \sum_{Bbt} A^{Aa}_{Bb}(t)\,\tilde{x}_{Bb}(t) + \tilde{\varepsilon}_{Aa}(0),$$

with the bias constraint,

$$A^{Aa}_{**}(*) = 1$$

Note that the sum now extends outside the class, A. Also, the global mean, ξ, does not appear: the model tells us how to compute it from the experience data. The result is identical to equation (6), except that where ξ appears in (6) we have now

$$\hat{\xi} = \frac{1}{\zeta_{**}(*)} \sum_{Bbt} \zeta_{Bb}(t)\,\tilde{x}_{Bb}(t),$$

where

$$\zeta_{Bb}(t) = \frac{K_B Z_{Bb}(t)}{K + K_B Z_{B*}(*)} .$$

Another natural extension of the model is to take into account the problem raised in the last section: that the identity of an individual risk changes as time elapses and that the relevance of experience decays with age. Let us suppose that this decay is exponential in time. The indicated covariance model becomes

$$E\left[(\tilde{x}_{Aa}(t)-\xi)(\tilde{x}_{Bb}(u)-\xi)\right] = \delta_{AB}\left[\frac{1}{K} + \delta_{ab}\left(\frac{1}{K_A} + \frac{\delta_{tu}}{W_{Aa}(t)}\right)\right] e^{-\lambda|t-u|},$$

$$\lambda > 0.$$

This model is much more difficult to solve (and to apply in practice) than the model of equation (1). At this writing, the general solution is not available. We can, however, exhibit the partial credibilities in a simple, one-class model using two periods of data:

$$Z(1) = \frac{b\,W(1)\left[1 + \frac{W(2)}{K}(1-b^2)\right]}{K + W(1)\left[1 + \frac{W(2)}{K}(1-b^2)\right] + W(2)},$$

$$Z(2) = \frac{b^2 W(2)}{K + W(1)\left[1 + \frac{W(2)}{K}(1-b^2)\right] + W(2)},$$

where $b = e^{-\lambda}$.

This form is less than appealing, but not out of the question in a computerized experience rating system. The form becomes progressively more complex as more periods of data are introduced. It is seen that, as $b \to 1$ these expressions approach the more familiar partial credibilities defined with equation (5).

IV. Estimation

As we have remarked, our basic model, with the linear Ansatz, prescribes the form of the credibilities at all levels of aggregation and even tells us how to compute the most efficient linear estimates of the aggregates at each level. (Our results apply to a simple, cellular scheme of classification. More complex schemes, involving several class differentials, require separate study.) We have, however, to estimate the homogeneity parameters, and our preferred forms make that task singularly awkward. Regardless of difficulties, this task of estimation cannot be bypassed. Reliance on judgment produces an untuned credibility scheme that can do more harm than good. The administration of an experience rating plan is two expensive and time consuming to leave its risk-control aspect out of reckoning since it is just this that justifies the trouble and expense.

In this section we shall discuss several aspects of the estimation problem, starting with the algebraic estimation scheme of Bühlmann and Straub. We shall then discuss a more elementary approach: that of non-linear estimation by searching on the parameter space. We shall then conclude with some brief remarks on alternative estimation schemes associated with the name of the statistician, Charles Stein.

A. Algebraic Estimation

We use the term algebraic estimation to denote the use of statistics which can be computed directly and combined to yield a solution for the parameter in question. Following the method of Buhlmann and Straub in their 1972 paper (1), we may define the statistics, for our model defined in equation (1) of Section II,

$$\tilde{S}_A^{(1)} = \sum_{at} W_{Aa}(t)\left[\tilde{x}_{Aa}(t) - \bar{x}_{Aa}(*)\right]^2,$$

and

$$\tilde{S}_A^{(2)} = \sum_{at} W_{Aa}(t)\left[\tilde{x}_{Aa}(t) - \bar{x}_{A*}(*)\right]^2,$$

with

$$\bar{x}_{Aa}(*) = \frac{1}{W_{Aa}(*)} \sum_t W_{Aa}(t)\,\tilde{x}_{Aa}(t) \quad ,$$

and

$$\bar{x}_{A*}(*) = \frac{1}{W_{A*}(*)} \sum_{at} W_{Aa}(t)\,\tilde{x}_{Aa}(t),$$

for each class, A, and the global statistics,

$$\tilde{T}^{(1)} = \sum_A \tilde{S}_A^{(2)} \, ,$$

$$\tilde{T}^{(2)} = \sum_{Aat} W_{Aa}(t)\left[\tilde{x}_{Aa}(t) - \bar{x}_{**}(*)\right]^2.$$

