Combining Credibility and GLM for Rating of Multi-Level Factors

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Abstract

A *multi-level factor* (MLF) is a rating factor with a large number of levels, each of which we want to rate separetely even though many of them do not have a sufficient amount of data. Examples include Car model, Geographic Zone and Company (in experience rating).

Rating of MLFs is a standard situation for employing credibility theory. Traditional credibility theory models MLFs as random effects, but does not treat the situation where there are also ordinary rating factors (like Sex and Age class) alongside with the MLF. The aim of this paper is to show how such a situation can be handled by combining credibility theory and GLM. The method can be seen as an extension of the classical Buhlmann-Straub approach.

The method is presented via an example of experience rating in bus insurance, while the theory and more general results are given in Ohlsson and Johansson (2003).

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1 Multi-level factors and credibility

In non-life insurance rating, it is customary to use Generalized Linear Models (GLMs) to estimate price relativities for a number of rating factors under a multiplicative model. The rating factors are either categorical with a few levels (e.g. Sex) or a grouping of a continous variable (e.g Age group, Mileage class). In case enough data is not available for some group, one can merge groups to get more reliable estimates, *nota bene* at the price of a less detailed tariff However, for rating factors with a large number of levels without an inherent ordering there is no simple way to form groups with sufficient data We introduce the term *multi-level factor* (MLF) for such rating factors and next give some examples.

Example 1.1 (Car model) In private motor car insurance it is well known that the model of the car is an important rating factor, both for third-party liability, hull and theft. In Sweden there is a common basic grouping where car models that are technically very close to each other are put in the same class. Nevertheless, we are left with several thousands of car model classes, some of which represent popular cars with sufficent data available, whereas most classes have moderate or sparse data. Even after taking into account auxiliary variables like weight, effect and brand the car model remains an important rating factor. There is no sensible way to group the models *a priori* and there is not enough data to do a relevant posterior grouping. Hence, car model is a typical MLF. □

Example 1.2 (Experience rating) Using the *customer* as a rating factor is another important example of an MLF. In the commercial lines it is important to base the rating to some extent on the individual claims experience, even though there is often not suffient data for separate rating of each company. This is the classical situation for which North-American actuaries like Whitney and Mowbay introduced credibility estimators in the early 1900's. In the private lines, Lemaire (1995) and others use (European type) credibility estimators for the construction of optimal bonus/malus systems, with the customer as MLF. □

Example 1.3 (Geographic area) In order to get risk homogenous geographic areas one often has to use a very fine subdivision of the country, based on for instance ZIP codes. Neighbouring areas can have quite different risk profiles and hence a prior grouping can be very hard to achieve and we are again left with an MLF. □

As already indicated in example 1.2, a way to solve the rating problem for MLFs is to use *credibility theory*. However, classical Bühlmann-Straub credibility theory does not treat the important situation where we have ordinary rating factors (non-MLFs) besides the MLF This is the problem considered in the present paper, and the proposed solution is to use a combination of GLM and credibility as will now be outlined.

2 Extended Credibility Predictors

For simplicity, we will describe the method in terms of a simple example from bus insurance. For a general treatment we refer to Ohlsson and Johansson (2003).

We consider data for 1993-1998 from the former Swedish insurance company Wasa on 624 bus companies. Here we have just two ordinary rating factors *Age* (with five classes of bus age) and *Zone* (a standard subdivision of Swedish parishes into seven zones). Note that geographic area is not used as an MLF here, as would be the case if we operated on the parishes themselves. Our MLF is the company itself and hence we are looking for a proper experience rating in the presence of the ordinary rating factors Age and Zone

As is standard in insurance applications of GLMs, we perform a separate analysis of *claim frequency* and *average claim cost*. For briefness, we will only consider claim frequency here. Hence let Y_{ijk} be the observed claim frequency for the buses of company k in Age class i operating in Zone j, i.e. Y_{ijk} is the number of claims divided by the exposure weight w_{ijk} measured in policy years. The multiplicative tariff contains a base rate μ, plus factors α_i for Age and β_j for Zone. In credibility theory, the MLF is modelled by a random risk parameter, which suggests that we should introduce a stochastic factor (random effect) U_k for Company in our model in order to get credibility-like results. (This idea was introduced in the actuarial literature by Nelder and Verall, 1997.) The multiplicative model for the expected claim frequency hence becomes

$$E[Y_{ijk}|U_k = u_k] = \mu\alpha_i\beta_j u_k \tag{1}$$

where $\alpha_{i_0} = 1$ and $\beta_{j_0} = 1$ for some base classes i_0 and j_0 and $E(U_k) = 1$, since U_k should be a pure random effect — the systematic part being taken care of by μ. Furthermore, the U_k's are supposed to be independent and identically distributed with common variance $a \doteq \mathrm{Var}[U_k]$. Note that $\mu_{ij} \doteq E[Y_{ijk}] = \mu\alpha_i\beta_j$.

