TITLE: RELATIVITY PRICING THROUGH ANALYSIS OF VARIANCE

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I Introduction

The ideas for this paper were the outgrowth of considerations of the Construction-Protection relativity question in Commercial Property (Fire) Insurance. A literature search on this subject in the Proceedings suggests Bailey's method from Bailey & Simon's paper "Two Studies in Automobile Insurance Ratemaking" (Proceedings Vol. XLV II).

Using that paper as a base line, the only problem existing in their work is the conceptual model. Their model was either additive or multiplicative. Because of the direct similarities between the basic data that we have to investigate and the mathematical statistics formulation of an Analysis of Variance problem, this paper will purpose a model which contains both additive and multiplicative (interaction) terms.

Section II will develop the mathematical two-way model with estimation solutions and present a statistical test for the inclusion/exclusion of the interaction term.

Section III will give numerical examples, one from Commercial Auto and one from Commercial Fire. These examples, while on small data sets calculated by hand, will show the ease and simplicity of the approach and the accuracy of the final result.

Section IV will extend the model to three-way classifications and suggest the possible extension to n-way modelling situations.

Section V, the summary, will bring together all the theoretical and practical considerations that will justify this model under most situations. The topic of credibility and some possible alternatives will be discussed.

II The Two-Way Model

In any relativity problem there are a few questions to be determined before any work can proceed. One of these questions is dimensions. Are we going to consider only two dimensions, or will we need three or more? In Commercial Auto, we could consider such things as age, sex, territory, type of vehicle, etc. all at the same time and use an n-way approach.

In Commercial Fire, the question was between Construction and Protection and a two-way classification was sufficient. Higher order models could be considered, although, at any time.

A second question, is that of model specification. Will the model be additive? multiplicative? or some combination of both? Presented here is the specification of a two-way additive and multiplicative model, for which, ultimately, the multiplicative term can be tested for statistical significance.

The object of this exercise is to estimate the relativities, r_{ij}, where $i = 1, \ldots p$ (row effects - or α effects) and $j = 1, \ldots q$ (column effects - or β effects) with the possibility of the interaction term, $\alpha\beta_{ij}$ in all cells. p is the number of levels of the row effect as q is the number of levels of the column effects. In the numerical example to follow, there are three Protection groups (q = 3) and three Construction groups (p = 3). The actual loss ratio data appears in Section III.

The basic model for r_{ij} appears below:

$$r_{ij} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + e_{ij} \qquad (1)$$

with the constraints

$$\sum_i n_{ij}\alpha_i = \sum_j n_{ij}\beta_j = \sum_i n_{ij}\alpha\beta_{ij} = \sum_j n_{ij}\alpha\beta_{ij} = 0$$

In an Analysis of Variance (ANOVA) context μ is the grand mean or some overall base line measure, α_i, β_j and $\alpha\beta_{ij}$ are the column, row and interaction effects, respectively and e_{ij} is the error term (the error term puts the equation in balance such that the estimates of the various terms while not exactly the relativities sought will vary by the "random" error term).

At this point, we shift away from the usual ANOVA perspective of trying to determine if the α, β, and $\alpha\beta$ terms significantly reduce the inherent variation or "explain" the modelling situation, to the estimation of the α, β and $\alpha\beta$ terms which are of more interest to us, at the moment.

As you can see, the constraints following Equation (1) have a term n_{ij} in them. This term is a weighting factor. I have used it as a surrogate for credibility in the examples to follow such that in the Auto data, n_{ij} is the appropriate Earned Car years, and in the Fire data, n_{ij} is the premium.

