SOME INEQUALITIES FOR STOP-LOSS PREMIUMS

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1. *A certain family of premium calculation principles*

In this paper any given risk S (a random variable) is assumed to have a (finite or infinite) mean. We enforce this by imposing $E[S^-] < \infty$.

Let then $v(t)$ be a twice differentiable function with

$$v'(t) > 0, \; v''(t) \geq 0, \; -\infty < t < +\infty$$

and let z be a constant with $0 \leq z \leq 1$.

We define the premium P as follows

$$P = \sup\{Q \mid -\infty < Q < +\infty, \; E[v(S - zQ)] > v((1 - z)Q)\} \quad (1)$$

or equivalently

$$P = \sup\{Q \mid -\infty < Q < +\infty, \; v^{-1} o \, E[v(S - zQ)] > (1 - z)Q\}. \quad (2)$$

Notation: $v^{-1}(\infty) = \infty$.

The definitions (1) and (equivalently) (2) are meaningful because of the

Lemma: a) $E[v(S - zQ)]$ exists for all $Q \in (-\infty, +\infty)$.

b) The set $\{Q \mid -\infty < Q < +\infty, E[v(S - zQ)] > v((1 - z)Q)\}$ is *not* empty.

Proof: a) $E[v^-(S - zQ)] \leq v^-(0) \cdot P[S \geq zQ] + v'(0) \int_{S < zQ} (zQ - S) dP(S)$

$$\leq v^-(0) \cdot P[S \geq zQ] + v'(0)[zQ + E(S^-)] < \infty$$

b) Because of a) $E[v(S - zQ)]$ is always finite or equal to $+\infty$

If $v(-\infty) = -\infty$ then $E[v(S - zQ)] > v((1 - z)Q)$ is satisfied for sufficiently small Q. The left hand side of the inequality is a nonincreasing continuous function in P (strictly decreasing if $z > 0$), while the right hand side is a nondecreasing continuous function in Q (strictly increasing if $z < 1$).

If $v(-\infty) = c$ finite then $E[v(S - zQ)] > c$

(otherwise S would need to be equal to $-\infty$ with probability 1) and again $E[v(S - zQ)] > v((1 - z)Q)$ is satisfied for sufficiently small Q.

From the lemma we conclude the following useful

Corrolary: There are two cases to be distinguished

a) *finite case*: There exists Q^* (finite) with

$$E[v(S - zQ^*)] = v((1 - z)Q^*) \tag{1*}$$

or equivalently

$$v^{-1}\circ E[v(S - zQ^*)] = (1 - z)Q^* \tag{2*}$$

then $P = Q^*$.

b) *infinite case*: Otherwise $P = +\infty$.

Proof: From the proof of the lemma it is obvious that Q^* under a) coincides with the supremum defining P.

Our premium calculation principle is determined by the choice of the function v and the constant z satisfying the above conditions. It satisfies the following very desirable postulates: For any risk S, for which the premium P exists,

$$\boxed{\begin{array}{l} P_1 : P \geq E[S] \\ P_2 : P \leq \text{Max}\,[S] \end{array}}$$

Here Max $[S]$ denotes the right hand end point of the range of S.

Proof: For P_1 we start with equation (2) and make use of Jensen's inequality: P is the least upper bound of the set of Q's for which

$$(1 - z)Q < v^{-1}\circ E[v(S - zQ)].$$

By Jensen's inequality

$$v^{-1}\circ E[v(S - zQ)] \geq v^{-1}\circ v(E[S - zQ]) = E[S] - zQ.$$

The set of Q's for which

$Q < E[S]$ is hence a subset and its supremum $E[S]$ can not exceed the supremum P of the bigger set.

