OPTIMAL CONSUMPTION AND INSURANCE: A CONTINUOUS-TIME MARKOV CHAIN APPROACH

BY

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ABSTRACT

Personal financial decision making plays an important role in modern finance. Decision problems about consumption and insurance are in this article modelled in a continuous-time multi-state Markovian framework. The optimal solution is derived and studied. The model, the problem, and its solution are exemplified by two special cases: In one model the individual takes optimal positions against the risk of dying; in another model the individual takes optimal positions against the risk of losing income as a consequence of disability or unemployment.

KEYWORDS

Personal finance, multi-state model, stochastic control, financial decision making, mortality-disability-unemployment risk.

1. INTRODUCTION

Optimal personal financial decision making plays an important role in modern financial mathematics and economics. Merton (1969, 1971) introduced continuous-time consumption-investment problems that have been developed further and generalized since then. These generalizations are often based on modifications of financial market models or individual preferences. In this article we focus on the consumption decision along with the introduction of insurance decisions of various types and, for simplicity, we assume that there is only one financial investment opportunity that is risk-free, but this assumption can be relaxed.

Richard (1975) generalized Merton's results to the case where the individual has an uncertain life-time, income while being alive, and, in addition to the asset allocation and the consumption, decides continuously on a life insurance

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sum. The uncertain life-time was modelled by an age-dependent mortality intensity. Actually, the idea of involving the life insurance decision in the personal decision making of an individual with an uncertain life-time dates further back to Yaari (1965) who studied the problem in a discrete time setting.

The same year in which Merton (1969) first published his ideas, Hoem (1969) demonstrated that the continuous-time finite-state Markov chain is an inevitable tool in the construction of general life insurance products and the modelling of general life insurance risk. The finite-state Markov chain has been studied in the context of life insurance and vice versa since then by Hoem (1988), Norberg (1991) and many others. It provides a model for various kinds of risk connected to an individual's life. One important example is the risk of losing income due to disability or unemployment.

Richard (1975) studied the consumption and life insurance decisions in a survival model where the saving takes place on a private account. We generalize this in two directions. First we model the life insurance risk in a multi-state framework such that e.g. insurance decisions with respect to disability and unemployment can be studied. This reflects the variety and complexity of real life financial decisions and insurance markets. This is the primary contribution of this article. Second we allow for saving in the insurance company. Richard (1975) concludes his article by noting that 'rich, old' people optimally should be sellers of insurance while consuming their wealth. In his article this life insurance is sold although the policy holder has not saved anything in the insurance company. In practice this is not possible since the maximum life insurance sum the policy holder can sell, is exactly the savings in the company. Taking this sum to be equal to the savings in the company is exactly what happens when the policy holder holds a life annuity. We allow for saving in the insurance company by letting wealth consist of both the balance of a private account and the balance of an account in the insurance company. This is the secondary contribution of this article.

In Section 2, we briefly summarize the results of Richard (1975). Section 3 presents the general setup in a multi-state framework. In Section 4, we formalize an optimization problem and present its solution. Furthermore, we study and provide economic interpretations of the optimal decisions. Sections 5 and 6 analyze two important cases: The survival model is studied in 5, nesting Richard's model as a special case. The disability/unemployment model is studied in Section 6 providing further results and insights.

2. On Richard's Result

We start out by briefly recalling the ideas of Richard (1975) that are the central reference for our studies. As in the remainder of the article, we assume that there is only a deterministic investment opportunity. This section is intended to make the reader familiar with our patterns of thinking and our notation in a simple illustrative example before we consider the general case.

We model the death of an individual by a mortality intensity μ and denote by N and I the processes indicating whether the individual is dead and alive, respectively, i.e. $N(t) = 1$ [the individual is dead at time t] and $I(t) = 1$ [the individual is alive at time t]. Note that in this section $N = 1 - I$, but this will not be the case when we generalize N and I in the next section.

We work with three different payment processes A, B, and C playing completely different roles. The payment process A denotes the accumulated labor income earned at the rate $a(t)$ as long as the individual is alive. The rate is assumed to be time-dependent such that the individual's income distribution over the life-cycle can be taken into account (e.g. no income after retirement). The payment process B models the accumulated payments exchanged with a life insurance company. At any time point t, the policy holder decides upon the life insurance sum $b(t)$ that he wishes to hold over $(t, t + dt]$. This triggers a premium payment $b(t)\mu^*(t)\,dt$, where μ^* is the mortality rate used by the insurance company for pricing. This may or may not be different from μ. For simplicity, in this section, the insurance company does not build up reserves. The payment process C corresponds to the accumulated consumption consumed at the rate $c(t)$ as long as the individual is alive. The policy holder chooses the consumption rate. Upon death he (or rather his inheritors) consumes the death sum b paid out by the insurance company plus the remaining wealth. If the individual survives until the final time n, then he consumes himself the remaining wealth.

The wealth process, starting at x_0 at time 0, is denoted by X and earns a constant interest rate r. The capital gains are assumed to come from investing in the bond market which exactly gives the short term of interest as long as this is deterministic. We can formalize this setup by the following system of stochastic differential equations where $\varepsilon(t,n) = 1\,[t \geq n]$:

$$
\begin{aligned}
dA(t) &= a(t)I(t)\,dt,\ A(0) = 0,\\
dB(t) &= -b(t)I(t)\mu^*(t)\,dt + b(t)\,dN(t),\ B(0) = 0,\\
dC(t) &= c(t)I(t)\,dt + (X(t-) + b(t))\,dN(t) + X(t-)I(t)\,d\varepsilon(t,n),\ C(0) = 0,\\
dX(t) &= rX(t)\,dt + dA(t) + dB(t) - dC(t)\\
&= rX(t)\,dt + I(t)(a(t) - b(t)\mu^*(t) - c(t))\,dt - X(t-)(dN(t)\\
&\quad + I(t)\,d\varepsilon(t,n)),\\
X(0) &= x_0.
\end{aligned}
$$

The individual measures utility of his consumption-insurance decisions through the accumulated utility process U formalized by

$$
dU(t) = I(t)u^0(t,c(t))\,dt + u^{01}(t, X(t-) + b(t))\,dN(t) + I(t)\Delta U^0(t, X(t-))\,d\varepsilon(t,n).
$$

Here, $u^0(t,c(t))$ is the utility rate at time t of consuming at rate $c(t)$, $u^{01}(t, X(t-) + b(t))$ is the (inheritors') utility of consuming the death sum and the wealth just prior to death, and $\Delta U^0(t, X(t-))$ is the utility of consuming the wealth upon termination. Note that the notation Δ is reasonable here because U is the process of accumulated utility. Upon termination, the 'lump utility' measured is a discontinuity in this accumulated utility process and the difference operator identifies this amount of utility. After his death or termination, whatever occurs first, the wealth process equals zero. The individual chooses the consumption and insurance sum in order to maximize utility from future consumption, so we are interested in

$$\sup E\left[\int_0^n dU(t)\right].$$

We consider the particular case of power utility where consumption at rate c and consumption upon death or termination is measured by the same power utility function $\frac{1}{\gamma}(\cdot)^\gamma$ but weighted differently by the coefficients $w^0(t)^{1-\gamma}$, $w^{01}(t)^{1-\gamma}$, and $\Delta W^0(t)^{1-\gamma}$. These coefficients are, without loss of generality, taken to the power $(1-\gamma)$ for later notational convenience. Thus,

$$\begin{aligned}
u^0(t,x) &= \frac{1}{\gamma} w^0(t)^{1-\gamma} x^\gamma, \\
u^{01}(t,x) &= \frac{1}{\gamma} w^{01}(t)^{1-\gamma} x^\gamma, \\
\Delta U^0(t,x) &= \frac{1}{\gamma} \Delta W^0(t)^{1-\gamma} x^\gamma.
\end{aligned}$$