Using the defining equation of the model, we find,

$$E[\tilde{S}_A^{(1)}] = \sigma^2\left\{N_{A*}(*) - 2 + \sum_{at}\left(\frac{W_{Aa}(t)}{W_{Aa}(*)}\right)^2\right\},$$

$$E[\tilde{S}_A^{(2)}] = \sigma^2\left\{\frac{W_{A*}(*)}{K_A}\left[1 - \sum_a\left(\frac{W_{Aa}(*)}{W_{A*}(*)}\right)^2\right] + N_{A*}(*) - 2 + \sum_a\left(\frac{W_{Aa}(*)}{W_{A*}(*)}\right)^2\right\},$$

$$E[\tilde{T}^{(1)}] = \sum_A E[\tilde{S}_A^{(2)}],$$

$$E[\tilde{T}^{(2)}] = \sigma^2\left\{\frac{W_{**}(*)}{K}\left[1 - \sum_A\left(\frac{W_{A*}(*)}{W_{**}(*)}\right)^2\right] + \sum_A \frac{1}{K_A}\left[W_{A*}(*) - \sum_a\left(\frac{W_{Aa}(*)}{W_{A*}(*)}\right)^2\right] + N_{**}(*) - 1\right\}.$$

The important thing to notice about this system of expressions is that it can be solved for the K_A's and K in terms of the expectations and the exposures and risk counts (the N's). Replacing the expectations by the actual sums of squares then gives asymptotically unbiased estimators for the parameters, K and K_A, (all A).

For ease of computation, these estimators are unmatched. They do, however, possess some drawbacks. First, estimation of the K's involves taking ratios of sums of squares, and small sample bias may be substantial. This problem can be circumvented only by introducing a hypothesis for the detailed form of the distributions - something we would rather avoid. Second, the estimation scheme involves computing the aggregates in a way we have already seen to be suboptimal. Third, and most important from a practical viewpoint, the solution for the K's involves taking the difference of sums of squares with appropriate (positive) coefficients. This can, and sometimes does, lead to a negative estimate for $\frac{\sigma^2}{K}$ or $\frac{\sigma^2}{K_A}$, as should never be since these are components of variances and must be positive to make sense. One recourse, in such a situation is just to set $\frac{\sigma^2}{K}$ to zero ($K \to \infty$), a solution corresponding to perfect homogeneity. Considering the extreme ease of computation in this scheme, along with the countervailing drawbacks, it seems sensible, in the age of the electronic computing machine, to use these estimates as starting values in a more accurate iterative scheme. This brings us to our next topic.

B. Non-linear Estimation

On this subject, there is always much less to be said than to be done. This is the brute force method for estimating non-linear models when short cuts prove unacceptable. Given a model for the individual risk residuals, the method is applied in two stages: first define a loss function, usually a linear combination of the squared residuals, which we wish to minimize; second, vary the parameters of the model, following an efficient search algorithm, until no further improvement in the loss function can be achieved. Both stages are fraught with difficulty, and a discussion of search algorithms is entirely beyond the scope of this paper. Although refinements are possible, and perhaps desirable, a choice for the loss function of the form,

$$\tilde{\mathcal{L}} = \sum_{Aa} W_{Aa}(o)\, \tilde{\varepsilon}_{Aa}(o \mid \hat{K})^2$$

will usually serve. (Here $\hat{K}$ refers to estimated values of K and all the K_A, one for each class.) For the purpose of evaluating the efficiency of the model, $\tilde{\mathcal{L}}$ may be compared to

$$\tilde{\mathcal{L}}_o = \sum_{Aa} W_{Aa}(o) \left(\tilde{x}_{Aa}(o) - \hat{\xi} \right)^2$$

where $\hat{\xi}$ is the assumed aggregate value or the estimated value defined in Section III-C. The efficiency can be defined as

$$E = 1 - \frac{N}{N-n-1} \left(\frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{L}}_o} \right),$$

where N is the number of risks and n is the number of classes. Small sample corrections have been applied to account for the number of parameters (K's) that must be estimated. If the estimated value, $\hat{\xi}$, is used, it is preferable to minimize $\tilde{\mathcal{L}}/\mathcal{L}$.