For P_2 we start with equation (2*) (only the case $\text{Max}\,[S] < \infty$ needs to be proved) and get

$$\begin{aligned}(1 - z)P &= v^{-1} \circ E[v(S - zP)]\\ &\leq v^{-1} \circ \text{Max}\,[v(S - zP)]\\ &= v^{-1} \circ v(\text{Max}\,[S - zP])\\ &= \text{Max}\,[S] - zP\end{aligned}$$ q.e.d.

Remarks:

1) If $z = 1$, we obtain the *principle of zero utility*,
$$P = \sup\{Q \mid E[u(Q - S)] < u(0)\}$$
by setting $u(t) = -v(-t)$.

2) If $z = 0$, we obtain the *mean value principle*,
$$P = v^{-1} \circ E[v(S)].$$

3) In the case where the function v is linear or exponential, the premium calculation principle does not depend on the value of z.

2. *Partial Ordering among risks*

Let $G(x)$, $H(x)$ be any distributions on the real line. Then we say that $G < H$, if

$$(PO) \int_t^\infty (x - t)\, dG(x) \leq \int_t^\infty (x - t)\, dH(x), \; -\infty < t < \infty.$$

Condition (b) simply means that for any retention limit t the *net stoploss premium* for a risk whose cdf is G is not higher than the one for a risk whose cdf is H. We do allow the case where the integrals become infinite. Integration by parts leads to the following equivalent condition:

$$(PO') \int_t^\infty [1 - G(x)]\, dx \leq \int_t^\infty [1 - H(x)]\, dx.$$

The equivalence of (PO) and(PO') in the case of infinite integrals is e.g. proved in Feller II, page 150.

Let us now consider two stop-loss arrangements based on risks with cdf G and H, respectively. Let P_α^G, P_α^H denote the corresponding stop-loss premiums (α = retention limit). For example, P_α^H is obtained as the least upper bound of the set of Q's for which

$$v((1-z)Q) < v(-zQ)\,H(\alpha) + \int_\alpha^\infty v(t-\alpha-zQ)\,dH(t) \tag{3}$$

and in the finite case as the unique solution of

$$v[(1-z)P_\alpha^H] = v(-zP_\alpha^H)\,H(\alpha) + \int_\alpha^\infty v(t-\alpha-zP_\alpha^H)\,dH(t) \tag{3*}$$

The importance of the partial ordering introduced in this section becomes evident in the following theorem:

Theorem 1: Suppose $G < H$ Then $P_\alpha^G \leq P_\alpha^H$, $-\infty < \alpha < +\infty$

Proof: If $P_\alpha^H = \infty$ nothing is to be proved. We therefore assume P_α^H finite which implies $\int_t^\infty [1-H(x)]\,dx < \infty$ for all $t \in (-\infty, +\infty)$.

If we integrate in equation (3*) twice by parts, we obtain:

$$\begin{aligned} v((1-z)P_\alpha^H) &= v(-zP_\alpha^H) + \int_\alpha^\infty v'(t-\alpha-zP_\alpha^H)\,[1-H(t)]\,dt \\ &= v(-zP_\alpha^H) + v'(-zP_\alpha^H)\int_\alpha^\infty [1-H(t)]\,dt \\ &\quad + \int_\alpha^\infty v''(t-\alpha-zP_\alpha^H)\int_t^\infty [1-H(x)]\,dx\,dt. \end{aligned}$$

Now we estimate the last two terms from below, replacing H by G and using condition (PO'). By reversing the last step (integration by parts) we arrive at

$$v[(1-z)P_\alpha^H] \geq v(-zP_\alpha^H) + \int_\alpha^\infty v'(t-\alpha-zP_\alpha^H)\,[1-G(t)]\,dt$$

and therefore $P_\alpha^G \leq P_\alpha^H$ q.e.d.

We postpone examples to sections 3 and 4 and conclude this section with some useful lemmas. Their content is essentially that the partial ordering is preserved under mixing and under convolution.