Continuing with the model, the least squares solution calls for the minimization of

$$Z = \sum_{ij} n_{ij} (r_{ij} - \mu - \alpha_i - \beta_j - \alpha\beta_{ij})^2 \qquad (2)$$

which is the minimization of the squared error with appropriate weights, i.e. from (1) $e^2_{ij} = (r_{ij} - \alpha_i - \beta_j - \mu - \alpha\beta_{ij})^2$

To find estimates of α, β, μ, and $\alpha\beta$, Z must be differentiated with respect to each term and set equal to zero. Therefore:

$$\frac{dZ}{d\mu} = 0 \text{ yields } \hat{\mu} = \frac{\sum_{ij} n_{ij} r_{ij}}{\sum_{ij} n_{ij}} \qquad (3)$$

$$\frac{dZ}{d\alpha} = 0 \text{ yields } \hat{\alpha}_i = \frac{\sum_{j} n_{ij} r_{ij}}{\sum_{j} n_{ij}} - \hat{\mu}$$

$$= \hat{A}_i - \hat{\mu} \qquad (4)$$

$$\frac{dZ}{d\beta} = 0 \text{ yields } \hat{\beta}_j = \frac{\sum_{i} n_{ij} r_{ij}}{\sum_{i} n_{ij}} - \hat{\mu}$$

$$= \hat{B}_j - \hat{\mu} \qquad (5)$$

$$\frac{dZ}{d\alpha\beta} = 0 \text{ yields } \hat{\alpha\beta}_{ij} = r_{ij} - \hat{A}_i - \hat{B}_j + \hat{\mu} \qquad (6)$$

Because there is only one observation per cell in this analysis, Equation (6) will give back r_{ij} as the estimate in all cases. This situation takes the modelling approach to an illogical conclusion, and we need to back off for a second to reconsider our position.

If we only consider, for the moment, the estimate given by

$$\hat{r}'_{ij} = \mu + \hat{\alpha}_i + \hat{\beta}_j \qquad (7)$$

we have a strictly additive model, and now we can consider the residual of this equation with r_{ij} from Equation (1). Substituting the estimates in Equation (1), and assuming a zero error, we get

$$\hat{r}'_{ij} - r_{ij} = \alpha\beta_{ij} \qquad (8)$$

Now define the residuals, AB_{ij}, as being equal to

$$AB_{ij} = \hat{r}'_{ij} - r_{ij} = \varepsilon_i \delta_j \qquad (9)$$

where ε_i and δ_j are the multiplicative column and row effects, respectively.

We can now solve this model, and then reformulate our original model to get final estimates for the r_{ij}'s. The new overall model will be

$$\hat{r}_{ij} = \hat{A}_i + \hat{B}_j - \hat{\mu} + \hat{\varepsilon}_i \hat{\delta}_j \qquad (10)$$

The least square solution proceeds as before for the model

$$AB_{ij} = \varepsilon_i \delta_j + e_{ij} \qquad (11)$$

Minimizing $Z = \sum_{ij} n_{ij}(AB_{ij} - \varepsilon_i\delta_j)^2$ and differentiating and setting equal to zero yields the following:

$$\hat{\varepsilon}_i = \frac{\sum_j n_{ij} AB_{ij}\delta_j}{\sum_j n_{ij}\delta_j^2} \qquad (12)$$

$$\hat{\delta}_j = \frac{\sum_i n_{ij} AB_{ij}\varepsilon_i}{\sum_i n_{ij}\varepsilon_i^2} \qquad (13)$$

Now we have a complete model for r_{ij} and the estimates are calculated from the data as follows

$$r_{ij} = \hat{A}_i + \hat{B}_j - \hat{\mu} + \hat{\varepsilon}_i\hat{\delta}_j \qquad (14)$$

$$\mu = \frac{\sum_{ij} n_{ij} r_{ij}}{\sum_{ij} n_{ij}} \qquad \hat{A}_i = \frac{\sum_j n_{ij} r_{ij}}{\sum_j n_{ij}} \qquad \hat{B}_j = \frac{\sum_i n_{ij} r_{ij}}{\sum_i n_{ij}}$$

$$\hat{\varepsilon}_i = \frac{\sum_j n_{ij} AB_{ij}\delta_j}{\sum_j n_{ij}\delta_j^2} \qquad \hat{\delta}_j = \frac{\sum_i n_{ij} AB_{ij}\varepsilon_i}{\sum_i n_{ij}\varepsilon_i^2}$$

$$AB_{ij} = r_{ij} - \hat{A}_i - \hat{B}_j + \hat{\mu}$$

It will be shown in the examples that these estimates are always as good as the Baily & Simon approach and there are theoretical reasons to believe that a more general model, by definition, has to be a better approach. It is also true that under this modelling situation the following is true

$$\frac{\sum_i n_i \hat{A}_i}{\sum_i n_i} = 1$$

$$\frac{\sum_j n_j \hat{B}_j}{\sum_j n_j} = 1$$

which is a very nice intuitive result, i.e. the sum (weighted) of the marginal relativities is one.

