SOME COMMENTS ON THE SPARRE ANDERSEN MODEL IN THE RISK THEORY

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SUMMARY

The Sparre Andersen model assumes that the interclaim times and the amounts of claims are independent random variables, the former identically distributed according to a distribution function $K(t)$, $t \geq 0$, $K(0) = 0$, the latter identically distributed according to a distribution function $P(y)$ $-\infty < y < \infty$. As is well known, the Poisson risk process corresponds to the particular case $K(t) = 1 - e^{-\beta t}$. In the present paper it is pointed out that another particular case, viz. $K(t) = \varepsilon(t - h)$, corresponding to a fixed (and thus — strictly speaking—nonrandom) interclaim time, h, has interesting applications. Thus, the ruin problem considered by Giezendanner, Straub and Wettenschwiler in a paper to the 1972 International Congress of Actuaries in Oslo can be formulated by means of this particular case. The same can be said about the earlier model brought forward by Ammeter in his 1948 paper in Skandinavisk Aktuarietidskrift.

About the contents of the paper the following further information may be given. The general Sparre Andersen model is first presented and then the ruin formulas are given for the case with a positive gross risk premium. Thereafter, a modified and more direct method for deriving certain necessary auxiliary functions is illustrated by examples including i.a. the Giezendanner—Straub—Wettenschwiler model. The rest of the paper contains a discussion from the point of view of the Sparre Andersen model of (i) the discrete (equidistant) inspection of a Poisson process for ruin, (ii) the Ammeter model and analogous models, and (iii) the Giezendanner—Straub—Wettenschwiler model.

1. INTRODUCTION

E. Sparre Andersen in a paper to the 1957 International Congress of actuaries in New York [2] proposed a generalization of the classical (Poisson) risk theory. Instead of assuming just exponentially distributed independent interocurrence (interclaim) times, he introduced a more general distribution function but retained the assumption of independence. His model, therefore, can be characterized in the following way.

Let T_1, T_2, ... be the interclaim times. Thus T_n is the time between the $(n-1)$th and the nth claim. T_1 is the time between the zero point and the first claim. The amount of the nth claim is denoted by Y_n. It is now assumed that T_1, Y_1, T_2, Y_2, ... are independent stochastic variables, such that T_1, T_2, .. are identically distributed $K(t)$, $t \geq 0$, $K(0) = 0$ and Y_1, Y_2, .. are identically distributed $P(y)$, $-\infty < y < \infty$. Furthermore, it is assumed that the means of $K(t)$ and $P(y)$ exist and are finite. We denote them by k_1 and p_1 respectively. The gross risk premium per time unit is assumed to be independent of time and have the value c.

If $X(t)$ denotes the risks reserve at time t and $u \geq 0$ denotes the initial risk reserve we have

$X(t) = u + ct - Y(t)$, where $Y(t) = 0$ if $T_1 > t$ and $Y(t) = \sum_{\nu=1}^{n} Y_\nu$ if $T_1 + T_2 + .. + T_n \leqq t < T_1 + T_2 + .. + T_{n+1}$.

We can also write

$$X(t) = u + \sum_{\nu=1}^{n} (cT_\nu - Y_\nu) + c\,(t - \sum_{\nu=1}^{n} T_\nu).$$

Our main interest is in the probability of ruin during a finite or infinite time i.e. the probability of $X(\tau) < 0$ for some τ such that $0 < \tau \leqslant t$ where t is a fixed finite time, or the probability of $X(\tau) < 0$ for some $\tau > 0$.

The ruin problem for an infinite time was already considered by Sparre Andersen himself. The ruin problem for a finite time has been treated by the present author in a series of papers [11]-[14] and together with Nils Wikstad [15]. However, also other authors have treated the indicated problem. We mention among them the following authors: Brans [3], Dreze [6], Takács [9] and Segerdahl [8]. *Note* that Sparre Andersen in his New York paper introduced some limitations on $K(t)$ including i.a. absolute continuity. These limitations seem to have been caused by a desire to be able to define an intensity function $K'(t)/(1 - K(t))$. So far as I know, these limitations were not used by Sparre Andersen in his treatment of the ruin problem. In the papers by the present author (and in the papers by Brans, Dreze and Takács) there are no such limitations assumed in the general presentation of the model. If for special purposes limitations are needed they are explicitly stated. The occurrence process for the claims is thus assumed to be a general renewal process on the positive half axis.

In the present paper, where *we only treat the case with a positive gross risk premium*, i.e. $c > 0$, we first recall some general formulas pertaining to the ruin problem in this case. Thereafter we specialize to the case with only non-negative risk sums, i.e. we assume $P(0-) = 0$. For various assumptions as to $K(t)$ we give simplified forms of the essential Wiener-Hopf auxiliary functions in the ruin problem by a slightly modified method.

At last, we show how the discrete model proposed by Giezendanner, Straub and Wettenschwiler and also the older Ammeter model fit in the Sparre Andersen model. The former authors i.a. consider a case where their model can be said to be a discretization of a Poisson model. It is interesting to note how the relevant functions in this case of their model behave when one let their model converge to the corresponding Poisson model.

2. The Ruin Formulas when $c > 0$

From now on we assume that the gross risk premium per time unit is positive i.e. we assume that $c > 0$. We bring some of the relevant ruin formulas in this case from the papers [11] and [12].

As a consequence of the Remark (added in proof) at the end of Part II of [12] *the indicated formulas are valid without any restrictions on $K(t)$ and $P(y)$*. In particular, it is not assumed that the classical constant $R > 0$ does exist.

By $\Psi(u, t)$ we denote the probability of ruin in the interval $(0, t]$ when the initial risk reserve is $u \geq 0$. By definition $\Psi(u, 0) = 0$. The probability of ruin in the unlimited future we denote by $\Psi(u) = \Psi(u, \infty)$. The following fundamental integral equation is satisfied by the function $\Phi(u, t) = 1 - \Psi(u, t)$

$$\Phi(u, t) = \int_0^t dK(v) \int_{-\infty}^{u+cv} \Phi(u + cv - x, t - v)\, dP(x) + \int_t^\infty dK(v) \qquad (2.1)$$

(see [11] p. 40).