From the presentation of the problem, we go directly to the result by Richard which is also obtained as a special case of our results. The optimal consumption rate and the optimal death sum can be expressed in terms of three crucial deterministic functions, f, g, and h, which are defined in the following way: Introducing the notation,

$$\begin{aligned}
\tilde{\mu}(t) &= \mu(t)^{\frac{1}{1-\gamma}} \mu^*(t)^{-\frac{\gamma}{1-\gamma}}, \\
\delta &= \frac{\gamma}{1-\gamma}, \\
\tilde{r}(t) &= -\delta r - \delta(\mu^*(t) - \mu(t)) + \mu(t) - \tilde{\mu}(t),
\end{aligned}$$

we have that

$$\begin{aligned}
f(t) = &\int_t^n e^{-\int_t^s (\tilde{r}(\tau)+\tilde{\mu}(\tau))\,d\tau} \left(w^0(s) + \tilde{\mu}(s) w^{01}(s)\right) ds \\
&+ e^{-\int_t^n (\tilde{r}(\tau)+\tilde{\mu}(\tau))\,d\tau} \Delta W^0(n),
\end{aligned} \tag{1a}$$

$$g(t) = \int_t^n e^{-\int_t^s (r(\tau)+\tilde{\mu}^*(\tau))\,d\tau} a^0(s)\,ds, \tag{1b}$$

$$h(t) = \left(\frac{\mu(t)}{\mu^*(t)}\right)^{1/(1-\gamma)}. \tag{1c}$$

These functions have the following interpretations: The function f measures the expected value of the future utility weights where $\tilde{r}$ and $\tilde{\mu}$ can be interpreted as utility-adjusted interest and mortality rates. The function g measures the financial value of future income also referred to as human wealth. Finally, h is the ratio between the objective mortality and the pricing mortality relative to risk aversion and is, thus, a measure of how cheap the coverage appears to the individual.

With these functions in place we can formalize the optimal controls c and b as functions of time and wealth,

$$\begin{aligned} c(t,x) &= \frac{w^0(t)}{f(t)}(x+g(t)), \\ x+b(t,x) &= \frac{w^{01}(t)}{f(t)}h(t)(x+g(t)). \end{aligned}$$

The optimal consumption rate is a fraction of the total wealth measured as the wealth X plus the human wealth g. The fraction relates utility of consumption today w^0 to utility of consumption in the future f. The optimal death sum forms the sum consumed upon death $X(t-)+b(t,X(t-))$ as the total wealth held prior to death $X(t-)+g(t)$ multiplied by two factors: One factor measures utility of consumption upon death w^{01} against the expected utility of consumption in the future prior to death f; another factor h^{01} measures of how cheap the term insurance appears to the individual.

In the rest of the article we present two main generalizations to the results above:

- We formalize the insurance risk model by a finite state Markov chain such that more general insurances like e.g. disability insurance can be taken into consideration.
- We allow for building up reserves in the insurance company.

3. The Model and the Decision Processes

We take as given a probability space $(\Omega, \mathcal{F}, P)$. On this probability space is defined a process $Z=(Z(t))_{0\le t\le n}$ taking values in a finite set $\mathcal{J}=\{0,\ldots,J\}$ of possible states and starting, by convention, in state 0 at time 0. We define the $J+1$-dimensional counting process $N=(N^k)_{k\in\mathcal{J}}$ by

$$N^k(t) = \#\{s\in(0,t],\, Z(s-)\neq k,\, Z(s)=k\},$$

counting the number of jumps into state k until time t. Assume that there exist deterministic functions $\mu^{jk}(t)$, $j,k\in J$, such that N^k admits the stochastic intensity process $(\mu^{Z(t-)k}(t))_{0\le t\le n}$ for $k\in J$, i.e.

$$M^k(t) = N^k(t) - \int_0^t \mu^{Z(s)k}(s)\,ds$$

constitutes a martingale for $k \in J$. Then Z is a Markov process. For each state we introduce the indicator process indicating sojourn, $I^j(t) = 1[Z(t) = j]$, and the functions $\mu^{jk}(t)$, $j, k \in J$ and the intensity process $\mu^{Z(t-)k}(t)$ are connected by the relation $\mu^{Z(t)k}(t) = \sum_{j:j \neq k} I^j(t)\mu^{jk}(t)$.

The reader should think of Z as the state of life of an individual in a certain sense of personal financial decision making which will be described in this section. An important example to have in mind is the three state model illustrated in Figure 1. The absorbing state 2 is the state of being dead. The individual can jump between two states of being alive, 0 and 1, with certain age-dependent intensities, possibly 0. From each of these states the individual can jump into the state of being dead with an age- and state-dependent intensity. Two examples of states 0 and 1 are the following: A disability model where 0 is the state of ability/activity and 1 is the state of disability, and an unemployment model where 0 is the state of employment and 1 is the state of unemployment.

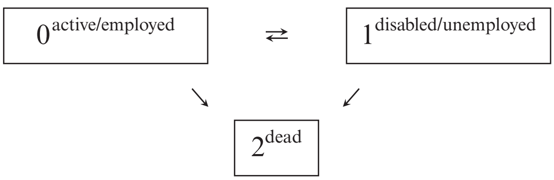

FIGURE 1: Disability/Unemployment model.

We now introduce the following three payment processes:

1. An income process $A = (A(t))_{0 \le t \le n}$ representing the accumulated income of the individual. The income process is assumed to follow the dynamics

$$dA(t) = a^{Z(t)}(t)\,dt + \sum\nolimits_{k:k \neq Z(t-)} a^{Z(t-)k}(t)\,dN^k(t),$$

where $a^j(t)$ and $a^{jk}(t)$ are assumed to be deterministic functions. Here, $a^j(t)$ is the rate of income given that the individual is in state j at time t and $a^{jk}(t)$ is the lump sum income given that the individual jumps from state j to state k at time t. By income we think primarily of labor income but other types of income could be taken into account.

2. An insurance payment process $B = (B(t))_{0 \le t \le n}$ representing the accumulated insurance net payments from the insurance company to the policy holder. The insurance payment process is assumed to follow the dynamics

$$dB(t) = dB^{Z(t)}(t) + \sum\nolimits_{k:k \neq Z(t-)} b^{Z(t-)k}(t)\,dN^k(t).$$

where $B^j(t)$ is a sufficiently regular adapted process specifying accumulated payments during sojourns in state j and $b^{jk}(t)$ is a sufficiently regular predictable process specifying payments due upon transitions from state j to state k. Note that adaptedness and predictability relate to the filtration $\mathcal{F}$ generated by Z and, thus, no other stochastic components enter into these processes. We assume that each B^j decomposes into an absolutely continuous part and a discrete part, i.e.

$$dB^j(t) = b^j(t)dt + \Delta B^j(t),$$

where $\Delta B^j(t) = B^j(t) - B^j(t-)$, when different from 0, is a jump representing a lump sum payable at time t if the policy holder is then in state j. Positive elements of B are called benefits whereas negative elements are called premiums.

3. A consumption process $C = (C(t))_{0 \le t \le n}$ representing the accumulated consumption of the individual. The consumption process is assumed to follow the dynamics

$$dC(t) = c^{Z(t)}(t)dt + \sum\nolimits_{k : k \neq Z(t-)} c^{Z(t-)k}(t)dN^k(t).$$

Here, $c^j(t)$ is the rate of consumption given that the individual is in state j at time t and $c^{jk}(t)$ is the lump sum consumption at time t given that the individual jumps from state j to state k at time t. The processes $c^j(t)$ and $c^{jk}(t)$ are decision processes chosen at the discretion of the individual.