If there are many classes in the model, it may be desirable to minimize in stages, guessing at the aggregates, optimizing K_A within each class, combining these results to find a value of K, recomputing the aggregates, and iterating until the loss function ceases to improve. Such an approach was suggested by Morris and Van Slyke in their recent paper. It is impossible to specify more detail outside a practical situation, but a simplified treatment using data from Bühlmann and Straub's 1972 paper is described in Appendix C.

C. Stein Estimators and Small Sample Corrections

In recent years, some very interesting developments have taken place in the statistical community which have an important bearing on actuarial credibility concepts. We refer to work in estimation theory initiated by Charles Stein (2). There is not space for thorough discussion, but the basis of these estimation techniques lies in shrinking the individually estimated means for an inhomogeneous ensemble toward some prior estimate or guess at the true means. The shrinkage factor varies according to the expected variance of the individual member and is chosen to minimize a quadratic loss function summarizing the deviations of the entire ensemble.

If all this sounds familiar, it is no wonder. If the center of the shrinkage is taken to be an estimate of the common aggregate mean, the procedure satisfies the definition of an actuarial credibility technique. Further the explicit goal of the estimation, minimization of quadratic loss, is identical to the goals of Bayesian credibility analysis. Indeed, a recent paper under review by the adventurous and innovative Subcommittee on Credibility at ISO, treats a scheme nearly identical to that proposed in Section III of the present work. The paper is that of Morris and Van Slyke (4), and their formal results are identical to ours except that the individual risk credibility turns out to have the form,

$$Z_a = 1 - \frac{n-3}{n}\,\frac{K}{K+W_a}$$

where n is the number of risks in the sample. As n becomes large, this approaches the usual Bayesian result. The disturbing point is that the small-sample correction is not derivable from their formalism, but must be introduced freehand, mainly to compensate bias introduced due to the use of maximum likelihood estimation assuming normal distributions.

It is interesting to compare with the Bayes-Mayerson formulation that we used in Section III. In this paper we let the minimal risk condition dictate the computation of aggregates, and the small-sample corrections did not appear. If, on the other hand, we had used straight premium weighting, as in the

Buhlmann Straub procedure, the model would have bristled with small-sample corrections of the clumsiest sort. These corrections are eliminated because they are built into the aggregates which are not fixed but slide around as the homogeneity parameters are varied. Since Morris and Van Slyke compute their aggregates in the same way, their small-sample correction must arise entirely from the method of estimation. Their method of maximum likelihood assuming normal distributions imposes a level of hypothesis which one would like to avoid in insurance statistics, if at all possible. It seems that a quest is in order for a method of estimation which does not have these drawbacks. We propose the method advanced earlier in this section as a possible candidate, deserving more study.

V. Conclusion

In conclusion, we shall review our main points, remark on some connections with financial theory, and close with a brief discussion on the systematics of practical use of these ideas.

Some of the lessons we have learned deserve underlining:

- A credibility scheme for pricing that is well-tuned for risk control can have significant impact on the marketing, underwriting, and financial aspects of an insurance business, on marketing through improved individual equity, on underwriting by allowing greater flexibility in the choice of insured and more rational review of individual risk experience, on financial by decreasing uncertainty in underwriting results consequently allowing more lucrative employment of the investment portfolio.

- An untuned credibility scheme cannot be expected to do any of these things optimally, or even reliably. In particular, the "classical" claim count credibility used in classification ratemaking is very rigid and difficult to tune and poorly adapted to risk control requirements.

- A credibility scheme cannot be expected to follow trends accurately, nor to prescribe the most efficient system of classification. These must be treated separately, though the credibility scheme may produce technical corrections to the global and class aggregates.

- Our proposed model, extended to estimate the overall aggregate corrections, dictatesthe splitting out of relativities from the top to the bottom of the process, including class credibilities, and has the convenient nested structure always assumed in class and experience rating.

- Although the question of estimation is far from settled. The recent work of Morris and Van Slyke shows that it is possible, by an iterative scheme, to go beyond the limitations of the Buhlmann-Straub estimators. Extensive lore on non-linear estimation techniques is available as a guide in such work.

- A great deal of credit is due to the ISO Subcommittee for bringing these ideas close to practical application.