For $\Phi(u) = 1 - \Psi(u)$ we get, $(t \to \infty)$,

$$\Phi(u) = \int_0^\infty dK(v) \int_{-\infty}^{u+cv} \Phi(u + cv - x)\, dP(x) \qquad (2.2)$$

or

$$\Phi(u) = \int_{-\infty}^{u} \Phi(u - t)\, dF(t) \qquad (2.3)$$

where $$F(t) = \int_0^\infty P(t + cv)\, dK(v)$$

(see [11] p. 39).

These equations can be solved by the Wiener-Hopf method. (This method was—in connection with ruin problems—first used by Cramér [5] when he treated the Poisson case.)

The following Laplace-Stieltjes transforms are introduced

$$k(z) = \int_0^\infty e^{zt}\, dK(t),\ Re(z) \leqq 0$$

$$p(s) = \int_{-\infty}^\infty e^{sy}\, dP(y),\ Re(s) = 0$$

$$\overline{\Psi}(u, z) = \int_0^\infty e^{zt}\, d_t\, \Psi(u, t),\ u \geqq 0,\ Re(z) \leqq 0$$

$$\overline{\Psi}(u, z) = 0,\ u < 0,\ Re(z) \leqq 0$$

$$\bar{\varphi}(s, z) = 1 - \int_{0-}^\infty e^{su} d_u\, \overline{\Psi}(u, z),\ Re(s) \leqq 0,\ Re(z) < 0$$

$$A(s, z) = \exp\left[\int_{0+}^\infty e^{su} d_u\, M(u, z) + \tfrac{1}{2}\Delta M(0, z)\right],\ Re(s) \leqq 0,\ Re(z) < 0 \tag{2.4}$$

$$B(s, z) = \exp\left[-\int_{-\infty}^{0-} e^{su} d_u\, M(u, z) - \tfrac{1}{2}\Delta M(0, z)\right],\ Re(s) \geqq 0,\ Re(z) < 0 \tag{2.5}$$

where

$$M(x, z) = \sum_{n=1}^\infty (1/n) \int_0^\infty e^{zv}(P^{n*}(x + cv) - 1)\, dK^{n*}(v) \tag{2.6}$$

$$\Delta M(0, z) = M(0+, z) - M(0-, z)$$

Note that $\Psi(u) = \overline{\Psi}(u, 0)$ (also for $u < 0$ by definition) and that

$$\frac{B(s, z)}{A(s, z)} = 1 - k(z - cs)\, p(s),\ Re(s) = 0,\ Re(z) < 0 \tag{2.7}$$

which latter relation is the Wiener-Hopf factorization relevant in the present case. The auxiliary functions $A(s, z)$ and $B(s, z)$ are, for fixed z with $Re(z) < 0$, in the half-planes $Re(s) \leqq 0$, $Re(s) \geqq 0$, respectively, continuous and, together with $1/A(s, z)$ and $1/B(s, z)$

respectively, bounded. In the interior of the respective halfplanes the functions are analytic and regular.

Note also the following relations

$$\lim_{s \to -\infty} A(s, z)) = \exp\left(\tfrac{1}{2}\, \Delta M(0, z)\right) \tag{2.8}$$

$$\lim_{s \to +\infty} B(s, z) = \exp\left(-\tfrac{1}{2}\, \Delta M(0, z)\right) \tag{2.9}$$

The fundamental relation is now

$$\bar{\varphi}(s, z) = \frac{A(s, z)}{A(0, z)}, \quad Re(s) \leqq 0, Re(z) < 0. \tag{2.10}$$

Note that $1 - \bar{\varphi}(s, z)$ is the double Laplace-Stieltjes transform of $\Psi(u, t)$.

From the relation (2.10) we may obtain $\Psi(u, t)$ by a double use of the Lévy inversion formula, duly adapted to the present case, or by some other inversion procedure. *Note* that the case $Re(z) = 0$ may be obtained by continuity. In particular, $\Psi(u) = \bar{\Psi}(u, 0)$ is got by only one inversion.

Note also the following formula, following from the definition of $\bar{\varphi}(s, z)$ and (2.10) letting $s \to -\infty$

$$\bar{\Psi}(0, z) = 1 - \frac{\exp(\frac{1}{2}\, \Delta M(0, z))}{A(0, z)} \tag{2.11}$$

3. Simplified Formulas for the Function $A(s,z)$ when $P(0-)=0$ and $K(t)$ have Some Simple Forms

The formulas (2.4) and (2.5) in combination with (2.6) may seem rather cumbersome. However, simplified formulas for $A(s, z)$ and $B(s, z)$ may sometimes be obtained directly from the factorization formula (2.7). From the properties of $A(s, z)$ and $B(s, z)$ which are described immediately after the formula (2.7) but before the formulas (2.8) and (2.9) it is obvious that if we find some other functions $A_0(s, z)$ and $B_0(s, z)$ satisfying (2.7) and having the mentioned properties then there is a constant $\varkappa(z) \neq 0$ such that

$$A(s, z) = \varkappa(z)\, A_0(s, z)$$
$$B(s, z) = \varkappa(z)\, B_0(s, z)$$

Thus

$$\bar{\varphi}(s, z) = \frac{A(s, z)}{A(0, z)} = \frac{A_0(s, z)}{A_0(0, z)}$$

so it is not necessary to determine $\varkappa(z)$. However, for some purposes it may be helpful to do so. This may be performed by means of the formulas (2.8) and (2.9).