These payment process affect, together with an interest rate r, the following three notions of wealth:

1. The *personal wealth* is accounted for on a *bank account* of the individual and accounts for all three payment processes A, B, and C in the sense that labor income and insurance benefits are accounted for as income and consumption is accounted for as outgo. The bank account has the following dynamics,

$$\begin{aligned} dX(t) &= rX(t)dt + dA(t) + dB(t) - dC(t), \\ X(0) &= x_0. \end{aligned} \tag{2}$$

2. The *institutional wealth* is accounted for on an *insurance account* of the individual and accounts for the payment process B, jumps in the reserve upon transition, and transition risk premia. The insurance account has the following dynamics,

$$\begin{aligned} dY(t) &= rY(t)dt - dB^{Z(t)}(t) - \sum_{k : k \neq Z(t)} \mu^{Z(t)k*}(t)\left(b^{Z(t)k}(t) + y^{Z(t)k}(t)\right)dt \\ &\quad + \sum_k y^{Z(t-)k}(t)dN^k(t), \\ Y(0) &= 0. \end{aligned} \tag{3}$$

where y^{jk} is a sufficiently regular predictable process specifying the account jump upon transition from state j to state k. Upon termination, the account prior to termination is simply paid out as a lump sum, i.e. $\Delta B^{Z(n)}(n) = Y(n-)$ such that the account closes at 0, i.e. $Y(n) = 0$. The risk premia are calculated on the basis of a set of pricing transition intensities $\mu^{jk*}(t)$, $j \neq k$, $j, k \in \mathcal{J}$. We show in Appendix A that the insurance account equals the traditionally defined reserve for future net benefits,

$$Y(t) = E^*\left[\int_t^n e^{-r(s-t)} dB(s) \middle| \mathcal{F}(t)\right]. \tag{4}$$

The asterisk decoration of E^* means that the expectation is taken with respect to a valuation measure P^* characterized by the transition intensities $\mu^{jk*}(t)$, $j \neq k, j, k \in \mathcal{J}$. We assume that these are deterministic such that Z is Markovian under P^*.

3. The *total wealth* is now obtained by simply adding up the institutional wealth and the personal wealth. This gives the following dynamics,

$$\begin{aligned} d(X(t) + Y(t)) = {} & r(X(t) + Y(t))dt + dA(t) - dC(t) \\ & + \sum_{k:k \neq Z(t-)} \left(b^{Z(t-)k}(t) + y^{Z(t-)k}(t)\right) dM^{Z(t)k^*}(t). \end{aligned}$$

These dynamics have the following interpretation. Firstly, the total wealth earns interest at rate r. Secondly, the income process and the consumption process affect the total wealth directly. Thirdly, upon a transition from j to k the total wealth increases by $b^{jk}(t) + y^{jk}(t)$. From this amount, $b^{jk}(t)$ is paid from the insurance institution to the individual and added to the bank account. The amount, $y^{jk}(t)$ is also paid from the insurance company to the individual but kept by the insurance company by adding it to the insurance account. For this total wealth increment of $b^{jk}(t) + y^{jk}(t)$, the individual pays a natural premium at rate $\mu^{jk*}(b^{jk}(t) + y^{jk}(t))$.

In this paper we consider a decision problem where, at time t, the policy holder decides on $dB^{Z(t)}(t)$, $b^{Z(t)k}(t)$, $y^{Z(t)k}(t)$, $c^{Z(t)}(t)$ and $c^{Z(t-)k}(t)$ for all $k \neq Z(t)$. This is really an unconventional construction and to a reader with a life insurance background, this may look like a very awkward decision problem. Deciding on $dB^{Z(t)}(t)$, $b^{Z(t)k}(t)$, $c^{Z(t)}(t)$ and $c^{Z(t-)k}(t)$ may seem reasonable but what does it mean that the policy holder decides on the reserve jump $y^{Z(t)k}(t)$?

In practice the policy holder decides on a set of future payments, i.e. $dB^j(s)$, $b^{jk}(s)$, $s > t$, $j \neq k$, and on the basis of these, the insurance company calculates the reserve jumps $y^{Z(t)k}(t)$, $k \neq Z(t)$. But this means that the policy holder in practice indirectly decides on the reserve jumps through specification of the future payments.

In the decision problem studied here we consider $y^{Z(t)k}(t)$, $k \neq Z(t)$ as the decision variable. For a chosen set of reserve jumps, the policy holder now needs to calculate a future payment stream leading to these reserve jumps. This is then the future payment stream he should optimally demand from the insurance company. There is typically a continuum of future payment processes leading to the right reserve jump so the future payments are not uniquely determined by this procedure. However, he can take any one of these as long as it leads to the right reserve jumps. The future payments are superseded anyway by $dB^{Z(t)}(t)$ and $b^{Z(t)k}(t)$ as the future turns into the present.

From the dynamics of the accounts, there are three important points to make that all have to do with the ambiguity of our problem formulation:

- In the dynamics of the total wealth, X and Y appear through their sum only. Thus, if they do so also in the objective function of the decision problem, one can replace the two processes by their sum and reduce the number of state processes. In our objective function introduced in the next section, X and Y will appear through their sum only. However, we still choose to work with the two state processes in order to be able to solve various kinds of constrained problems. E.g., to keep insurance business separated from banking (loaning) business, one could have the constraint that $Y(t) \geq 0$ for all t. By working with two state processes, it is possible to extract solutions to such and similar constrained problems directly from our results below.
- In the dynamics of total wealth, $b^{Z(t-)k}(t)$ and $y^{Z(t-)k}(t)$ appear through their sum only. Thus, if they do so also in the objective function of the decision problem, one can replace the two processes by their sum and reduce the number of decision processes. In our objective function introduced in the next section, $b^{Z(t-)k}(t)$ and $y^{Z(t-)k}(t)$ will not appear at all. However, we still choose to work with the two processes in order to be able to solve various kinds of constrained problems. E.g., to prevent people from selling insurances on their own lives, one could have the constraint that $b^{jk}(t) \geq 0$. By working with two decision processes, it is possible to extract solutions to such and similar constrained problems directly from our results below.
- In the dynamics of the total wealth, the continuous insurance payment rate $b^{Z(t)}(t)$ does not appear at all. Thus, if it also does not appear in the objective function of the decision problem, one can disregard this process and reduce the number of decision processes. In our objective function introduced in the next section, $b^{Z(t)}(t)$ will not appear at all. However, we still choose to work with this as a decision process in order to be able to solve various kinds of constrained problems.

Below we solve the unconstrained problem in general. The point is, however, that since we are working with 'too many' controlled processes (with no constraints on accounts, one of the accounts X and Y is redundant) or 'too many' control processes (with no constraints on the payments, b^j and one of the variables b^{jk} and y^{jk} are redundant) we have directly solved a series of relevant constrained control problems. This is seen in the examples where we look at special cases where X or Y are constrained to be zero. A final remark on constrained versus unconstrained problems is that the solution to the unconstrained problem is very often a crucial element in the solution to the constrained problem. Therefore, the understanding of the unconstrained problem and its solution is important, not only in its own right, but also as a first step towards solving constrained problems.

4. The Control Problem and Its Solution

In this section we present the control problem and its solution. We introduce a process of accumulated utility with dynamics given by

$$dU(t) = u^{Z(t)}\big(t, c^{Z(t)}(t)\big)dt + \sum\nolimits_{k:k \neq Z(t-)} u^{Z(t-)k}\big(t, c^{Z(t-)k}(t)\big)dN^k(t)$$
$$+ \Delta U^{Z(t-)}(t, X(t-), Y(t-))\, d\varepsilon(t,n).$$

Here, $u^j(t,c)$ is a deterministic utility function that measures utility of the consumption rate c given that the individual is in state j at time t and $u^{jk}(t,c)$ is a deterministic utility function that measures utility of the lump sum consumption c given that the individual jumps from state j to state k at time t. Finally, $\Delta U^j(n,x,y)$ is a deterministic function which measures utility of the terminal lump sum payout from the two accounts x and y given that the individual is in state j at time n. We assume that the individual chooses a consumption-insurance process to maximize utility in the sense of

$$\sup E\left[\int_0^n dU(t)\right].$$

where the supremum is taken over b^j, b^{jk}, y^{jk}, c^j, c^{jk}, $j \neq k$.