A final consideration, is that it is possible to statistically test the significance of the interaction term and thereby in some cases reduce the complexity of the model. This testing, and the results of the testing, may not always be appropriate or useful but they are available.

The test, mathematically derived in the Appendix, is an "F-Test" with 1 and $\left[(p-1)\ (q-1)\ -1\right]$ degrees of freedom. Any standard statistical text on Analysis of Variance will clarify the above statement for the statistical novice, e.g. The Analysis of Variance, Henry Scheffé; John Wiley & Sons, Inc., 1959.

The test statistic follows:

$$F = \frac{\left[(p-1)(q-1) - 1\right]\left[\sum_{ij} n_{ij}\ \hat{\varepsilon}_i\hat{\delta}_j\ AB_{ij}\right]^2}{\left[\sum_{ij} n_{ij}\varepsilon_i^2\delta_j^2\right]\left[\sum_{ij} n_{ij}\ AB_{ij}^2\right] - \left[\sum_{ij} n_{ij}\varepsilon_i\delta_j\ AB_{ij}\right]^2} \tag{15}$$

When the above value is larger than the table F, the interaction term is said to be significant at the α - level, where $(1 - \alpha)$ is the confidence (usually 95%; therefore $\alpha = .05$) of the test.

III Numerical Examples

The first numerical example is taken from Baily & Simon's "Two Studies in Automobile Insurance Ratemaking" (Proceedings Vol. XLVII) using Merit Rating and Class as the proposed discriminate variables on which relativities are sought. Using their data on page 15, the following matries are available for our exercise.

r_{ij}'s (Cell loss ratios ÷ total loss ratio)

	Merit Rating Class			
Class	A	X	Y	B
1	.786	1.016	1.115	1.358
5	1.071	1.079	1.410	1.642
3	1.212	1.285	1.450	1.885
2	1.269	1.747	1.519	1.784
4	2.050	2.192	2.412	2.853

n_{ij} (# of Earned Car Years)

Merit Rating Class

Class	A	X	Y	B
1	2758	131	164	274
5	64	4	5	9
3	247	16	20	38
2	131	7	10	22
4	157	18	21	57

With the above data, using Equation (14) and the subsequent definitions we will proceed with the calculation of the r_{ij}'s.