It is perhaps enlightening to examine our results in the light of some aspects of financial theory which came to the fore in last year's call paper program. In his prize-winning paper, (5), Robert Butsic brought forward the distinction between systematic and non-systematic risk: non-systematic risk is subject to the law of large numbers and can be reduced, relatively, by increasing the book of business; systematic risk cannot. The significance of this distinction to pricing is that an insurer can load its rates to cushion against systematic risk since it affects companies of all sizes in the same way. In a market with efficient price competition, however, it is not possible to load rates against non-systematic risk since smaller companies will price themselves out of the market. Inspection of the covariance assumptions of our model shows that it combines both elements: inhomogeneity is a key element of systematic risk, and credibility theory is aimed directly at controlling it. The lesson is clear: well-tuned pricing credibilities can give a company a real market advantage by allowing it to run safely on slimmer margins. The result is a more financially efficient operation from which all parties benefit.

Some final clarifications are due on how such ideas can be put into practice. We shall draw a broad outline, omitting - unfortunately but necessarily - some important questions of detail regarding data consistency and other matters.

First, tuning a credibility scheme requires data; where are they to come from? The answer is that companies and bureaus are awash in such data. The main problem is that they throw it away too soon - before the feedback loop is complete. A proper credibility scheme for ratemaking and individual risk pricing must be an information system and not just a set of assumptions, however apt.

A complete feedback loop implies correlation of premium and loss data on the individual risk level. This is done in the operation of all experience rating plans. However, these data must also be kept around for tuning the model for the next rating cycle and testing the stability of the optimal parameter values. Such tuning will yield aggregate corrections, class relativities, and a simple experience rating formula, using the K_A tabulated for each class, A, in which all complications are kept behind the scenes and away from the rating clerk's desk.

I wish to thank my colleague Glenn Meyers for keeping me up to date on happenings at ISO and developments in the literature. Special thanks go to Barbara Dudman for typing the manuscript.

Appendix A: Detailed Stochastic Models

Bayesian models are usually presented in terms of underlying variables, which can be observed only indirectly through Bayesian inference. We have not done so here because a variety of models leads to the covariance structure which determines the credibilities, and there is no need to commit oneself to a particular model. To illustrate this we show two simple models, one additive, one multiplicative which lead to the same structure.

1. Additive Model

Let $\tilde{x}_{Aa}(t) = \xi + \tilde{\xi}_A + \tilde{\eta}_{Aa} + \tilde{\zeta}_{Aa}(t)$, where

ξ is the aggregate mean;

$\tilde{\xi}_A$ is the systematic departure of class A's experience from the aggregate.

$\tilde{\eta}_{Aa}$ is the departure of risk a's experience from the mean of class, A.

$\tilde{\zeta}_{Aa}(t)$ is the random fluctuation producing risk a's actual experience in period, t.

The random variables , $\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}$, have unconditional mean, zero, and are mutually independent. Their covariance structure is

$$E[\tilde{\xi}_A \tilde{\xi}_B] = \frac{\sigma^2}{K}\delta_{AB}, \quad E[\tilde{\eta}_{Aa}\tilde{\eta}_{Bb}] = \frac{\sigma^2}{K_A}\delta_{AB}\delta_{ab},$$

$$E[\tilde{\zeta}_{Aa}(t)\tilde{\zeta}_{Bb}(u)] = \frac{\sigma^2}{W_{Aa}(t)}\delta_{AB}\delta_{ab}\delta_{tu}.$$

These assumptions lead to

$$E\left[\left(\tilde{x}_{Aa}(t)-\xi\right)\left(\tilde{x}_{Bb}(u)-\xi\right)\right] = \sigma^2 \delta_{AB}\left[\frac{1}{K} + \delta_{ab}\left(\frac{1}{K_A} + \frac{\delta_{tu}}{W_{Aa}(t)}\right)\right],$$

our model from Section III.

2. Multiplicative Model

Let us apply exactly the same symbols and assumptions to a different model,

$$\tilde{x}_{Ac}(t) = \xi\left(1+\tilde{\xi}_A\right)\left(1+\tilde{\eta}_{Ac}\right)\left(1+\tilde{J}_{Ac}(t)\right)$$

The same covariance for this model is

$$\sigma^2 \xi^2 \delta_{AB}\left\{\frac{1}{K} + \delta_{ab}\left(1+\frac{\sigma^2}{K}\right)\left[\frac{1}{K_A} + \left(1+\frac{\sigma^2}{K_A}\right)\frac{\delta_{tu}}{W_{Aa}(t)}\right]\right\}.$$

This has the same algebraic structure as the additive model and, after redefinition of the parameters, will lead to the same credibility formula.