We specify further the utility functions appearing in the utility process. We are interested in solving the problem for an individual with preferences represented by the power utility function in the sense of

$$u^j(t,c) = \frac{1}{\gamma} w^j(t)^{1-\gamma} c^\gamma,$$
$$u^{jk}(t,c) = \frac{1}{\gamma} w^{jk}(t)^{1-\gamma} c^\gamma,$$
$$\Delta U^j(t,x,y) = \frac{1}{\gamma} \Delta W^j(t)^{1-\gamma} (x+y)^\gamma.$$

Here, $w^j(t)$ is the non-negative weight on power utility of the consumption rate c given that the individual is in state j at time t, $w^{jk}(t)$ is the non-negative weight on power utility of the lump sum consumption c given that the individual jumps from state j to state k at time t. Finally, $\Delta W^j(t)$ the non-negative weight on power utility of lump sum consumption given that the individual is in state j at time t. It is convenient to think of these weight functions as stemming from a weight process with dynamics given by

$$dW(t) = w^{Z(t)}(t)dt + \sum\nolimits_{k:k \neq Z(t-)} w^{Z(t-)k}(t)\,dN^k(t) + \Delta W^{Z(t)}(t)\,d\varepsilon(t,n).$$

Calculations in Appendix B show that the optimal consumption and insurance strategies are given by the following feed-back functions for $c^j(t)$, $c^{jk}(t)$, and $b^{jk}(t) + y^{jk}(t)$,

$$c^j(t,x,y) = \frac{w^j(t)}{f^j(t)}(x + y + g^j(t)), \tag{5a}$$

$$c^{jk}(t,x,y) = \frac{w^{jk}(t)}{f^j(t)} h^{jk}(t)\,(x + y + g^j(t)), \tag{5b}$$

$$\begin{aligned} b^{jk}(t,x,y) + y^{jk}(t,x,y) &= \frac{f^k(t) + w^{jk}(t)}{f^j(t)} h^{jk}(t)\,(x + y + g^j(t)) \\ &\quad - (a^{jk}(t) + x + y + g^k(t)), \end{aligned} \tag{5c}$$

where f, g, and h satisfy

$$g_t^j(t) = rg^j(t) - a^j(t) - \sum\nolimits_{k:k \neq j} \mu^{jk*}(t)(a^{jk}(t) + g^k(t) - g^j(t)), \tag{6}$$

$$g^j(n) = 0,$$

$$f_t^j(t) = \tilde{r}^j(t) f^j(t) - w^j(t) - \sum\nolimits_{k:k \neq j} \tilde{\mu}^{jk}(t)\,(w^{jk}(t) + f^k(t) - f^j(t)), \tag{7}$$

$$f^j(n) = \Delta W^j(n),$$

$$h^{jk}(t) = \left(\frac{\mu^{jk}(t)}{\mu^{jk*}(t)}\right)^{1/(1-\gamma)}, \tag{8}$$

with

$$\tilde{\mu}^{jk}(t) = \mu^{jk}(t) h^{jk}(t)^{\gamma} = \mu^{jk^*}(t) h^{jk}(t),$$

$$\delta = \frac{\gamma}{1-\gamma},$$

$$\tilde{r}^j(t) = -\delta r - \delta(\mu^{j\cdot *}(t) - \mu^{j\cdot}(t)) + \mu^{j\cdot}(t) - \tilde{\mu}^{j\cdot}(t).$$

The solution to the system of ordinary differential equations for g has the Feynman-Kac representation

$$g^j(t) = E^*_{t,j}\left[\int_t^n e^{-r(s-t)} dA(s)\right]$$
$$= \int_t^n e^{-r(s-t)} \sum_k p^*_{jk}(t,s)\left(a^k(s) + \sum_{l:l\neq k} \mu^{kl*}(s)\, a^{kl}(s)\right) ds. \tag{9}$$

Thus, $g^j(t)$ is the conditional expected present value of the future income process where the expectation is taken under P^*. This is, in other words, the financial value of the future income.

The solution to the system of ordinary differential equations for f has the Feynman-Kac representation

$$f^j(t) = \tilde{E}_{t,j}\left[\int_t^n e^{-\int_t^s \tilde{r}^{Z(\tau)}(\tau)\, d\tau} dW(s)\right]. \tag{10}$$

Thus, $f^j(t)$ is the conditional expected value of the future weight process where expectation is taken under an artificial measure $\tilde{P}$ under which N^k admits the intensity process $\tilde{\mu}^{Z(t)k}(t)$. This is, in other words, an artificial financial value of the future weights in the sense that we apply an artificial stochastic interest rate process and an artificial valuation measure.

We now take a closer look at the optimal controls. First we give interpretations of them as they appear in (5). In all three formulas appear the sum $x + y + g^j(t)$. This can be interpreted as the total wealth of the individual given that he is in state j at time t. This total wealth consists of personal wealth x, institutional wealth y, and human wealth g^j. Recall that g^j is the financial value of future income given that the individual is in state j. Furthermore, in (5c) appears the sum $a^{jk}(t) + x + y + g^k(t)$. This can be interpreted as the total wealth of the individual upon transition from state j to state k at time t before the effect of insurance. This wealth consists of the lump sum income upon transition $a^{jk}(t)$ and then again of personal wealth x, institutional wealth y, and human wealth $g^k(t)$. Here the human wealth is measured given that the individual is in state k at time t. We emphasize that this is the wealth before a possible insurance sum is paid out or a reserve jump has been added to the institutional wealth. With these interpretations of total wealth in mind we can now interpret the three control functions:

- The optimal continuous consumption rate in (5a) is a fraction of total wealth. The fraction $w^j(t)/f^j(t)$ measures the utility of present consumption against utility of consumption in the future. Recall that $f^j(t)$ is an artificial value of the future weights.
- The optimal lump sum consumption upon transition in (5b) is also a fraction of wealth. The fraction $h^{jk}(t)\, w^{jk}(t)/f^j(t)$ consists of two elements. The

fraction $w^{jk}(t)/f^j(t)$ measures the utility of consumption upon transition against utility of future consumption. However, future consumption is calculated given that the individual is in state j at time t – and not given that he is in state k – since the transition risk is partly 'insured away'. The price of this insurance is, together with the individual attitude towards risk, hidden in the factor h^{jk}.

- The optimal insurance sum plus reserve jump upon transition in (5c) can be interpreted as a protection of wealth. In the optimal decision one should not distinguish between an insurance sum that is a sum added to personal wealth, and a reserve jump that is a sum added to institutional wealth. The optimal insurance sum plus reserve jump measures the difference between a fraction $h^{jk}(t)(f^k(t)+w^{jk}(t))/f^j(t)$ of present wealth $x+y+g^j(t)$ and wealth upon transition $a^{jk}(t)+x+y+g^k(t)$. If the fraction $h^{jk}(t)(f^k(t)+w^{jk}(t))/f^j(t)$ is 1 then this difference reduces to $-(a^{jk}(t)+g^k(t)-g^j(t))$ which is minus the human wealth sum at risk. Thus, this is really the wealth that is potentially lost upon transition and which should be protected by an opposite insurance position. However, in the calculation of the optimal protection two further considerations should be taken into account: 1) The utility of future wealth in case of no transition is measured against the utility of future wealth in case of transition in the ratio $(f^k(t)+w^{jk}(t))/f^j(t)$. If utility of future wealth given a transition is lower than without transition, i.e. $(f^k(t)+w^{jk}(t))/f^j(t)<1$, then one should underinsure ones wealth under risk, $-(a^{jk}(t)+g^k(t)-g^j(t))$; 2) If the protection is 'expensive', i.e. $h^{jk}<1$, then one should also underinsure ones wealth under risk in order to 'pick up' some of this market price of risk.