$$\hat{\mu} = \frac{\sum_{ij} n_{ij} \; r_{ij}}{\sum_{ij} n_{ij}} = 1.006$$

$$\hat{A}_1 = \frac{\sum_{j} n_{1j} \; r_{1j}}{\sum_{j} n_{1j}} = 0.8584$$

$$\hat{A}_2 = \frac{\sum_{j} n_{2j} \; r_{2j}}{\sum_{j} n_{2j}} = 1.1547$$

$$\hat{A}_3 = \frac{\sum_{j} n_{3j} \; r_{3j}}{\sum_{j} n_{3j}} = 1.3101$$

$$\hat{A}_4 = \frac{\sum_{j} n_{4j} \; r_{4j}}{\sum_{j} n_{4j}} = 1.4750$$

$$\hat{A}_5 = \frac{\sum_{j} n_{5j} \; r_{5j}}{\sum_{j} n_{5j}} = 2.2710$$

$$\hat{B}_1 = \frac{\sum_i n_{i1} r_{i1}}{\sum_i n_{i1}} = 0.9007$$

$$\hat{B}_2 = \frac{\sum_i n_{i2} r_{i2}}{\sum_i n_{i2}} = 1.1912$$

$$\hat{B}_3 = \frac{\sum_i n_{i3} r_{i3}}{\sum_i n_{i3}} = 1.2940$$

$$\hat{B}_4 = \frac{\sum_i n_{i4} r_{i4}}{\sum_i n_{i4}} = 1.6955$$

AB_{ij}

Class	A	X	Y	B
1	.0329	-.0276	-.0314	-.1899
5	.0216	-.2609	-.0327	-.2022
3	-.0028	-.2103	-.1481	-.1146
2	-.1007	.0868	-.2440	-.3805
4	-.1157	-.2642	-.1470	-.1075

Finally, using an iteration procedure,

$\hat{\varepsilon}_1 =$	1.046	$\hat{\delta}_1 =$	0.0171
$\hat{\varepsilon}_2 =$	1.1858	$\hat{\delta}_2 =$	-0.0506
$\hat{\varepsilon}_3 =$	0.7623	$\hat{\delta}_3 =$	-0.0589
$\hat{\varepsilon}_4 =$	1.6931	$\hat{\delta}_4 =$	-0.1862
$\hat{\varepsilon}_5 =$	0.5848		

And the final solutions look like

	$\hat{r}_{ij}$					
	A	X	Y	B	Class Marginals	$\hat{\varepsilon}_i$
1	.7710	.9907	1.0848	1.3531	0.8584	1.0460
5	1.0697	1.2799	1.3729	1.6234	1.1547	1.1858
3	1.2178	1.4567	1.5532	1.8577	1.3101	0.7623
2	1.3987	1.5745	1.6633	1.8492	1.4750	1.6931
4	2.1757	2.4266	2.5246	2.8516	2.2710	0.5848
Merit Rating Marginals	0.9007	1.1912	1.2940	1.6955	$\hat{\mu} = 1.006$	
$\hat{\delta}$	0.0171	-0.0506	-0.0589	-0.1862		

Bailey & Simon showed four different models in their paper, using the Customary method (1), and straight multiplication (2), an additive model (3), and a scaled additive model (4). They used three measures to show the "goodness" of their models; Balance, Average Error, and χ^2. The results of the above three measures, with their definitions are listed below including a fourth measure $\hat{\sigma}^2$, the weighted square error, for Bailey & Simon's four methods and ANOVA.

$$\text{Balance} = \frac{\Sigma n_{ij}\, \hat{r}_{ij}}{\Sigma n_{ij}\, r_{ij}}$$

$$\text{Average Error} = \frac{\Sigma n_{ij}\, |\, r_{ij} - \hat{r}_{ij}\,|}{\Sigma n_{ij}\, r_{ij}}$$

$$\chi^2 = K\ \Sigma\left[\frac{n_{ij}(r_{ij} - \hat{r}_{ij})^2}{\hat{r}_{ij}}\right] \qquad \text{Where } K = \frac{1}{200}$$

$$\hat{\sigma}^2 = \frac{\Sigma n_{ij}\,(r_{ij} - \hat{r}_{ij})^2}{\Sigma n_{ij}}$$

	Method 1	Method 2	Method 3	Method 4	ANOVA
Balance	1.0103	1.0011	1.0006	0.9983	0.9999
Avg. Error	.0401	.0317	.0098	.0111	0.0256
χ^2	.1021	.0363	.0104	.0083	0.0276
$\hat{\sigma}^2$	.0136	.0030	.0009	.0006	0.0020

When one reviews the table, it is difficult to pick a clearly better model from Method 3, Method 4, and ANOVA. It appears that for this data base, an additive and a multiplicative model, Methods 3 and Method 4, respectively, fit the data equally well. Apart from this area of confusion is the ability of ANOVA to fit all types of data with equal accuracy. To further advocate this new model, is the ease of calculation of the estimates and the property stated earlier, that the weighted marginals sum to one,

$$\frac{\sum_i n_i \hat{A}_i}{\sum_i n_i} = 1$$

$$\frac{\sum_j n_j \hat{B}_j}{\sum_j n_j} = 1$$

A final compelling consideration is the interaction term. For this data set, the calculated F value is 22.41 which is significant at the 5% level. Therefore, statistically, an interaction term is appropriate and also intuitive. It is very easy to believe that there is most likely some interaction between Class and Merit rating.