At this point we emphasize that one way to obtain a couple $A_0(s, z)$, $B_0(s, z)$ is to take the logarithm of the right side of (2.7) and apply a Cauchy integral formula. It turns out that in this case $\varkappa(z) \equiv 1$ so we obtain $\log A(s, z)$ and $\log B(s, z)$ as the left open halfplane value and the right open halfplane value, respectively, of a Cauchy integral

$$\frac{1}{2\pi i}\, vp \int_{-i\infty}^{i\infty} \frac{H(s', z)}{s' - s}\, ds'$$

where $H(s', z) = -\log\,(1 - k(z - cs')\, p(s'))$. Note that in the general case it is necessary to interpret the Cauchy integral as a principal value at infinity. Details of the deduction are found in [12] pp. 19-21. In the same paper the indicated formula was used, to obtain—by modification of the integration path—simple formulas for $A(s, z)$ and $B(s, z)$ in some cases where we know something about the zeros of $1 - k(z - cs)\, p(s)$.

In the present paper, however, we apply a more direct deduction of $A_0(s, z)$, $B_0(s, z)$ without use of complex integration. Some knowledge about the zeros of $1 - k(z - cs)\, p(s)$ is however exploited. Instead of doing a systematic study by means of this direct method we illustrate the power of the method in three examples. We begin by an example, which is a particular case of the next example.

Example 1

$K(t) = 1 - e^{-\beta t}$, $P(0 -) = 0$.

Here $k(z) = \dfrac{1}{1 - z/\beta}$ and $p(s) = \int_{0-}^{\infty} e^{sy}\, dP(y)$ are analytic and regular in the open halfplane $Re(s) < 0$ and continuous in the closed one ($Re(s) \leqslant 0$).

Thus, for $Re(z) < 0$,

$$g(s, z) = 1 - k(z - cs)\, p(s) = 1 - \frac{\beta p(s)}{\beta + cs - z}$$

is meromorphic in $Re(s) < 0$ with one simple pole at $s = (z - \beta)/c$.

It is easy to see that there is also one and only one zero in $Re(s) < 0$ (use Rouché's theorem in the circle with radius β/c and center in $(z - \beta)/c$, cf. [10] p. 38). Denote this unique zero by $s_1(z)$. Let us now construct a $B_0(s, z)$ as simple as possible. Since $B(s, z) = A(s, z)\, g(s, z)$ according to formula (2.7) we see that $B(s, z)$ is meromorphic in the halfplane $Re(s) < 0$ with a simple pole at $s = (z - \beta)/c$ and a simple zero at $s_1(z)$. Since there are no more poles or zeros and $B(s, z)$ and $1/B(s, z)$ are bounded and zerofree in $Re(s) \geqslant 0$ the obvious suggestion is

$$B_0(s, z) = \frac{s - s_1(z)}{s - (z - \beta)/c} = \frac{c(s - s_1(z))}{\beta + cs - z}$$

This corresponds to

$$A_0(s, z) = \frac{c(s - s_1(z))}{\beta + cs - \beta\, p(s) - z}$$

Since $A_0(s, z)$ is bounded and zerofree (analytic and regular in $Re(s) < 0$ and continuous in $Re(s) \leqslant 0$) together with $1/A_0(s, z)$ in $Re(s) \leqslant 0$ we have

$$B(s, z) = \varkappa(z)\, B_0(s, z)$$
$$A(s, z) = \varkappa(z)\, A_0(s, z)$$

However, $\varkappa(z) \equiv 1$ which is seen in the following way. From (2.8) and (2.9) we see that

$$A(-\infty, z) = \exp\left(\tfrac{1}{2}\, \Delta M(0, z)\right)$$
$$B(+\infty, z) = \exp\left(-\tfrac{1}{2}\, \Delta M(0, z)\right)$$

However, it is immediately seen that

$$A_0(-\infty, z) = 1 = B_0(+\infty, z)$$

Thus we must have

$$\exp\left(\tfrac{1}{2}\, \Delta M(0, z)\right) = \varkappa(z)$$
$$\exp\left(-\tfrac{1}{2}\, \Delta M(0, z)\right) = \varkappa(z)$$

and the only solution hereof which is compatible with (2.6) is

$$\Delta M(0, z) = 0, \varkappa(z) = 1$$

We thus have

$$A(s, z) = \frac{c(s - s_1(z))}{\beta + cs - \beta p(s) - z} \tag{3.1}$$

$$B(s, z) = \frac{c(s - s_1(z))}{\beta + cs - z} \tag{3.2}$$

The formulas (3.1) and (3.2) for $\beta = 1$ were deduced in [10] p. 38 by complex integration in the slightly more restrictive case $P(0) = 0$. Of course, also the complex integration method works if only $P(0 -) = 0$. The assumption of a positive probability for zero risk sums may seem unrealistic. However it may be a way to take care of very small risk sums. Note also the well-known fact that such a process may be reduced to another Poisson risk process without zero risk sums but with another intensity. Such a reduced process has the same $A(s, z)$ as the unreduced process.

Example 2

$$K(t) = 1 - \sum_{\nu=1}^{n} b_\nu e^{-\beta_\nu t}, \sum_{\nu=1}^{n} b_\nu = 1. P(0 -) = 0.$$

(It is not necessary that all $b_\nu > 0$ as e.g. convolutions of simple but different exponential distributions show. The β_ν's are assumed to lie in the halfplane $Re(\beta_\nu) > 0$ and be disjoint. The non-real among them must lie in complex-conjugate pairs. Among the β_ν's with minimum real part one must be real.)

The properties of $p(s)$ are as in Example 1.

The function $k(z) = \int_0^\infty e^{zt}\, dK(t) = \sum_{\nu=1}^{n} \frac{b_\nu}{1 - s/\beta_\nu}$ is meromorphic with simple poles at $s = \beta_\nu$, $\nu = 1, \ldots, n$.