Now, take a closer look at the controls c^j and c^{jk}. For fixed $Z(t)=j$, we can study the optimally controlled processes X^j and Y^j that solve the following ordinary differential equations

$$\frac{d}{dt}X^j(t) = rX^j(t)+a^j(t)+b^j(t)-c^j(t),$$

$$\frac{d}{dt}Y^j(t) = rY^j(t)-b^j(t)-\sum_{k:k\neq j}\mu^{jk*}(t)(b^{jk}(t)+y^{jk}(t)).$$

Since X^j and Y^j evolve deterministically, we can study the state-wise controls $c^j(t,X^j(t),Y^j(t))$ and $c^{jk}(t,X^j(t),Y^j(t))$ as functions of time. With a slight abuse of notation we denote these deterministic functions by $c^j(t)$ and $c^{jk}(t)$. Furthermore, we consider the optimal wealth upon transition before consumption which is given by

$$\begin{aligned} q^{jk}(t,x,y) &= b^{jk}(t,x,y)+y^{jk}(t,x,y)+a^{jk}(t)+x+y+g^k(t) \\ &= \frac{f^k(t)+w^{jk}(t)}{f^j(t)}h^{jk}(t)(x+y+g^j(t)). \end{aligned}$$

Also $q^{jk}(t, X(t), Y(t))$ can be studied as a function of time, denoted accordingly by $q^{jk}(t)$. From (5) we can, by using

$$\begin{aligned} c_t^j(t) &= \frac{\partial}{\partial t} c^j(t, X^j(t), Y^j(t)) \\ &\quad + \frac{\partial}{\partial x} c^j(t, X^j(t), Y^j(t)) \frac{dX^j(t)}{dt} \\ &\quad + \frac{\partial}{\partial y} c^j(t, X^j(t), Y^j(t)) \frac{dY^j(t)}{dt} \end{aligned}$$

and similar formulas for c^{jk} and q^{jk}, derive the following simple exponential differential equations for $c^j(t)$, $c^{jk}(t)$, and $q^{jk}(t)$,

$$c_t^j(t) = c^j(t) \left(\frac{1}{1-\gamma} \left(r + \mu^{j\cdot *}(t) - \mu^{j\cdot}(t) \right) + \frac{w_t^j(t)}{w^j(t)} \right),$$

$$c_t^{jk}(t) = c^{jk}(t) \left(\frac{1}{1-\gamma} \left(r + \mu^{j\cdot *}(t) - \mu^{j\cdot}(t) \right) + \frac{w_t^{jk}(t)}{w^{jk}(t)} + \frac{h_t^{jk}(t)}{h^{jk}(t)} \right),$$

$$q_t^{jk}(t) = q^{jk}(t) \left(\frac{1}{1-\gamma} \left(r + \mu^{j\cdot *}(t) - \mu^{j\cdot}(t) \right) + \frac{f_t^k(t) + w_t^{jk}(t)}{f^k(t) + w^{jk}(t)} + \frac{h_t^{jk}(t)}{h^{jk}(t)} \right).$$

By the definition of h in (8) and introducing $\mu^{jk*}(t) = (1 + \Gamma^{jk}(t))\mu^{jk}(t)$, we can calculate that $h_t^{jk}(t)/h^{jk}(t) = -\Gamma_t^{jk}(t)/((1-\gamma)(1 + \Gamma^{jk}(t)))$. If we define the weights according to the usual impatience factor, i.e. $w^j(t)^{1-\gamma} = \exp(-\iota t)$ we can furthermore calculate that $w_t^j(t)/w^j(t) = -\iota/(1-\gamma)$. Plugging in these relations, we get the following simple differential equations for the optimal controls $c^j(t)$ and $c^{jk}(t)$,

$$c_t^j(t) = c^j(t) \frac{1}{1-\gamma} \left(r - \iota + \mu^{j\cdot *}(t) - \mu^{j\cdot}(t) \right), \tag{11}$$

$$c_t^{jk}(t) = c^{jk}(t) \frac{1}{1-\gamma} \left(r - \iota + \mu^{j\cdot *}(t) - \mu^{j\cdot}(t) - \frac{\Gamma_t^{jk}(t)}{1 + \Gamma^{jk}(t)} \right).$$

5. The Survival Model

In this section we specialize the results in Section 4 in the case of a survival model. We study optimal consumption and insurance decisions of an individual who has utility of consumption while being alive including utility of lump sum consumption upon termination. Furthermore he (or rather his inheritors) has utility of consumption upon death before termination. In Figure 2, we have illustrated a set of income process coefficients and a set of utility weight coefficients.

0^{alive}	$\overset{\mu}{\rightarrow}$	1^{dead}
$w^0 \neq 0,\ \Delta W^0 \neq 0$	$w^{01} \neq 0$	$w^1 = \Delta W^1 = 0$
$a^0 \neq 0,\ \Delta A^0 = 0$	$a^{01} = 0$	$a^1 = \Delta A^1 = 0$

FIGURE 2: Survival model with income and utility weights.

All statewise coefficients are zero in the state 'dead'. This means that there is no income and no utility of consumption in that state. Weights on utility of consumption in the state 'alive' are specified by the coefficients w^0 and ΔW^0, and weight on utility of a lump sum payment upon death is specified by w^{01}. Income is specified by the rate a^0 and other income coefficients are set to zero such that there is no lump sum income upon death or upon survival until termination. We start out by specifying the functions f and g for this special case.

According to (7) $f^1 = 0$ and f^0 is characterized by

$$\begin{aligned} f_t^0(t) &= -w^0(t) + f^0(t)\,\tilde{r}(t) - \tilde{\mu}(t)\,(w^{01}(t) - f^0(t)), \\ f^0(n) &= \Delta W^0(n), \\ \tilde{r} &= -\delta r - \delta(\mu^*(t) - \mu(t)) + \mu(t) - \tilde{\mu}(t). \end{aligned}$$

This differential equation has the solution presented as f in (1a).

According to (6) $g^1 = 0$ and g^0 is characterized by

$$\begin{aligned} g_t^0(t) &= rg^0(t) - a^0(t) + \mu^*(t)\,g^0(t), \\ g^0(n) &= 0. \end{aligned} \tag{12}$$

This differential equation has the solution presented as g in (1b).

We can now specify the optimal controls in terms of f^0 and g^0 and get

$$\begin{aligned} c^0(t,x,y) &= \frac{w^0(t)}{f^0(t)}\left(x + y + g^0(t)\right), \\ c^{01}(t,x,y) &= \frac{w^{01}(t)}{f^0(t)}\,h^{01}(t)\left(x + y + g^0(t)\right), \end{aligned}$$

$$\begin{aligned} b^{01}(t,x,y) + y^{01}(t,x,y) &= \frac{w^{01}(t)}{f^{0}(t)} h^{01}(t)\left(x + y + g^{0}(t)\right) - x - y \\ &= c^{01}(t,x,y) - x - y. \end{aligned} \tag{13}$$

Upon death the benefit $b^{01}(t,x,y)$ is paid out and $c^{01}(t,x,y)$ is consumed. If x and y are the accounts just prior to death, these accounts upon death will then be $x + b^{01}(t,x,y) - c^{01}(t,x,y)$ and $y + y^{01}(t,x,y)$, respectively. But according to (13) these accounts are the same with opposite signs. We restrict ourselves to the case where the insurance account is set to zero upon death by choosing $y^{01}(t,x,y) = -y$. Then, by (13), $x + b^{01}(t,x,y) - c^{01}(t,x,y) = 0$, such that also the bank account is set to zero then. Both accounts remain zero until termination.

We also specify the simple exponential differential equation characterizing the statewise consumptions,

$$c_t^{0}(t) = c^{0}(t)\,\frac{1}{1-\gamma}\left(r - \imath + \mu^{*}(t) - \mu(t)\right), \tag{14}$$

$$c_t^{01}(t) = c^{01}(t)\,\frac{1}{1-\gamma}\left(r - \imath + \mu^{*}(t) - \mu(t) - \frac{\Gamma_t(t)}{1+\Gamma(t)}\right). \tag{15}$$

Example 1. ***No insurance account.***

The special case where there is no insurance account is basically what is considered in Section 2. By fixing the premium $-b$ as the natural premium for the optimal death sum,

$$-b(t) = \mu^{*}(t)\,b^{1}(t),$$

and realizing from (3) and $y^{01}(t,x,y) = -y$ that $Y(t) = 0$ for all t, we can put $y = 0$ in all controls and skip the dependence on y. This gives exactly the controls presented in Section (2). The formulas are identical to those by Richard (1975, (42, 43)). In comparison we mention that Richard (1975) uses the following notation (Richard notation $\equiv$ notation here): $a \equiv f^{1-\gamma}$, $b \equiv g$, $h \equiv w^{1-\gamma}$, $m \equiv (w^{1})^{1-\gamma}$, $\lambda \equiv \mu$, $\mu \equiv \mu h^{\gamma-1}$.