Lemma 1: Let (G_n), (H_n) be sequences of distributions, and let (p_n) be a discrete probability distribution. If $G_n < H_n$ for all n, then

$$\sum_n p_n G_n < \sum p_n H_n.$$

Proof: Apply monotone convergence theorem

Lemma 2: If $G < H$, then

$$G * F < H * F.$$

Proof: To establish the validity of condition (PO'), we observe that

$$\int_t^\infty [1 - G * F(x)]\, dx$$

$$= \int_t^\infty \int_{-\infty}^\infty [1 - G(x - s)]\, dF(s)\, dx$$

and by Fubini's theorem

$$= \int_{-\infty}^\infty \int_{t-s}^\infty [1 - G(y)]\, dy\, dF(s).$$

The last expression shows that we obtain an upper bound if we replace G by H. q.e.d.

Lemma 3: If $G_i < H_i$, $(i = 1, 2, \ldots, n)$, then

$$G_1 * G_2 * \ldots * G_n < H_1 * H_2 * \ldots * H_n.$$

Proof: Repeated application of Lemma 2 leads to

$$\begin{aligned} & G_1 * G_2 * G_3 * \ldots * G_n \\ < \; & H_1 * G_2 * G_3 * \ldots * G_n \\ < \; & H_1 * H_2 * G_3 * \ldots * G_n \\ < \; & H_1 * H_2 * H_3 * \ldots * G_n \qquad \text{etc.} \end{aligned}$$

q.e.d.

3. *Application 1: Dangerous Distributions*

Definition: A distribution H is called ***more dangerous*** than a distribution G if (A) the first moments say μ_G, μ_H exist and $\mu_G \leq \mu_H$ and if (B) there is a constant β such that

$$G(x) \leq H(x) \text{ for } x < \beta$$
$$G(x) \geq H(x) \text{ for } x \geq \beta.$$

Example 1: Let G be unimodal with $G(a-) = 0$, $G(b) = 1$ for $-\infty < a < b < \infty$. Let c, d be numbers such that $c \leq a$, $b \leq d$ and $(c + d)/2 \geq \mu_G$. Then the uniform distribution over the interval (c, d) is more dangerous than G.

Example 2: Let F be a distribution with $F(a-) = 0$, $F(b) = 1$ for $-\infty < a < b < \infty$. Let

$$G(x) = \begin{cases} 0 \text{ for } x < \mu_F \\ 1 \text{ for } x \geq \mu_F \end{cases}$$

and

$$H(x) = \begin{cases} 0 \text{ for } x < a \\ \dfrac{b - \mu_F}{b - a} \text{ for } a \leq x < b \\ 1 \text{ for } x \geq b. \end{cases}$$

Then F is more dangerous than G, and H is more dangerous than F.

Theorem 2: If H is more dangerous than G, then $G < H$.

Proof: Condition (PO') is obviously satisfied if $t \geq \beta$. If $t < \beta$, its validity can be seen as follows:

$$\int_t^\infty [1 - G(x)]\, dx - \int_t^\infty [1 - H(x)]\, dx$$
$$= \int_t^\infty [H(x) - G(x)]\, dx$$
$$\leq \int_{-\infty}^\infty [H(x) - G(x)]\, dx = \mu_G - \mu_H \leq 0. \qquad \text{q.e.d.}$$

Illustration 1: Let $S = S_1 + S_2 + \ldots + S_n$ be a sum of n independent risks. If we replace each of these risks by a more dangerous risk, the stop-loss premium for the sum of these new risks will be at least as high as the stop-loss premium for S (use Theorems 1, 2 and Lemma 3).