Thus for $Re(z) < 0$ the function $g(s, z) = 1 - k(z - cs)\, p(s)$ is meromorphic as a function of s in the halfplane $Re(s) < 0$. There are n simple poles at $s = (z - \beta_\nu)/c$. It is also easy to see that there are n zeros in the said halfplane (note that

$$\frac{1}{2\pi i} \int_C d_s \log g(s, z) = 0$$

where C is a halfcircle to the left of the imaginary axis with a sufficiently large diameter lying on the said axis). Let the zeros be $s_{1\nu}(z)$, $\nu = 1, \ldots, n$. Since $B(s, z) = A(s, z)\, g(s, z)$ we see that $B(s, z)$ in $Re(s) < 0$ must have the $s_{1\nu}(z)$'s as zeros and the points $(z - \beta_\nu)/c$, $\nu = 1, \ldots, n$, as poles. At the same time $B(s, z)$ and $1/B(s, z)$ must be bounded and continuous in $Re(s) \geqslant 0$ (analytic and regular in $Re(s) > 0$). The obvious suggestion of a function with these properties is

$$B_0(s, z) = \prod_{\nu=1}^{n} \frac{s - s_{1\nu}(z)}{s - (z - \beta_\nu)/c} = \prod_{\nu=1}^{n} \frac{c(s - s_{1\nu}(z))}{\beta_\nu + cs - z}$$

The corresponding $A_0(s, z)$ is clearly

$$A_0(s, z) = \frac{B_0(s, z)}{1 - p(s) \sum\limits_{\mu=1}^{n} b_\mu \beta_\mu / (\beta_\mu + cs - z)} =$$

$$= \frac{\prod\limits_{\nu=1}^{n} c(s - s_{1\nu}(z))}{\prod\limits_{\nu=1}^{n} (\beta_\nu + cs - z) - p(s) \sum\limits_{\mu=1}^{n} b_\mu \beta_\mu \prod\limits_{\substack{\nu=1 \\ \nu \neq \mu}}^{n} (\beta_\mu + cs - z)}$$

It is immediately clear that $A_0(s, z)$ has the properties required for such a function.

We thus have

$$A(s, z) = \varkappa(z)\, A_0(s, z)$$
$$B(s, z) = \varkappa(z)\, B_0(s, z)$$

Exactly as in Example 1 it is found that $\Delta M(0, z) = 0$, $\varkappa(z) \equiv 1$ so we have

$$A(s, z) = A_0(s, z)$$
$$B(s, z) = B_0(s, z)$$

The same result was deduced in [12] by means of complex integration though under the restriction $P(0) = 0$. The remarks at the end of Example 1 pertaining to the case with a positive probability for zero risk sums carry over, *mutatis mutandis*, to the present example.

Example 3

$K(t) = \varepsilon(t-h),\ h > 0, dP(chj = a_j, j=0, 1, \ldots, \sum_{j=0}^{\infty} a_j = 1,\ a_j \geqq 0,\ a_0 > 0$

This is the discrete model presented by Giezendanner, Straub and Wettenschwiler [7]. Note that the gross risk premium per time unit enters in the definition of $P(y)$.

Obviously $p(s) = \sum_{j=0}^{\infty} a_j e^{schj}$ is a periodic function of s with the period $2\pi i/(ch)$. Moreover, the function is analytic and regular in $Re(s) < 0$ and continuous in $Re(s) \leqslant 0$. Since we have $k(z) = e^{hz}$ we find that

$$g(s, z) = 1 - k(z - cs)\, p(s) = 1 - e^{hz} \sum_{j=0}^{\infty} a_j\, e^{sch\,(j-1)} \tag{3.3}$$

also is a periodic function of s with the same period $2\pi i/(ch)$. It is also analytic and regular in $Re(s) < 0$ and continuous in $Re(s) \leqq 0$. What can we say about the zeros of $g(s, z)$ for $Re(s) < 0$, $Re(z) < 0$? We introduce the variable $w = e^{chs}$ and study the equation

$$1 - e^{hz} \sum_{j=0}^{\infty} a_j\, w^{j-1} = 0 \tag{3.4}$$

Corresponding to $Re(s) < 0$ we have $|\, w \,| < 1$. We multiply by w and get the equivalent equation

$$\sum_{j=0}^{\infty} a_j\, w^j = e^{-hz}\, w \tag{3.5}$$

(Note that $w = 0$ neither is a root of (3.4) nor of (3.5) since we have assumed that $a_0 > 0$).

An application of Rouché's theorem now shows that (3.5) has just one root in $|\, w \,| < 1$. In fact take

$$f(w) = e^{-hz}\, w,\ \ g(w) = e^{-hz}\, w - \sum_{j=0}^{\infty} a_j\, w^j$$

then on $|\, w \,| = 1$ we have for $Re(z) < 0$

$$|\, f(w) - g(w) \,| = |\, \sum_{j=0}^{\infty} a_j\, w^j \,| \leqq 1 < |\, f(w) \,|$$

Thus $f(w)$ and $g(w)$ have the same number of zeros in $|\, w \,| < 1$ However, $f(w)$ has just the zero $w = 0$ so also $g(w)$ has one and only one root, say $w_1(z)$, in $|\, w \,| < |$ and this one cannot lie in $w = 0$ as we pointed out above. Note that $0 < w_1(z) < 1$, when z is negative real.

Thus the function (3.3) has just the simple zeros in $Re(s) < 0$

$$s_{1\nu}(z) = \frac{1}{ch} \log w_1(z) + \frac{2\pi i\nu}{ch}, \ \nu = 0, \pm 1, \pm 2, \ldots.$$

where we determine $\log w_1(z)$ so that it is real for z negative real and varies continuously with z.

Obviously, these zeros are also the zeros of $1 - e^{-ch(s - s_{10}(z))}$. Since this function is analytic and regular in the whole s-plane and is bounded and also bounded away from zero in the right halfplane $Re(s) \geqslant 0$, we suggest the choice

$$B_0(s, z) = 1 - e^{-ch(s - s_{10}(z))}$$

Then we have

$$A_0(s, z) = \frac{1 - e^{-ch(s - s_{10}(z))}}{1 - e^{hz} e^{-chs} \sum_{j=0}^{\infty} a_j e^{chjs}}$$

Clearly $A_0(s, z)$ is together with $1/A_0(s, z)$ analytic and regular in $Re(s) < 0$. In $Re(s) \leqq 0$ they are continuous and bounded.