Conditionally on U_k, Y_{ijk} is assumed to follow a (w-weighted) Poisson

GLM with mean given by (1) and hence with variance $\mathrm{Var}[Y_{ijk}|U_k = u_k] = \sigma^2 \mu_{ij} u_k / w_{ijk}$, where σ^2 is the overdispersion parameter (in GLMs usually called ϕ and possibly set to one beforehand).

As in traditional credibility theory, we look for estimators (or rather predictors) $\hat{U}_k$ of U_k which are optimal in the meaning of mean square error, i.e. which minimize $E[(\hat{U}_k - U_k)^2]$. In Section 2.3 of Ohlsson and Johansson (2003) it is shown — as a special case — that the solution to this problem is given, under certain conditions, by the credibility formula

$$\hat{u}_k = z_k \, \overline{u}_k + (1 - z_k) \cdot 1 \tag{2}$$

where

$$\overline{u}_k = \frac{\sum_{i,j} w_{ijk} y_{ijk}}{\sum_{i,j} w_{ijk} \mu_{ij}} \tag{3}$$

and the *credibility factor* z_k is

$$z_k \doteq \frac{\sum_{i,j} w_{ijk} \mu_{ij}}{\sum_{i,j} w_{ijk} \mu_{ij} + \sigma^2 / a} \tag{4}$$

where $\sum_{i,j}$ extends over all tariff cells (i,j) where company k has at least one bus. Note that $E[\overline{U}_k|U_k] = U_k$, and that $\overline{u}_k$ is the ratio between the number of claims by company k and the corresponding expectated value in a tariff with just Age and Zone as rating factors. Hence, $\hat{u}_k$ is a credibility weighted average of our empirical experience of the company, $\overline{u}_k$, and the number 1 — the latter implying rating of the company's buses by their tariff values for Age and Zone only.

Note that we get high credibility if we have large exposure in terms of expected number of claims $w_{ijk}\mu_{ij}$ or if the variance between companies a is large compared to the within company variance σ^2.

Note also that in the case of very high credibility, i.e. $z_k \approx 1$, equation

(2) becomes

$$\sum_{i,j} w_{ijk}\mu_{ij}\hat{u}_k = \sum_{i,j} w_{ijk}y_{ijk} \tag{5}$$

which we recognise as the estimating equations for the ML estimates in a Poisson GLM (equivalent to the method of marginal totals). Hence, the credibility estimator in the case of high credibility is nothing but the ordinary GLM estimator. This is an appealing property since high credibility occurs when there is enough data for the company k.

Remark. If we disregard the ordinary rating factors Age and Zone, so that $\mu_{ij} = \mu$, it is not hard to see that $\mu \cdot \hat{u}_k$ reduces to the ordinary Bühlmann-Straub credibility estimator, see Ohlsson and Johansson (2003), Section 2.3.1. □

The proof of (2)–(4), in a general setting, is given in Ohlsson and Johansson (2003). It is based on an extension of the famous theorem by Jewell (1974) on exact credibility. A fundamental assumption is that some suitable transformation of U_k follows the natural conjugate prior distribution of the GLM distribution (here: of the Poisson distribution). In a forthcoming paper we will show how $\overline{u}_k$ can alternatively be derived without distributional assumptions as the optimal *linear* predictor — in analogy with the original result by Bühlmann and Straub

2.1 Estimation of variance parameters

It remains to estimate the variance parameters σ^2 and a. We use an approach with unbiased estimators based on sums of squares, similar to the one proposed in classical credibility theory (see e.g. Goovaerts and Hoogstad, 1987, p. 48). The derivation is given in Ohlsson and

Johansson (2003) — here we just present the results in our special case. Let

$$\hat{\sigma}_k^2 = \frac{1}{n_k - 1} \sum_{i,j} w_{ijk} \mu_{ij} \left(\frac{Y_{ijk}}{\mu_{ij}} - \overline{u}_k \right)^2$$

where n_k are the number of tariff cells (i, j) where we have $w_{ijk} > 0$.

For each k this gives a separate unbiased estimator of σ^2. As an overall estimator we suggest

$$\hat{\sigma}^2 = \frac{\sum_k (n_k - 1)\hat{\sigma}_k^2}{\sum_k (n_k - 1)} \tag{6}$$

Next we present an unbiased estimator of $a = \text{Var}[U_k]$.

$$\hat{a} = \frac{\sum_k \tilde{w}_k (\overline{u}_k - 1)^2 - K\hat{\sigma}^2}{\sum_k \tilde{w}_k} \tag{7}$$

where $\tilde{w}_k = \sum_{i,j} w_{ijk}\mu_{ij}$ and K is the number of companies Note that this estimator is unbiased only when μ_{ij} is assumed known. In practice we estimate μ_{ij} in a GLM and hence the estimators are not strictly unbiased.