The next example comes from Commercial Fire.

The data and estimates follow:

LOSS RATIOS

Construction	Protection 4-8	1,9	2,3	Total
1	0.569	0.477	0.558	
2,3	0.432	0.420	0.558	
4-6	0.445	0.463	0.369	
Total			0.474	

Relative Loss Ratios (r_{ij})

1.200	1.006	1.177
.911	.886	1.177
.939	.977	.778

Premiums (n_{ij}) (in millions of dollars)

320.4	54.2	37.5
677.0	62.7	100.2
194.3	17.1	40.9

Estimates: $\hat{\mu} = 1.000$ $\qquad \hat{B}_1 = 0.993$
$\hat{A}_1 = 1.172$ $\qquad \hat{B}_2 = 0.946$
$\hat{A}_2 = 0.941$ $\qquad B_3 = 1.086$
$A_3 = 0.915$

AB_{ij}

.035	-.112	-.081
-.023	-.001	.150
.031	.116	-.223

$\hat{\varepsilon}_1 = -3.4738$ $\qquad$ $\hat{\delta}_1 = -.0060$
$\hat{\varepsilon}_2 = 4.2795$ $\qquad$ $\hat{\delta}_2 = .0035$
$\hat{\varepsilon}_3 = -6.2127$ $\qquad$ $\hat{\delta}_3 = .0340$

$\hat{r}_{ij}$

	4-8	1,9	2,3	Construction Marginals	i
1	1.186	1.106	1.140	1.172	-3.4738
2,3	0.908	0.902	1.173	0.941	4.295
4-6	0.945	0.839	0.790	0.915	-6.2127
Protection Marginals	0.993	0.946	1.086	$\hat{\mu} = 1.000$	
$\hat{\delta}_j$	-.0060	.0035	.0340		

$F = 15.83$

The χ^2 for the above data is proportional to 1,016,570. Using a multiplicative model for the Bailey & Simon approach results in a χ^2 proportional to 5,994,534.

The F value (F = 15.83) is statistically significant even for this small data set. Therefore, the interaction term has remained, which is theoretically pleasing.

IV Three-Way Classification

The above two-way classification model can be extended to an n-way model, with perhaps reasonable complications in the mathematics. Presented here, are the assumptions and estimates for a reduced three-way Classification ANOVA model.

With a two-way model, the interaction term is fairly straight forward. In a three-way model there are available three, two-way interaction terms (pair-wise) and a single three-way interaction term in the complete specification of an ANOVA Model. Since our objective is to find "good" estimates of the relativities (r_{ijk}), we can imbed all of the different types of interactions into a single interaction term. This being the case, the model is as follows:

$$r_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + \alpha\beta\gamma_{ijk} + e_{ijk} \tag{16}$$

with the constraints

$$\sum_i n_{ijk}\alpha_i = \sum_j n_{ijk}\beta_j = \sum_k n_{ijk}\gamma_k = \sum_i \alpha\beta_{ijk}\, n_{ijk} = \sum_j \alpha\beta\gamma_{ijk}\, n_{ijk} = \sum_k \alpha\beta\gamma_{ijk}\, n_{ijk} = 0$$

Using the least squares estimation technique here as in the two-way model, the following estimates result:

$$\hat{r}_{ijk} = \hat{A}_i + \hat{B}_j + \hat{C}_k - 2\hat{\mu} + \hat{\varepsilon}_i\hat{\delta}_j\hat{\phi}_k$$

$$\hat{A}_i = \frac{\sum_{jk} n_{ijk}\, r_{ijk}}{\sum_{jk} n_{ijk}} \qquad \hat{B}_j = \frac{\sum_{ik} n_{ijk}\, r_{ijk}}{\sum_{ik} n_{ijk}}$$