Thus we have $A(s, z) = \varkappa(z)\, A_0(s, z), B(s, z) = \varkappa(z)\, B_0(s, z)$

In order to determine $\varkappa(z)$ we note that

$$\lim_{s \to -\infty} A_0(s, z) = (1/a_0)\, e^{chs_{10}(z) - hz}$$

$$\lim_{s \to \infty} B_0(s, z) = 1$$

From (2.8) and (2.9) we now conclude that

$$\exp\left(\tfrac{1}{2}\, \Delta M(0, z)\right) = \varkappa(z) \exp\left(ch\, s_{10}(z) - hz\right)/a_0$$

$$\exp\left(-\tfrac{1}{2}\, \Delta M(0, z)\right) = \varkappa(z)$$

The only solution hereof, compatible with (2.6), is

$$\varkappa(z) = \sqrt{a_0} \exp\left(\frac{h}{2} z - \frac{ch}{2} s_{10}(z)\right) \tag{3.6}$$

$$\Delta M(0, z) = -\log a_0 + ch\, s_{10}(z) - hz \tag{3.7}$$

We thus get

$$A(s, z) = \sqrt{a_0} \exp\left(\frac{h}{2} z - \frac{ch}{2} s_{10}(z)\right) \frac{1 - e^{-ch(s - s_{10}(z))}}{1 - e^{hz} e^{-chs} \sum_{j=0}^{\infty} a_j e^{chjs}} \tag{3.8}$$

$$B(s, z) = \sqrt{a_0} \exp\left(\frac{h}{2} z - \frac{ch}{2} s_{10}(z)\right) \left(1 - e^{-ch(s - s_{10}(z))}\right) \tag{3.9}$$

Giezendanner, Straub and Wettenschwiler mainly consider their model as a discretization of an underlying Poisson risk process. It may in such a case be interesting to see how formulas (3.8) and (3.9) behave when the discretization parameter h tends to zero. It turns out—not surprisingly—that the $A(s, z)$ according to (3.8) tends towards the $A(s, z)$ according to (3.1). Contrary to this, $B(s, z)$ according to (3.9) and $g(s, z)$ according to (3.3) tend to zero. A division by h before the limiting process, however, brings them in a closer contact with the corresponding functions for the Poisson process.

In fact, considering a Poisson process such that the distribution function of the accumulated claim amounts up to the epoch t is

$$F(y, t) = \sum_{\nu=0}^{\infty} e^{-\beta t} \frac{(\beta t)^\nu}{\nu!} Q^{\nu *}(y), Q(0) = 0 \tag{3.10}$$

the authors define two sets of quantities a_n, $n = 0, 1, \ldots.$ (see [7] p. 647) representing a lower and an upper discretization of $F(y, h)$

I $\quad a_0 = F(0, h),\ a_j = F(chj, h) - F(ch(j-1), h),\ j = 1, 2, \ldots$

i.e. $\quad a_0 = e^{-\beta h},\ a_j = \sum\limits_{\nu=0}^{\infty} e^{-\beta h} \dfrac{(\beta h)^\nu}{\nu!} [Q^{\nu *}(chj) - Q^{\nu *}(ch(j-1))]$

II $\quad a_0 = F(ch-, h),\ a_j = F(ch(j+1)-, h) - F(chj-, h),\ j = 1, 2, \ldots.$

i.e. $\quad a_0 = \sum\limits_{\nu=0}^{\infty} e^{-\beta h} \dfrac{(\beta h)^\nu}{\nu!} Q^{\nu *}(ch-),\ a_j = \sum\limits_{\nu=0}^{\infty} e^{-\beta h} \dfrac{(\beta h)^\nu}{\nu!}$

$$[Q^{\nu *}(ch(j+1)-) - Q^{\nu *}(chj-)]$$

In both cases I and II it is easily found that

$$g(s, z) = 1 - e^{-hz-chs} \sum_{j=0}^{\infty} a_j e^{chjs} = h(\beta + cs - z - \beta q(s)) + o(h),\ h \to 0$$

where

$$q(s) = \int_0^\infty e^{sy}\, dQ(y)$$

It is also found in both cases that

$$B(s, z) = h\, c(s - s_1(z)) + o(h)$$

and lastly that

$$A(s, z) = \frac{(s - s_1(z))}{\beta + cs - z - \beta q(s)} + o(1) \text{ as } h \to 0 \qquad (3.11)$$

Here $s_1(z)$ is the zero appearing in the Poisson case. Clearly (3.11) shows that in the limit we get the $A(s, z)$ belonging to the Poisson case.

In passing, we note that the authors also, for $\beta h < 1$, consider the more manageable choices

III $a_0 = 1 - \beta h,\ a_j = \beta h(Q(chj) - Q(ch(j-1))),\ j = 1, 2, \ldots$

IV $a_0 = 1 - \beta h(1 - Q(ch -)),\ a_j = \beta h(Q(ch(j+1)-) - Q(chj -)),\ j = 1, 2, \ldots$

Of course, the relation (3.11) is still valid.

4. The Equidistant Inspection for Ruin of a Poisson Process

If we have a Poisson risk process with the intensity β and the claim distribution $Q(y)$, $y > 0$, and wish to consider the probability of ruin at the equidistant points νh, $\nu = 1, 2, \ldots, n$, we can as well consider the probability of ruin during the time interval $(0, nh]$ for a Sparre Andersen process with $K(t) = \varepsilon(t - h)$, $P(y) = F(y, h)$ where $F(y, h) = \sum_{\nu=0}^{\infty} e^{-\beta h} \frac{(\beta h)^\nu}{\nu!} Q^{\nu *}(y)$, $Q(0) = 0$. As is well-known the ruin can only occur at the epochs of "claims" i.e. at points νh, $\nu = 1, 2, \ldots, n$. The amounts of "claims" are distributed according to $P(y) = F(y, h)$. Note that $Y = 0$ has a positive probability $F(0, h) = e^{-\beta h}$. In this case it is thus meaningful to speak about zero risk sums.