2.2 An algorithm

Nelder and Verall (1997), using a different approach based on hierarchical likelihood, suggested iteration between GLM parameter estimation for ordinary rating factors and prediction of random effects u_k. In our example iteration is also reasonable since there might be confounding between the MLF and the ordinary rating factors (e.g. a good company might operate mainly in one zone and this might lower the factor for that zone). We get the following algorithm for simultaneous rating of ordinary factors and MLFs.

(0) Initially, let $\hat{u}_k = 1$ for all k.

(1) Estimate μ, α_i and β_j in a Poisson GLM, using a log-link and having $\log(\hat{u}_k)$ as *offset*-variable. This yields $\hat{\mu}_{ij} = \hat{\mu}\hat{\alpha}_i\hat{\beta}_j$.

(2) Compute $\hat{\sigma}^2$ and $\hat{a}$, using $\hat{\mu}_{ij}$ from Step 1.

(3) Compute $\hat{u}_k$ for $k = 1, 2, \ldots, K$, using the estimates from Step 1 and 2.

(4) Return to Step 1 with the offset-variable $\log(\hat{u}_k)$ from Step 3.

Repeat Step 1-4 until convergence, which in our experience often takes just a few iterations.

Notice that the computation of ML estimates in ordinary GLMs requires iteration between the estimating equations for the different rating factors. Together with the observations made in connection with equation (5) this means that in case of very high credibility the algorithm will be equivalent to the one for computing the ML estimates when all rating factors are treated as ordinary GLM factors.

3 Numerical results

In Table 1 we show the relativities for the ordinary rating factors Age and Zone, first after running a GLM with these covariates alone, then after 30 iterations of the algorithm with *Company* as MLF. We see that use of the algorithm results in a substantial change in the rating factors for Zone

Next, we list the credibility estimates for a selection of companies in Table 2.

Rating factor	*Level*	*Estimated relativities*	
		GLM only	*Algorithm*
Bus Age	0-2 yrs	2.64	3.05
	3-5 yrs	1.90	1.78
	6-8 yrs	1.77	1.78
	9-11 yrs	1.42	1.37
	12+ yrs	1.00	1.00
Zone	1	1.00	1.00
	2	1.82	1.03
	3	1.43	1.41
	4	1.32	0.94
	5	2.28	1.39
	6	1.44	1.08
	7	0.40	0.95

Table 1: *Estimated values for ordinary rating factors in bus insurance.*

In this rather simple example, the ordinary rating factors explain quite little of the variation in the data and consequently quite high credibility is given even to bus companies with a limited amount of data. The estimated values of the dispersion parameters are $\hat{\sigma}^2 = 1.12$ and $\hat{a} = 1.00$. Note that only the ratio $\hat{\sigma}^2/\hat{a} = 1.12$ enters into the formula for the credibility factor z_k — this value is quite low in our experience. Nevertheless, at the lower end of the table credibility is low and $\hat{u}_k$ is close to one, which means that one has to rely on the ordinary rating factors for these companies.

For an example of the use of the above algorithm in private car insurance, see Section 4 of Ohlsson and Johansson (2003)

k	$w_{\cdot k}$	$\overline{u}_k$	$\hat{u}_k$	z_k
1	219.81	1.81	1.80	0.98
2	269.40	0.90	0.90	0.98
3	181.84	1.36	1.35	0.97
4	102.60	1.16	1.15	0 97
⋮	⋮	⋮	⋮	⋮
201	12.01	0.00	0.45	0.55
202	7.33	0.75	0.86	0.55
203	5.13	0.75	0.86	0.54
204	9.52	2.31	1.70	0.54
⋮	⋮	⋮	⋮	⋮
401	1.89	2.96	1.46	0.23
402	2.43	0.00	0.77	0.23
403	1.90	0.00	0.77	0.23
404	3.22	0.00	0.78	0.22
⋮	⋮	⋮	⋮	⋮
601	0.11	0.00	0.98	0.02
602	0.17	0.00	0.99	0.01
603	0.08	0.00	0.99	0.01
604	0.14	0.00	0.99	0.01
⋮	⋮	⋮	⋮	⋮

Table 2: *Selected bus companies, k, with their number of policy years $w_{\cdot k}$, experience values $\overline{u}_k$, credibility predictors $\hat{u}_k$ and credibility factors z_k. The companies are ordered according to z_k.*

4 Concluding remarks

The method presented in this paper can be seen as either a way to work with random effects in GLMs or as a way to introduce fixed effects into the credibility framework. In any case, combination of GLM and credibility is a very useful and rather simple tool for simultaneous analysis of ordinary and multi-level factors, with many potential applications in different lines of business.

5 References

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