$$\hat{C}_k = \frac{\sum_{ij} n_{ijk}\, r_{ijk}}{\sum_{ij} n_{ijk}} \qquad \hat{\varepsilon}_i = \frac{\sum_{jk} n_{ijk}\, Z_{ijk}\, \delta_j\phi_k}{\sum_{jk} n_{ijk}\, \delta_j^2\, \phi_k^2}$$

$$\hat{\delta}_j = \frac{\sum_{ik} n_{ijk}\, Z_{ijk}\, \varepsilon_i\phi_k}{\sum_{ik} n_{ijk}\, \varepsilon_i^2\, \phi_k^2} \qquad \hat{\phi}_k = \frac{\sum_{ij} n_{ijk}\, Z_{ijk}\, \varepsilon_i\delta_j}{\sum_{ij} n_{ijk}\, Z_{ijk}\, \varepsilon_i^2\, \delta_j^2}$$

$$Z_{ijk} = r_{ijk} - \hat{A}_i - \hat{B}_j - \hat{C}_k + 2\mu$$

The above estimates, while not impossible, are more difficult to accomplish by hand than the two-way model, and the agony will increase with n, the number of dimensions. Conceptually an n-way model is a trivial extension from the above. I would advocate using only one interaction term for the same reasons as associated with the three-way model.

V Summary

The ANOVA model has shown in the Fire data its clear-cut superiority as measured by the χ^2 statistic of the Baily & Simon paper. With the Commercial Auto data, ANOVA was at least as good, but perhaps had a much better conceptual base.

In both examples, using the weights (n_{ij}'s) tends to mitigate credibility considerations, since the weights were chosen to be surrogate variables for the credibilities.

The ANOVA model is much more general, allowing for an n-way classification and with the test statistic a reasonable way to test the construction and formulation of the model.

The two-way model has been accepted by ISO, having been approved by the appropriate committees, and is currently being used by them for Construction-Protection relativities in Commercial Fire Insurance. It has been tested by an Ad-Hoc Subcommittee of ISO and, at least for Fire, has lived up to expectations.

APPENDIX

TEST OF SIGNIFANCE OF INTERACTION TERM

The basic model is

$$r_{ij} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + e_{ij} \qquad (1)$$

with $\sum_i n_{ij}\alpha_i = \sum_j n_{ij}\beta_j = \sum_i n_{ij}\alpha\beta_{ij} = \sum_j n_{ij}\alpha\beta_{ij} = 0$ as constraints.

We would like to make a statement about the significance of the interaction term ($\alpha\beta_{ij}$) that has been previously been estimated in the body of the paper. Therefore, let's try the following definition for $\alpha\beta_{ij}$:

$$\alpha\beta_{ij} = \lambda\epsilon_i\delta_j \qquad (2)$$

Under this formulation, we will attempt to test whether or not, statistically, that λ is equal to zero, thereby forcing our conclusion about the interaction term, $\alpha\beta_{ij}$.

Our ultimate objective is an F-test on the sum of squares of λ (SS_λ) which will be a comparison of that sum of squares to the remaining or balance sum of squares ($SS_{Balance}$), modified of course by their appropriate degrees of freedom (d.f.). These concepts and terms can easily be found in a standard statistical text dealing with Analysis of Variance.

Given Equation (2), we can now modify our model, with the appriate constraints of Eq. (1), to be

$$r_{ij} = \mu + \alpha_i + \beta_j + \lambda \varepsilon_i \delta_j + e_{ij} \tag{3}$$

To find a "least squares" estimate for λ, define Z to be the following

$$Z = \sum_{ij} n_{ij} (r_{ij} - \mu - \alpha_i - \beta_j - \lambda \varepsilon_i \delta_j)^2 \tag{4}$$

assuming α_i and β_j are known. We now differentiate Z and set it equal to zero and solve for λ.