Appendix B: Solution of Least-Squares Conditions

Our purpose here is to record calculations too involved to include in the main text: the derivation of our chief result, equations (5, 6) of Section III and the proof of our assertion of harmony between individual equity and aggregate risk control.

1. Derivation of Credibility Formula

Requiring the risk function, R_{Aa} , of Section III, equation (4) to be stationary to variations of the coefficients, A, leads to the equations,

$$\frac{1}{K} A^{Aa}_{*}(*) - \frac{1}{K} + \lambda = 0 , \qquad (1)$$

$$\frac{1}{K_A} A^{Aa}_{b}(*) - \frac{\delta_{ab}}{K_A} - \lambda + \mu_b = 0 , \qquad (2)$$

$$\frac{1}{W_{Ab}(t)} A^{Aa}_{b}(t) - \mu_b = 0. \qquad (3)$$

Solution:

$$(3) \longrightarrow A^{Aa}_{b}(t) = \mu_b W_{Ab}(t) ;$$

$$(2) \,\&\, (3) \rightarrow A^{Aa}_{b}(*) = \delta_{ab} + K_A(\lambda - \mu_b) = \mu_b W_{Ab}(*) ;$$

whence

$$\mu_b = \frac{\delta_{ab}}{K_A + W_{Ab}(*)} + \frac{K_A \lambda}{K_A + W_{Ab}(t)} ,$$

and

$$A^{Aa}_{b}(t) = \delta_{ab} Z_{Aa}(t) + K_A \lambda Z_{Ab}(t), \quad (4)$$

where

$$Z_{Aa}(t) = \frac{W_{Aa}(t)}{K_A + W_{Aa}(*)} ;$$

$$(1) \,\&\, (4) \longrightarrow A^{Aa}_{*}(*) = 1 - K\lambda = Z_{Aa}(*) + K_A \lambda Z_{A*}(*),$$

whence $\lambda = \left(1 - Z_{Aa}(*)\right) \dfrac{1}{K + K_A Z_{A*}(*)}$.

which on substitution into (4) gives

$$A_b^{Aa}(t) = \delta_{ab} Z_{Aa}(t) + \left(1 - Z_{Aa}(*)\right) \frac{K_A Z_{Ab}(t)}{K + K_A Z_{A*}(*)} .$$

This is equation (5) of Section III, which, with equation (3) of that section gives equation (6), the desired result.

2. Harmony of Equity and Aggregate Risk Control

Our assertion in the main text was that

$$E\left[\left(\sum_a W_{Aa}(o)\, \tilde{\varepsilon}_{Aa}(o)\right)^2\right]$$

is minimized by the same solution which minimizes

$$E\left[\ \tilde{\varepsilon}_{Aa}(o)^2\right],$$

so that there is no conflict between the two. This would be obvious except that these model residuals involve the same individual risk records through the classification aggregates and are not manifestly independent.

To proceed, we first define

$$\bar{A}_b^A(t) = \sum_a W_{Aa}(o) A_b^{Aa}(t),$$

so that the aggregate residual is

$$\sum_a W_{Aa}(o)\, \tilde{\varepsilon}_{Aa}(o) = \sum_a W_{Aa}(o)\left(\tilde{x}_{Aa}(o) - \xi\right) - \sum_{bt} \bar{A}_b^A(t)\left(\tilde{x}_{Ab}(t) - \xi\right)$$

and the expectation of its square is

$$\sigma^2\Big\{\frac{1}{K}W_{A*}(0)^2+\frac{1}{K_A}W_{A*}(0)^2+W_{A*}(0)-\frac{2}{K}W_{A*}(0)\bar{A}^A_*(*)-\frac{2}{K_A}\sum_b W_{Ab}(0)\bar{A}^A_b(*)$$

$$+\frac{1}{K}\bar{A}^A_*(*)^2+\frac{1}{K_A}\sum_b\bar{A}^A_b(*)^2+\sum_{bt}\frac{1}{W_{Ab}(t)}\bar{A}^A_b(t)^2$$

$$+\ 2\Lambda\Big(\bar{A}^A_*(*)-\sum_b\bar{A}^A_b(*)\Big)$$

$$+\ 2\sum_b M_b\Big(\bar{A}^A_b(*)-\sum_t\bar{A}^A_b(t)\Big).$$

giving the equations,

$$\bar{A}^A_b(t)=M_b\,W_{Ab}(t)$$

$$\bar{A}^A_b(*)=W_{Ab}(0)+K_A(\Lambda-M_b)$$

$$\bar{A}^A_*(*)=W_{A*}(0)-K\Lambda$$

These can be solved by the same steps used earlier in this appendix, using the sum constraints; the result is

$$\bar{A}^A_b(t)=W_{Ab}(0)Z_{Ab}(t)+\sum_a W_{Aa}(0)\big(1-Z_{Aa}(*)\big)\frac{K_A Z_{Ab}(t)}{K+K_A Z_{A*}(*)},$$

which is easily recognized as

$$\sum_a W_{Aa}(0)A^{Aa}_b(t)$$

where the A is the solution derived earlier for the individual risk problem.