Richard (1975) modelled the mortality such that the probability of survival until termination n, $\exp\left(-\int_t^n \mu\right)$, is zero for all t. This is obtained by $\mu \to \infty$ for $t \to n$. Furthermore, it is assumed that $\mu^{*}(t)/\mu(t) \to 1$ for $t \to n$. But then the last term of (1a) is zero and $\Delta W(n)$ is superfluous: If we know that we will not survive time n, the utility of consumption at time n plays no role for our decision. With $\Delta W(n) = 0$, (1a) and (1b) are identical to those by Richard (1975, (41,25)).

A problem, from a practical point of view, with this construction is that the optimal insurance sum may become negative. If the individual (and his inheritors) has relatively large utility from consuming while being alive compared to consuming upon death, he should optimally risk losing parts of his wealth as he *grows old. When there is no institutional wealth he does so by selling life insurance. But the way individuals sell life insurance in practice is instead by holding life annuities based on institutional wealth. Therefore, a much more realistic special case is given now in an example with no bank account.*

Example 2. ***No bank account.***

We can put the bank account equal to zero by specifying that income minus consumption goes directly into the insurance account, i.e.

$$B = C - A.$$

In this concrete case this corresponds to letting $-b^0(t) = a^0(t) - c^0(t)$*, i.e. the excess of income over consumption is paid as premium on the insurance contract, and* $b^{01}(t) = c^{01}(t)$*, i.e. upon death the insurance benefit is consumed (by the inheritors). Realize from (2) and (13) that then* $X(t) = 0$ *for all* t *and we can put* $x = 0$ *and skip the dependence on* x *in all controls, i.e.*

$$\begin{aligned} c^0(t,y) &= \frac{w^0(t)}{f^0(t)}\big(y + g(t)\big), \qquad (16) \\ c^{01}(t,y) &= \frac{w^{01}(t)}{f^0(t)} h(t)\big(y + g(t)\big) \\ &= b^{01}(t). \end{aligned}$$

Note that since now $b^{01} = c^{01}$*, the differential equation (15) holds also for the optimal death sum.*

Remark 3. *The optimal consumption rate (16) solves the problem of optimal design of a life annuity. If from time* t *there is no more income, i.e.* $g^0(t) = 0$*, the optimal life annuity rate is given by the fraction* $w^0(t)/f^0(t)$ *of the reserve. Assume that there is no utility from benefits upon death or termination, i.e.* $w^{01}(t) = \Delta W^0(n) = 0$*. If e.g.* $w^0(t) = 1$*, we simply get the optimal annuity rate as the reserve divided by a life annuity based on* $(\tilde{r}, \tilde{\mu})$*,* $\int_t^n \exp\left(-\int_t^s (\tilde{r}(\tau) + \tilde{\mu}(\tau))\, d\tau\right) ds$*. For e.g. the logarithmic investor* $(\gamma = 0)$ *with* $(w^0(t))^{1-\gamma} = e^{-\iota t}$*, we simply get the optimal annuity rate as the reserve divided by a life annuity based on* (ι, μ)*,* $\int_t^n \exp\left(-\int_t^s (\iota + \mu(\tau))\, d\tau\right) ds$*. The dynamics of the optimal life annuity rate is in general given by the differential equation (11),*

$$c_t^0(t) = c^0(t)\,\frac{1}{1-\gamma}\,(r - \iota + \mu^*(t) - \mu(t)),$$

that, if the insurance is priced fair, simplifies to

$$c_t^0(t) = c^0(t)\,\frac{r - \iota}{1 - \gamma}.$$

6. The Disability/Unemployment Model

In this section we specialize the results in Section 4 into the special case of a disability/unemployment model. We study the optimal consumption and insurance decisions of an individual who has utility of consumption as long as he is alive. The utility may change, however, as he jumps into a state where he looses his income. This state may be interpreted as a disability state or unemployment state. In Figure 2, we have illustrated a set of income process coefficients and a set of utility weight coefficients.

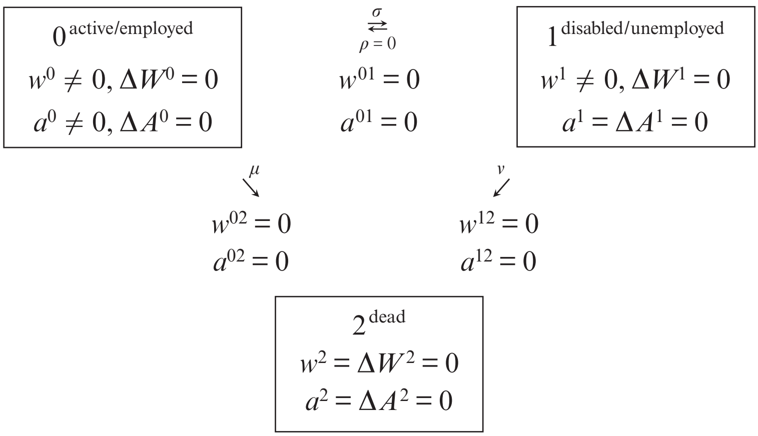

Figure 3: Disability/Unemployment model.

All statewise coefficients are zero in the state 'dead'. This means that there is no income and no utility of consumption in that state. Weight on utility of consumption in the state 'active/employed' is specified by the coefficient w^0. Lump sum consumption upon termination or transition between states gives no utility, i.e. $\Delta W^0 = \Delta W^1 = w^{01} = 0$. Weight on utility of consumption in the state 'disabled/unemployed' is specified by the coefficient w^1. The income in that state is set to zero, $a^0 = 0$. Letting $\rho = 0$ may be less realistic in an unemployment interpretation than in a disability interpretation but is nevertheless assumed here to obtain explicit solutions for f and g.

According to (7) $f^2 = 0$ and f^1 and f^0 are characterized by

$$
\begin{aligned}
f_t^1(t) &= (\tilde{r}^1(t) + \tilde{\nu}(t)) f^1(t) - w^1(t), \; f^1(n) = 0, \\
\tilde{r}^1(t) &= -\delta r - \delta(\nu^* - \nu) + \nu - \tilde{\nu}, \\
f_t^0(t) &= (\tilde{r}^0(t) + \tilde{\mu}(t) + \tilde{\sigma}(t)) f^0(t) - w^0(t) - \tilde{\sigma}(t) f^1(t), \; f^0(n) = 0, \\
\tilde{r}^0(t) &= -\delta r - \delta(\mu^*(t) + \sigma^*(t) - \mu(t) - \sigma(t)) + \mu(t) + \sigma(t) - \tilde{\mu}(t) - \tilde{\sigma}(t).
\end{aligned}
$$

This differential equation has the solution and Feynman-Kac representation, respectively,

$$\begin{aligned} f_1(t) &= \int_t^n e^{-\int_t^s \left(\tilde{r}^1(\tau)+\tilde{\nu}(\tau)\right) d\tau} w^1(s)\, ds \\ &= \tilde{E}_{t,1}\left[\int_t^n e^{-\int_t^s \tilde{r}^1(\tau)\, d\tau}\, dW(s)\right], \\ f_0(t) &= \int_t^n e^{-\int_t^s \left(\tilde{r}^0(\tau)+\tilde{\mu}(\tau)+\tilde{\sigma}(\tau)\right) d\tau} \left(w^0(s) + \tilde{\sigma}(s) f^1(s)\right) ds \\ &= \tilde{E}_{t,0}\left[\int_t^n e^{-\int_t^s \tilde{r}^{Z(\tau)}(\tau)\, d\tau}\, dW(s)\right]. \end{aligned}$$

We specify further this solution in the special case where $\sigma^* = \sigma$, $\nu^* = \mu^*$ and $\nu = \mu$. This means that disability/unemployment risk is priced by the objective measure and that the mortality risk is not changed when jumping into state 1. In that case we have that $\tilde{r}^1 = \tilde{r}^0 \equiv \tilde{r}$ and $\tilde{\sigma} = \sigma$. If we furthermore have that the utility is the same for the states 0 and 1, i.e. $w^0 = w^1 \equiv w$, we get that $f^0 = f^1$ is given by

$$\begin{aligned} f^0(t) &= \int_t^n e^{-\int_t^s \left(\tilde{r}(\tau)+\tilde{\mu}(\tau)\right) d\tau} w(s)\, ds \\ &= \tilde{E}_{t,0}\left[\int_t^n e^{-\int_t^s \tilde{r}(\tau)\, d\tau}\, dW(s)\right]. \end{aligned}$$

According to (6) $g^2 = g^1 = 0$ and g^0 is characterized in the same way as in Section 2 with μ^* replaced by $\mu^* + \sigma^*$.