$$\frac{dZ}{d\lambda} = -2 \sum_{ij} n_{ij} (r_{ij} - \mu - \alpha_i - \beta_j - \lambda \varepsilon_i \delta_j)(\varepsilon_i \delta_j) = 0$$

$$\hat{\lambda} = \frac{\sum_{ij} n_{ij} \varepsilon_i \delta_j (r_{ij} - \mu - \alpha_i - \beta_j)}{\sum_{ij} \varepsilon_i^2 \delta_j^2 n_{ij}} \tag{5}$$

Now consider the sum of squares for the weighted interaction term, $\alpha\beta_{ij}$ using the definition from (2).

$$\sum_{ij} \alpha\beta_{ij}^2 n_{ij} = \lambda^2 \sum_{ij} \varepsilon_i^2 \delta_j^2 n_{ij} \tag{6}$$

If we substitute our estimate of λ from (5) into (6) we have

$$\sum_{ij} n_{ij}\alpha\beta_{ij}^2 = \frac{\left[\sum_{ij} n_{ij}c_i\delta_j (r_{ij} - \mu - \alpha_i - \beta_j)\right]^2}{\sum_{ij} n_{ij}c_i^2\delta_j^2} \tag{7}$$

Furthermore, since we do not know α_i, β_j, ε_i or δ_j, if we put their estimates into (7) we have

$$SS_\lambda = \sum_{ij} n_{ij}\alpha\beta_{ij}^2 = \frac{\left[\sum_{ij} n_{ij}\hat{\varepsilon}_i\hat{\delta}_j AB_{ij}\right]^2}{\sum_{ij} n_{ij}\hat{\varepsilon}_i^2\hat{\delta}_j^2} \tag{8}$$

where the needed estimates are found in section II of the paper.

The final F statistic will take the form

$$\frac{SS_\lambda/\text{d.f.}}{SS_{\text{Balance}}/\text{d.f.}}$$

where the degrees of freedom (d.f.) for SS_λ are 1, and for SS_{Balance} is $\left[(p-1)\ (q-1) - 1\right]$. We need to define SS_{Balance} now.

What we are really doing here is looking at the residual sum of squares after fitting the additive terms and seeing if further, significant reduction can be made by fitting an interaction term. The residual sum of squares (SS_{Residual}) is

$$SS_{\text{Residual}} = \sum_{ij} n_{ij} (r_{ij} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j)^2 \tag{9}$$

$$= \sum_{ij} n_{ij}AB_{ij}^2$$

If we reduce this by SS_λ we have the balance left unexplained or $SS_{Balance}$. Subtracting (8) from (9) yields

$$SS_{Balance} = \Sigma n_{ij} AB_{ij}^2 - \frac{\left[\sum_{ij} n_{ij} \hat{\varepsilon}_i \hat{\delta}_j AB_{ij}\right]^2}{\sum_{ij} n_{ij} \hat{\varepsilon}_i^{\,2} \hat{\delta}_j^{\,2}}$$

$$= \frac{\left[\sum_{ij} n_{ij} \hat{\varepsilon}_i^{\,2} \hat{\delta}_j^{\,2}\right]\left[\sum_{ij} n_{ij} AB_{ij}^{\,2}\right] - \left[\sum_{ij} n_{ij} \hat{\varepsilon}_i \hat{\delta}_j AB_{ij}\right]^2}{\Sigma n_{ij} \varepsilon_i^{\,2} \delta_j^{\,2}}$$

Now dividing by the appropriate degrees of freedom and taking the ratio of SS_λ/d.f. to $SS_{Balance}$/d.f. gives us

$$F = \frac{\left[(p-1)\ (q-1) - 1\right] \left[\sum_{ij} n_{ij} \hat{\varepsilon}_i \hat{\delta}_j AB_{ij}\right]^2}{\left[\sum_{ij} n_{ij} AB_{ij}^{\,2}\right]\left[\sum_{ij} n_{ij} \hat{\varepsilon}_i^{\,2} \hat{\delta}_j^{\,2}\right] - \left[\sum_{ij} n_{ij} \hat{\varepsilon}_i \hat{\delta}_j AB_{ij}\right]^2}$$

which has a standard F distribution and can be compared to any table of F-values in any standard statistical text.