The crucial Wiener-Hopf function $g(s, z)$ takes here the form

$$g(s, z) = 1 - k(z - cs)\, p(s) = 1 - e^{h(z-cs)}\, e^{\beta h(q(s)-1)}$$
$$= 1 - e^{-h(\beta + cs - z - \beta q(s))}$$

So far we have not used the assumption $Q(0) = 0$ of (3.10). From now on, however, we emphasize this assumption. In order to perform the factorization

$$g(s, z) = \frac{B(s, z)}{A(s, z)}$$

we first look at the location of the zeros of $g(s, z)$ in the left half-plane $Re(s) < 0$.

These zeros are the zeros of the functions

$$\beta + cs - z - \beta q(s) - \frac{2\pi i}{h}\nu, \quad \nu = 0, \pm 1, \ldots .$$

i.e. $s_1\left(z + \frac{2\pi i}{h}\nu\right)$ if $s_1(z)$ denotes the unique zero in $Re(s) < 0$ of $\beta + cs - z - \beta q(s)$ i.e. the zero pertaining to the Poisson process if continuously inspected (cf Example 1).

As pointed out in Example 1 we always know that $s_1(z)$ is located in the interior of a circle with center at $(z - \beta)/c$ and with radius β/c. Thus

$$\left| s_1\left(z + \frac{2\pi i}{h}\nu\right) - \frac{z + (2\pi i/h)\,\nu - \beta}{c} \right| < \frac{\beta}{c} \tag{4.1}$$

so that the exponent of convergence of the $s_1\left(z + \frac{2\pi i}{h}\nu\right)$ is one.

Since, in the present case, $g(s, z)$ as a function of s is analytic and regular in $Re(s) < 0$ and in this region is of exponentid type ch (in $Re(s) \leqslant 0$ $g(s, z)$ is continuous), we conclude that $B(s, z)$ is an entire function in s of exponential type ch having the zeros

$$s_1\left(z + \frac{2\pi i}{h}\nu\right), \quad \nu = 0, \pm 1, \ldots\ldots .$$

The general appearence of such a function can be deduced from Hadamard's factorization theorem. Without going into details in the general case we concentrate on the particular case when $Q(y)$ is a discrete distribution function with its jumps at $y = ch, 2ch, \ldots$ i.e.

$$q(s) = \sum_{\mu=1}^{\infty} b_\mu e^{ch\mu s}, \sum_{\mu=1}^{\infty} b_\mu = 1, b_\mu \geqq 0, \mu = 1, 2, \ldots$$

Here $q(s)$ is a periodic function with the period $2\pi i/(ch)$. Thus

$$s_1\left(z + \frac{2\pi i}{h}\nu\right) = s_1(z) + \frac{2\pi i}{ch}\nu$$

so we get

$$B(s, z) = \varkappa(z)\ (1 - e^{-ch(s - s_1(z))})$$

and

$$A(s, z) = \varkappa(z)\ \frac{1 - e^{-ch(s - s_1(z))}}{1 - e^{-h(\beta + cs - z - \beta q(s))}}$$

Thus we are back in the model by Giesendanner, Straub and Wettenschwiler.

The weights $a_0, a_1, \ldots.$ to apply in their model may be expressed by $b_0, b_1, \ldots.$ if we use the identity

$$\sum_{j=0}^{\infty} a_j x^j = e^{-\beta h(1 - \sum_{\mu=1}^{\infty} b_\mu x^\mu)} \tag{4.2}$$

In Example 3 we applied the said model by following the authors' direct discretization of $F(y, h)$ by the choices I and II of $a_0, a_1 \ldots.$ In this section we are led to their model by a discretization of $Q(y)$ rather than of $F(y, h)$. It is obvious that a discrete $Q(y)$ entails a discrete $F(y, h)$. Of course, this discrete $F(y, h)$ must not be confused with the discretized versions according to I or II of the original $F(y, h)$ before the discretization of $Q(y)$. Needless to say, the direct discretization of $F(y, h)$ is superior in efficiency to a discretization via $Q(y)$.

5. The Ammeter Model and Analogous Models

Let us consider a risk process in *discrete* time $(0, h, 2h, 3h \ldots, nh, \ldots)$ such that for every nh, $n \geqslant 1$, we have a stochastic variable Y_n. We assume that the Y_n's are independent and identically distributed stochastic variables with the common distribution function $P(y)$. Our risk theoretic interpretation of this simple scheme is now that Y_n represents the total claim amount in the interval $((n - 1)h, nh)$ and our interest is in the possibility of ruin at points nh, $n = 1, 2, \ldots [t/n]$. The solution of this problem is obviously the same as the solution of the ruin problem for a Sparre Andersen process with $K(t) = \varepsilon(t - h)$ and the indicated $P(y)$. One instance was already considered in the previous section 4:

$$P(y) = \sum_{\nu=0}^{\infty} e^{-\beta h} \frac{(\beta h)^\nu}{\nu!} Q^{\nu *}(y) \tag{5.1}$$

Another instance is the model considered by Ammeter in his 1948 paper in Skandinavisk Aktuarietidskrift [1]. Here we have

$$P(y) = \sum_{\nu=0}^{\infty} P_\nu(h)\, Q^{\nu *}(y) \tag{5.2}$$

where

$$P_\nu(h) = \int_0^\infty e^{-xh} \frac{(xh)^\nu}{\nu!}\, dU(x)$$

with

$$U(x) = \frac{1}{\Gamma(\gamma)} \int_0^{x\gamma} e^{-v}\, v^{\gamma-1}\, dv \tag{5.3}$$

Other choices of the weight function $U(x)$ give further instances.