Assume now, as in the previous section, that the insurance account is set to zero upon death, i.e.

$$y^{02}(t,x,y) = y^{12}(t,x,y) = -y. \tag{17}$$

We can now specify the optimal controls in terms of f and g. The consumption upon transition is zero since we have no utility of consumption upon transition. But then the optimal life insurance sum just sets also the bank account to zero upon death,

$$\begin{aligned} c^{01} &= c^{02} = c^{12} = 0, \\ b^{02} &= b^{12} = -x. \end{aligned} \tag{18}$$

The more interesting controls are the consumption rates as active and disabled and the optimal protection against disability risk,

$$\begin{aligned} c^0(t,x,y) &= \frac{w^0(t)}{f^0(t)}\big(x+y+g^0(t)\big), \\ c^1(t,x,y) &= \frac{w^1(t)}{f^1(t)}(x+y), \\ b^{01}(t,x,y)+y^{01}(t,x,y) &= \frac{f^1(t)}{f^0(t)}h^{01}(t)\big(x+y+g^0(t)\big)-(x+y). \end{aligned} \tag{19}$$

We also specify the simple exponential differential equations characterizing the statewise consumptions,

$$\begin{aligned} c_t^0(t) &= c^0(t)\,\frac{1}{1-\gamma}\Big(r-\iota+\mu^*(t)-\mu(t)+\sigma^*(t)-\sigma(t)\Big), \\ c_t^1(t) &= c^1(t)\,\frac{1}{1-\gamma}\Big(r-\iota+\mu^*(t)-\nu(t)\Big). \end{aligned}$$

The optimal protection against loss of income is given in (19). There we see that for the special case with the same utility in states 0 and 1, i.e. $f\equiv f^0=f^1$, the optimal protection reduces to $h^{01}(t)\,(x+y+g^0(t))-(x+y)$. If furthermore, the price of this protection is calculated by the P-intensity, i.e. $h^{01}=1$, then the protection reduces to $g^0(t)$. Thus, under these circumstances the individual should fully protect the financial value of future income. If utility of consumption as disabled is lower than utility of consumption as active and/or if the protection is expensive in the sense of $h^{01}>1$, then one should underinsure the potential loss.

Example 4. ***No insurance account.***

We can put the insurance account equal to zero by specifying that the natural premium for the optimal death sum is the only payment to the insurance account, i.e.

$$\begin{aligned} -b^0(t) &= \sigma^*(t)\,b^{01}(t)+\mu^*(t)\,b^{02}(t), \\ -b^1(t) &= \mu^*(t)\,b^{12}(t), \\ y^{12}(t) &= 0. \end{aligned}$$

Realize from (3) and $y^{02}(t,x,y)=y^{12}(t,x,y)=-y$ *that then* $Y(t)=0$ *for all* t *and we can put* $y=0$ *and skip the dependence on* y *in all controls, i.e.*

$$c^0(t,x) = \frac{w^0(t)}{f^0(t)}\big(x+g^0(t)\big),$$

$$c^1(t,x) = \frac{w^1(t)}{f^1(t)}x,$$

$$b^{01}(t,x) = \frac{f^1(t)}{f^0(t)}h^{01}(t)\left(x + g^0(t)\right) - x.$$

The remark at the and of Example 2 applies again: The insurance sum may become negative, and in practice negative insurance sums are not obtained by an individual's selling of life insurance but by putting the wealth saved in the institution at risk through some annuity contract. Therefore, a much more realistic special case is given now in an example with no bank account.

Example 5. ***No bank account.***

We can put the bank account equal to zero by specifying that income minus consumption goes directly into the insurance account, i.e.

$$B = C - A.$$

In this concrete case this corresponds to letting $-b^0(t) = a^0(t) - c^0(t)$*, i.e. as active the excess of income over consumption is paid as premiums on the insurance contract;* $b^1(t) = c^1(t)$*, i.e. as disabled the annuity benefit is fully consumed; and* $b^{01}(t) = c^{01}(t) - a^{01}(t) = 0$*, i.e. there is no lump sum death benefit paid out. Realize from (2) that then* $X(t) = 0$ *for all* t*. We can then put* $x = 0$ *in all controls and skip the dependence on* x*, i.e.*

$$c^0(t,y) = \frac{w^0(t)}{f^0(t)}\left(y + g^0(t)\right),$$

$$c^1(t,y) = \frac{w^1(t)}{f^1(t)}y,$$

$$y^{01}(t,y) = \frac{f^1(t)}{f^0(t)}h^{01}(t)\left(y + g^0(t)\right) - y. \tag{20}$$

The question is now, what should the policy holder actually do in order to demand the optimal reserve jump $y^{01}(t,y)$*. We consider the case where the policy holder demands the optimal reserve jump by purchasing an optimal disability annuity. In general, we have that the disability annuity rate solves the equivalence principle upon transition*

$$y + y^{01}(t,y) = \int_t^n e^{-\int_t^s (r + \nu^*(\tau))\,d\tau} b^1(s)\,ds,$$

which by the optimization relation (20) leads to

$$\frac{f^1(t)}{f^0(t)} h^{01}(t) \left(y + g^0(t)\right) = \int_t^n e^{-\int_t^s (r + \nu^*(\tau))\, d\tau} b^1(s)\, ds.$$

If the disability annuity demanded is constant this leads to the optimal annuity rate demanded at time t

$$b^1(t) = \frac{f^1(t)}{f^0(t)} h^{01}(t) \frac{y + g^0(t)}{\int_t^n e^{-\int_t^s (r + \nu^*(\tau))\, d\tau} ds}.$$

This rate becomes particularly simple in the special case where preferences in the states 0 and 1 are equal and where insurance is priced fair, i.e. $h^{01}(t) f^1(t) / f^0(t) = 1$. *However in that case one could also come up with a very simple non-constant solution. The differential equation for* $Y^0(t) + g^0(t)$,

$$\begin{aligned}\frac{d}{dt}(Y^0(t) + g^0(t)) &= (r + \mu(t))\, Y^0(t) - c^0(t) + a^0(t) - \sigma(t)\, y^{01}(t, Y^0(t)) \\ &\quad + (r + \mu(t) + \sigma(t))\, g^0(t) - a^0(t) \\ &= (r + \mu(t))(Y^0(t) + g^0(t)) - c^0(t)\end{aligned}$$

should be equal to the differential equation for the value of the future annuity benefits,

$$\frac{d}{dt}\left(\int_t^n e^{-\int_t^s (r + \mu(\tau))\, d\tau} b^1(s)\, ds\right) = \left(r + \mu(t)\right) \int_t^n e^{-\int_t^s (r + \mu(\tau))\, d\tau} b^1(s)\, ds - b^1(t).$$

But these are the same exactly if

$$b^1(t) = c^0(t).$$

Thus the policy holder obtains the optimal reserve jump by demanding a disability annuity with a time dependent payment rate corresponding to his optimal consumption rate given that he is still in state 0. It is intuitively clear that he then gets full protection if the disability rate equals his optimal consumption in state 0 since this gives him the opportunity, in case of disability, to continue consuming 'as if nothing had happened'. If instead the policy holder is underinsured, i.e. $h^{01}(t) f^1(t) / f^0(t) < 1$, *because he has lower utility from consumption as disabled than as active and/or because the protection is expensive, then he would have to demand a correspondingly lower disability annuity.*