Illustration 2: Let S be a risk with a compound Poisson distribution, say with Poisson parameter λ and amount distribution $F(x)$. We assume that $F(0) = 0$ (only positive claims) and that $F(M) = 1$ for some $M > 0$ (a claim amount is at most M), and let μ denote the mean of F (i.e. the average claim amount). We compare S with the two compound Poisson risks S^μ, S^M with fixed claim amounts μ, M, respectively, and Poisson parameters λ, $\Lambda = \lambda(\mu/M)$, respectively. (Observe that $E(S^\mu) = E(S) = E(S^M)$.) From Example 2 (with $a = 0$, $b = M$), Lemmas 1, 3, and Theorems 1, 2 we obtain inequalities for the corresponding stop-loss premiums:

$$P^\mu_x \leq P_x \leq P^M_x.$$

In the case of net stop-loss premiums the second inequality has been proved by Gagliardi and Straub (Mitteilungen Vereinigung schweizerischer Versicherungsmathematiker, 1974, Heft 2).

4. *Application 2: Random sums of positive risks*

In this section we shall compare a distribution of the form

$$G = (1 - q)\, F^{*0} + qF,\ 0 \leq q \leq 1 \tag{4}$$

with one of the more general form

$$H = \sum_{n=0}^{\infty} p_n F^{*n} \tag{5}$$

where

$$0 \leq p_n \leq 1,\ \sum_{n=1}^{\infty} p_n = 1.$$

> *Theorem 3*: Suppose $F(0) = 0$
>
> If $\sum_1^\infty np_n = q$, then $G < H$, where G, H are given by (4), (5).

Proof: Firstly, we show that

$$F < \frac{n-1}{n} F^{*0} + \frac{1}{n} F^{*n}, \; n = 1, 2, \ldots \tag{6}$$

which is a special case of Theorem 3.

To show the validity of condition (PO) we introduce the independent random variables $X_1, X_2, \ldots X_n$ with common distribution F. Then condition (PO) is equivalent to

$$\sum_{i=1}^{n} E[(X_i - t)_+] \leq (n-1)(-t)_+ + E[(\sum_{i=1}^{n} X_i - t)_+].$$

But the corresponding inequality is satisfied for any outcomes of $X_1, X_2, \ldots, X_n$.

Secondly, we show that $G < H$ in the general case. Since

$$H = \sum_{n=1}^{\infty} n\,p_n \left[\frac{n-1}{n} F^{*0} + \frac{1}{n} F^{*n}\right] + (1-q)\,F^{*0}$$

$$G = \sum_{n=1}^{\infty} n\,p_n\,F + (1-q)\,F^{*0}$$

this follows from equation (6) and Lemma 1.

Illustration: *Individual versus collective model*: The individual model is described by n numbers q_i, $0 < q_i \leq 1$, and n distributions F_i with $F_i(0) = 0$. We have in mind a portfolio consisting of n components. Then q_i is the probability that a claim occurs in component i, and F_i is the distribution of its amount. Let

$$S^{\text{ind}} = S_1 + S_2 + \ldots + S_n$$

denote the total claims of the portfolio, where

$$\text{Prob}\,(S_i = 0) = 1 - q_i$$
$$\text{Prob}\,(S_i \leq x) = 1 - q_i + q_i F_i(x), \; x > 0$$

for $i = 1, 2, \ldots, n$. We assume that $S_1, S_2, \ldots, S_n$ are independent and denote the stop-loss premium for S^{ind} by P_α^{ind} ($\alpha =$ retention limit).

A collective model is assigned to the individual model in a well known fashion: Let S^{coll} denote the compound Poisson random variable with

$$\text{Poisson parameter} \quad \lambda = \sum_{i=1}^{n} q_i$$

$$\text{Amount distribution } F = \sum_{i=1}^{n} q_i/\lambda \; F_i.$$

Let P_α^{coll} denote the stop-loss premium for S^{coll}. By applying Theorem 3 to each of the n components (replacing S_i by a compound Poisson random variable with Poisson parameter q_i and amount distribution F_i), we recognize from Theorem 1 and Lemma 3 that $P_\alpha^{\text{ind}} \leq P_\alpha^{\text{coll}}$. Thus a cautious reinsurer will prefer the collective model to the individual model.