As is well known (see Bühlmann [4] pp. 74-75 or Thyrion [16]) the case of an infinitely divisible $U(x)$ (as e.g. (5.3)) admits *for fixed h* a transformation of h and Q to say h_1 and Q_1 such that

$$P(y) = \sum_{\nu=0}^{\infty} e^{-h_1} \frac{h_1^\nu}{\nu!} Q_1^{\nu *}(y)$$

Then the treatment of section 4 can be applied if we let $\beta = h_1/h$. In the case (5.3) Ammeter himself performed such a reduction of his ruin problem to the corresponding problem in the Poisson case.

6. Som Further Comments on the Example 3 of Section 3

Giezendanner, Straub and Wettenschwiler considered only the ruin problem for an infinite time period. They used a combinatorial result due to Takács in order to get a starting value in a recurrence scheme for the determination of $\Psi(u)$.

We will now try to retrieve the recurrence scheme and the starting value by means of the theory for the Sparre Andersen process. However, we will already now point out that there is a difference between our definition of ruin and the one adopted by Giezendanner, Straub and Wettenschwiler. They speak about ruin when $X(\tau) \leqslant 0$ while our definition is $X(\tau) < 0$. When at least one of $K(t)$ and $P(y)$ in the Sparre Andersen model is continuous the indicated difference in definition is irrelevant. In the present case both $K(t)$ and $P(y)$ are discontinuous and we must be more cautious. However, it is easy to make a final correction for the difference of definition so we start out with our own definition.

We consider the equation (2.3) i.e.

$$\Phi(u) = \int_{-\infty}^{u} \Phi(u - t)\, dF(t),\ \Phi(u) = 1 - \Psi(u),\ u \geqq 0$$

where

$$F(t) = \int_{0}^{\infty} P(t + cv)\, dK(v)$$

and recall our choice $K(t) = \varepsilon(t - h)$ and $dP(chj) = a_j, j = 0, 1, 2, \ldots, a_j \geqq 0, a_0 > 0, \sum_{j=0}^{\infty} a_j = 1$.

Thus $F(t) = P(t + ch)$ and

$$\Phi(u) = \int_{-\infty}^{u} \Phi(u - t)\, dP(t + ch) = \sum_{j=-1}^{H[u/ch]} \Phi\,(u - chj)\, a_{j+1} \qquad (6.1)$$

From the definition of $\Psi(u)$ it follows that $\Psi(u)$ and $\Phi(u)$ must be constant in every interval $\nu ch \leqq u < (\nu + 1)\, ch$ since the increment of $X(t)$ from nh to $(n + 1)h$ always is an integer multiple of ch. Let us now denote $\Phi(\nu ch)$ by Φ_ν. From (6.1) we get

$$\Phi_\nu = \sum_{j=-1}^{\nu} \Phi_{\nu-j}\, a_{j+1},\ \nu = 0, 1, 2, \ldots$$

i.e. $$\Phi_\nu = \Phi_{\nu+1}\, a_0 + \Phi_\nu\, a_1 + \sum_{j=1}^{\nu} \Phi_{\nu-j}\, a_{j+1}$$

Thus

$$a_0\Phi_{\nu+1} = (1 - a_1)\, \Phi_\nu - \sum_{j=1}^{\nu} a_{j+1}\, \Phi_{\nu-j},\ = 0, 1, 2, \ldots \qquad (6.2)$$

Since we have assumed $a_0 > 0$ we may obtain all Φ_ν recursively if we know $\Phi_0 = \Phi(0) = 1 - \Psi(0)$. For this purpose we exploit the formula (2.11)

$$\overline{\Psi}(0, z) = 1 - \frac{\exp\left(\frac{1}{2}\, \Delta M(0, z)\right)}{A(0, z)}$$

and formulas (3.7) and (3.8).

Since $$\Psi(0) = \lim_{z \to 0} \overline{\Psi}(0, z)$$

we seek $\lim\limits_{z \to 0} \Delta M(0, z)$ and $\lim\limits_{z \to 0} \dfrac{1}{A(0, z)}$.

We find from formula (3.7)

$$\text{that } \lim_{z\to 0} \Delta M(0, z) = -\log a_0 + chs_{10}(0)$$

where

$$s_{10}(0) = \lim_{z\to 0} s_{10}(z) = \lim_{z\to 0} \frac{1}{ch} \log w_1(z)$$

Regarding the equation (3.5) we easily see that

$$\lim_{z\to 0} \quad w_1(z) = 1 \text{ if } \alpha = \sum_{j=1}^{\infty} j\, a_j \leqq 1$$

but

$$0 < \lim_{z\to 0} \quad w_1(z) < 1 \text{ if } \alpha > 1$$

Taking first the case $\alpha \leqq 1$

we find

$$\lim_{z\to 0} \Delta M(0, z) = -\log a_0$$

and

$$\lim_{z\to 0} \frac{1}{A(0, z)} = \frac{1-\alpha}{\sqrt{a_0}} \qquad \text{(use formula (3.8))}$$

Thus, in the case $\alpha \leqq 1$

$$\Phi(0) = 1 - \Psi(0) = \frac{1-\alpha}{a_0}$$

Let us now consider the case $\alpha > 1$.

We get

$$\lim_{z\to 0} \frac{1}{A(0, z)} = 0$$

and thus

$$\Phi(0) = 0.$$

In the case $\alpha \geqq 1$ we thus have $\Phi(0) = 0$ and (6.2) shows that $\Phi(u) = 0$ for every $u \geqq 0$.

In the case $\alpha < 1$ we have $\Phi(0) = \dfrac{1-\alpha}{a_0} > 0$ and (6.2) gives $\Phi(u)$ for $u \geqq 0$ by recurrence.