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APPENDIX A

Proposition 6. *Let Y and $\tilde{Y}$ be defined by (3) respectively*

$$\tilde{Y}(t) = E^*\left[\int_t^n e^{-r(s-t)} dB(s) \middle| \mathcal{F}(t)\right], \tag{21}$$
$$\tilde{Y}(0) = -\Delta B^0(0).$$

If $\Delta B^j(n) = \tilde{Y}(n-)$ then

$$Y = \tilde{Y}.$$

We then have that

$$y^{Z(t-)k}(t) = E^*\left[\int_t^n e^{-r(s-t)} dB(s) \middle| \mathcal{F}(t) \cap \{Z(t) = k\}\right] - Y(t-). \tag{22}$$

Proof. *First realize that, according to (3), we have that*

$$\begin{aligned} e^{-r(n-t)}\tilde{Y}(n-) &= \tilde{Y}(t) + \int_t^{n-} d\left(e^{-r(s-t)}\tilde{Y}(s)\right) \\ &= \tilde{Y}(t) + \int_t^{n-} d\left(-re^{-r(s-t)}\tilde{Y}(s)\,ds + e^{-r(s-t)}d\tilde{Y}(s)\right). \end{aligned}$$

Plugging this relation into (21), using that $\Delta B^j(n) = \tilde{Y}(n-)$, and applying (3) now gives the result that the reserve equals $\tilde{Y}(t)$,

$$\begin{aligned} Y(t) &= E^*\left[\int_t^{n-} e^{-r(s-t)} dB(s) + e^{-r(n-t)}\tilde{Y}(n-) \,\middle|\, \mathcal{F}(t)\right] \\ &= \tilde{Y}(t) + E^*\left[\int_t^n e^{-r(s-t)} \sum_{k:k\neq Z(s-)} \left(b^{Z(s-)k}(s) + y^{Z(s-)k}(s)\right) dM^{k*}(s) \,\middle|\, \mathcal{F}(t)\right]. \end{aligned}$$

Here, M^{k*} is a martingale under P^* such that the last term vanishes.

We know that $\tilde{Y}$ upon transition of Z to k at time t equals $\tilde{Y}(t-) + y^{Z(t-)k}(t)$. We also know from the definition of Y that Y upon transition of Z to k at time t can be written as $E^*[\int_t^n e^{-r(s-t)} dB(s) \mid \mathcal{F}(t) \cap \{Z(t) = k\}]$. But if $Y = \tilde{Y}$ these observations together give (22).

■

APPENDIX B

For solution of the control problem we introduce a value function

$$V(t,x,y) = \sup E^{j}_{t,x,y}\left[\int_t^n dU(s)\right],$$

where $E^{j}_{t,x,y}$ denotes conditional expectation given that $X(t) = x,\ Y(t) = y$, and $Z(t) = j$. The HJB equation for this value function is as follows,

$$\begin{aligned}
V_t^j(t,x,y) = \inf\Big[&-\frac{1}{\gamma}w^j(t)^{1-\gamma}c^j(t)^{\gamma}\\
&-V_x^j(t,x,y)\left(rx + a^j(t) + b^j(t) - c^j(t)\right)\\
&-V_y^j(t,x,y)\left(ry - b^j(t) - \sum_{k:k\neq j}\mu^{jk*}(t)\left(b^{jk}(t) + y^{jk}(t)\right)\right)\\
&-\sum_{k:k\neq j}\mu^{jk}(t)\left(\frac{1}{\gamma}w^{jk}(t)^{1-\gamma}c^{jk}(t)^{\gamma} + V^k\left(t, x + x^{jk}(t), y + y^{jk}(t)\right) - V^j(t,x,y)\right)\Big]\\
V^j(n-,x,y) = \frac{1}{\gamma}&\Delta W^j(n)(x+y)^{\gamma}
\end{aligned}$$

with

$$x^{jk}(t) = a^{jk}(t) + b^{jk}(t) - c^{jk}(t).$$

We now guess that the HJB equation is solved by the following function with according derivatives,

$$\begin{aligned}
V^j(t,x,y) &= \frac{1}{\gamma}f^j(t)^{1-\gamma}(x+y+g^j(t))^{\gamma},\\
V_t^j(t,x,y) &= \frac{1-\gamma}{\gamma}f^j(t)^{-\gamma}f_t^j(t)(x+y+g^j(t))^{\gamma}\\
&\quad + f^j(t)^{1-\gamma}(x+y+g^j(t))^{\gamma-1}g_t^j(t),\\
V_x^j(t,x,y) &= f^j(t)^{1-\gamma}(x+y+g^j(t))^{\gamma-1},\\
V_y^j(t,x,y) &= f^j(t)^{1-\gamma}(x+y+g^j(t))^{\gamma-1}.
\end{aligned}$$

First we consider the first order conditions for the elements of the consumption process. The conditions for c and c^k become

$$\begin{aligned} c^j(t) &= \frac{w^j(t)}{f^j(t)}(x+y+g^j(t)), \\ c^{jk}(t) &= \frac{w^{jk}(t)}{f^k(t)+w^{jk}(t)}(a^{jk}(t)+b^{jk}(t)+x+y+y^{jk}(t)+g^k(t)). \end{aligned} \tag{23}$$

Here, one should note that b^{jk} and y^{jk} appear in the first place in the relation for c^{jk}.

Second we consider the first order conditions for the elements of the insurance contract. These conditions are given by the following relation

$$b^{jk}+y^{jk} = \frac{h^{jk}(t)\, f^k(t)}{f^j(t)}(x+y+g^j(t)) - (x+a^{jk}(t)-c^{jk}+y+g^k(t)). \tag{24}$$

Here, there are several points to make. First, there is no condition on b^j. Second, b^{jk} and y^{jk} are not uniquely determined since the first order condition only puts a condition on their sum. Third, c^{jk} appears on the right hand side. Since $b^{jk}+y^{jk}$ appears in c^{jk} and vice versa the optimal controls are linked together. However, we can separate them by solving the two equations (23) and (24) with respect to the two unknowns $c^{jk}(t)$ and $b^{jk}(t)+y^{jk}(t)$. The solution is

$$\begin{aligned} c^{jk}(t) &= \frac{w^{jk}(t)}{f^j(t)} h^{jk}(t)\,(x+y+g^j(t)), \\ b^{jk}+y^{jk} &= \frac{f^k(t)+w^{jk}(t)}{f^j(t)} h^{jk}(t)\,(x+y+g^j(t)) \\ &\quad -(a^{jk}(t)+x+y+g^k(t)). \end{aligned}$$

Plugging these control candidates and the value function and its derivatives into the HJB equation leads to the ordinary differential equations for f and g presented in (6) and (7). That these ordinary differential equations actually have solutions, see (9) and (10), verifies that the suggested value function is the right one and that the control candidates above are indeed the optimal ones.

In order to be sure that we have the right maximizing solution we also have to check the second order conditions. These are fulfilled if the optimally controlled process of total wealth $X(t)+Y(t)+g^{Z(t)}(t) \geq 0$. By plugging in the optimal candidate in the dynamics of X and Y we get the following dynamics,

$$\begin{aligned} \frac{d\left(X(t)+Y(t)+g^{Z(t)}(t)\right)}{X(t)+Y(t)+g^{Z(t)}(t)} &= \left(r - \frac{w^{Z(t)}(t)}{f^{Z(t)}(t)} \right) dt \\ &+ \sum_{k:k\neq Z(t-)} \left(\frac{f^k(t)+w^{Z(t-)k}(t)}{f^{Z(t-)}(t)} h^{Z(t-)k}(t) - 1 \right) dM^{Z(t-)k^*}(t), \end{aligned}$$

showing that the total wealth process is a geometric jump process. This process stays non-negative if only the jump size stays larger than or equal to –1, i.e. if only

$$\frac{f^k(t) + w^{Z(t-)k}(t)}{f^{Z(t-)}(t)}\, h^{Z(t-)k}(t) \geq 0.$$

This is a consequence of non-negative weights and non-negative intensities.