Appendix C: Estimation of Single Class Model with Data

It is desirable to test our model against data; however, data are hard to come by. We shall content ourselves by working with the tabulation presented in (1). This is a set of reinsurance data, gross premiums and excess loss ratios on a uniform, 'as-if' basis. We present these in the following table:

Treaty/Year:		4	3	2	1	0
1	w=	5.	6.	8.	10.	12.
	x=	0.0	0.0	4.2	0.0	7.7
2	w=	14.	14.	13.	11.	10.
	x=	11.3	25.0	18.5	14.3	30.0
3	w=	18.	20.	23.	25.	27.
	x=	8.0	1.9	7.0	3.1	5.2
4	w=	20.	22.	25.	29.	35.
	x=	5.4	5.9	7.1	3.1	5.2
5	w=	21.	24.	28.	34.	42.
	x=	9.7	8.9	6.7	10.3	11.1
6	w=	43.	47.	53.	61.	70.
	x=	9.7	14.5	10.8	12.0	13.1
7	w=	70.	77.	85.	92.	100.
	x=	9.0	9.6	8.7	11.7	7.0

In the above table, w stands for the premium and is used as a statistical weight, while x stands for excess loss ratio (in percents).

We have used the data from years 4 through 1 to predict experience in year zero. The collective consists of a single class and our model is

$$\tilde{x}_a(0) = \sum_{t=1}^{4} Z_a(t)\tilde{x}_a(t) + (1-Z_a(*))\hat{\xi} + \tilde{\varepsilon}_a(0),$$

$$Z_a(t) = \frac{W_a(t)}{K+W_a(*)} \quad ; \quad 1-Z_a(*) = \frac{K}{K+W_a(*)};$$

$$\hat{\xi} = \frac{1}{Z_*(*)} \sum_{\substack{a=1,7 \\ t=1,4}} Z_a(t)\tilde{x}_a(t).$$

The asterisk denotes summation on treaty, a=1, 7, or on year, t = 1, 4.

The model has a single parameter, K. Its structure, however, involving the aggregate estimator, $\hat{\xi}$, is too complex for the non-linear fitting modules of most statistical software packages. In this instance, though, it was a simple matter to construct a Fortran program to scan across prescribed parameter values and write out the value of the loss function,

$$\tilde{\mathcal{L}} = \sum_{a=1,7} W_a(0)\, \tilde{\mathcal{E}}_a(0)^2$$

and other statistics.

The results are shown in Exhibit 1 and are somewhat surprising. Buhlmann and Straub, using their algebraic estimators, arrived at a value, $K_{BS} \doteq 17.3$. Optimizing the loss function in our model gives a value, $K \doteq 0.1$, or, essentially, zero. This is the same as saying that the members of the collective are so diverse that the aggregate information is of no use, and each insured is rate should be determined from his experience alone. Statistically, the Bühlmann-Straub result is not much different from this since 17.3 is a good deal smaller than the cumulative premium on any one treaty. Unfortunately, the collective is very small, and any estimate will be highly uncertain.

Exhibit 1

Fitting of Single Class Model

Parameter, K	Loss Function	Sliding Mean
- 10.0	3631.13	--
- 1.0	3334.86	--
- 0.1	3332.94	--
0.0	3332.89	8.644
0.1	3332.87	8.646
1.0	3334.00	8.663
10.0	3425.40	8.787
∞	6431.49	9.496

Fitting Residuals

Treaty	K = 0.0	K = ∞	Weight
1	6.54	-1.80	12.
2	12.58	20.50	10.
3	0.31	-4.30	27.
4	1.80	-1.20	35.
5	2.17	1.60	42.
6	1.32	3.60	70
7	-2.83	-2.50	100.
Average	0.59	0.32	296.
Root Mean Square	3.36	4.66	

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