We now turn to the definition of ruin by condition $X(\tau) \leqq 0$ for some $\tau > 0$.

Obviously, the probability of such a ruin when $u > 0$, is $\Psi(u-)$ if $\Psi(u)$ is the probability of ruin according to our definition.

The two values $\Psi(u)$ and $\Psi(u-)$ differ only for $u = chj$. Thus according to the definition $(X(\tau) \leqq 0)$ we should have the probabilities of non-ruin at $u = ch\nu$, $\nu = 1, 2, \ldots$ to be $\Phi_{\nu-1}$ rather than Φ_ν. Denoting $\Phi_{\nu-1}$ by Q_ν we get the recurrence equations

$$a_0 Q_{\nu+2} = (1 - a_1)\, Q_{\nu+1} - \sum_{j=1}^{\nu} a_{j+1}\, Q_{\nu-j+1},\ \nu = 0, 1, 2, \ldots$$

or

$$\begin{cases} a_0\, Q_{\nu+1} = (1 - a_1) Q_\nu - \sum\limits_{j=1}^{\nu-1} a_{j+1}\, Q_{\nu-j},\ \nu = 1, 2, \ldots. \\ Q_1 = \dfrac{1-\alpha}{a_0} \end{cases}$$

which are the recurrence relations given by Giezendanner, Straub and Wettenschwiler [7] p. 646. However, we have not introduced Q_0 but this may be done by the obvious formula $Q_0 = a_0 Q_1$.

Note that it is possible—at least partly—to treat the ruin problem for a *finite* period by similar recurrence methods.

Consider first equation (2.1), letting $K(t) = \varepsilon(t-h)$.

We get for $0 \leqq t < h$

$\Phi(u, t) = 1$ (since only the second term of the righthand side gives a contribution)

and for $t \geqq h$

$$\Phi(u, t) = \int_{-\infty}^{u+ch} \Phi(u + ch - x, t - h)\, dP(x)$$
$$= \sum_{j=0}^{H[u/(ch)]+1} \Phi(u + ch - chj, t - h)\, a_j$$

Since $\Phi(u, t)$ is constant for $\nu ch \leqq u < (\nu + 1)\, ch$, $\mu h \leqq t < (\mu + 1)h$ we simplify the notation by denoting the said constant by $\Phi_{\nu,\mu}$. Thus

$$\Phi_{\nu,0} = 1$$
$$\Phi_{\nu,\mu} = \sum_{j=0}^{\nu+1} \Phi_{\nu+1-j,\mu-1}\, a_j, \quad \mu \geqq 1.$$

In practice, these relations may be used for small μ.

Another procedure is the following.

Corresponding to (2.1) we have the relation (see [11] p. 41) valid for $Re(z) \leqq 0$, where $\overline{\Phi}(u, z) = 1 - \overline{\Psi}(u, z)$, $u \geqq 0$,

$$\overline{\Phi}(u, z) = \int_0^\infty (1 - e^{zv})\, dK(v) + \int_0^\infty e^{zv}\, dK(v) \int_{-\infty}^{u+cv} \overline{\Phi}(u + cv - x, z)\, dP(x)$$

Taking $K(t) = \varepsilon(t - h)$ we get

$$\overline{\Phi}(u, z) = 1 - e^{zh} + e^{zh} \int_{-\infty}^{u+ch} \overline{\Phi}(u + ch - x, z)\, dP(x)$$

i.e.

$$\overline{\Phi}(u, z) = 1 - e^{zh} + e^{zh} \sum_{j=0}^{[u/ch]+1} \overline{\Phi}(u + ch - chj, z)\, a_j$$

Observing that $\overline{\Phi}(u, z)$ as a function of u is constant in $\nu ch \leqq u < (\nu + 1)\, ch$ and denoting this value by $\overline{\Phi}_\nu(z)$ we get

$$\begin{aligned}\overline{\Phi}_\nu(z) &= 1 - e^{zh} + e^{zh} \sum_{j=0}^{\nu+1} \overline{\Phi}_{\nu+1-j}(z)\, a_j \\ &= 1 - e^{zh} \\ &\quad + e^{zh}(\overline{\Phi}_{\nu+1}(z)\, a_0 + \overline{\Phi}_\nu(z)\, a_1 + \\ &\qquad + \sum_{j=2}^{\nu+1} \overline{\Phi}_{\nu+1-j}(z)\, a_j).\end{aligned}$$

Thus

$$\begin{aligned}a_0\overline{\Phi}_{\nu+1}(z) &= 1 - e^{-zh} \\ &\quad + (e^{-zh} - a_1)\, \overline{\Phi}_\nu(z) \\ &\quad - \sum_{j=2}^{\nu+1} \overline{\Phi}_{\nu+1-j}(z)\, a_j, \\ &\quad \nu = 0, 1, 2, \ldots.\end{aligned} \tag{6.3}$$

If we know the value of $\overline{\Phi}_0(z) = \overline{\Phi}(0, z)$ the relation (6.3) admits a recurrence determination of $\overline{\Phi}_\nu(z)$ in just the same way as Φ_ν. However, $\overline{\Phi}(0, z)$ is easily determined by (2.11), (3.7) and (3.8).

If we can exploit an inversion algorithm from $\overline{\Phi}_\nu(z)$ to $\Phi_{\nu,\mu}$ which only requires a very limited number of z's to be used, it may be manageable to determine $\Phi_{\nu,\mu}$ this way.

Addendum

The method used in section 3 for a direct derivation of the essential Wiener-Hopf auxiliary functions has for $P(0-) = 0$ been illustrated for various assumptions as to $K(t)$. Needless to say, the method may also be used, *mutatis mutandis*, in cases where $K(t)$ is completely general but instead some special assumption is made about $P(y)$. One instance is the case where $P(0) = 0$ and $p(s) = \int_0^\infty e^{sy}\, dP(y)$ is rational, treated in [12] pp. 25-28 by means of the slightly more cumbersome method of complex integration.

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