

PRICING GENERAL INSURANCE USING OPTIMAL CONTROL THEORY

BY

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ABSTRACT

Insurance premiums are calculated using optimal control theory by maximising the terminal wealth of an insurer under a demand law. If the insurer sets a low premium to generate exposure then profits are reduced, whereas a high premium leads to reduced demand. A continuous stochastic model is developed, which generalises the deterministic discrete model of Taylor (1986). An attractive simplification of this model is that existing policyholders should pay the premium rate currently set by the insurer. It is shown that this assumption leads to a bang-bang optimal premium strategy, which cannot be optimal for the insurer in realistic applications.

The model is then modified by introducing an accrued premium rate representing the accumulated premium rates received from existing and new customers. Policyholders pay the premium rate in force at the start of their contract and pay this rate for the duration of the policy. It is shown that, for two demand functions, an optimal premium strategy is well-defined and smooth for certain parameter choices. It is shown for a linear demand function that these strategies yield the optimal dynamic premium if the market average premium is lognormally distributed.

KEYWORDS

Competitive demand model, Optimal premium strategies, Maximum principle, Bellman equation.

1. INTRODUCTION

There is a considerable actuarial literature concerned with the development of a robust premium calculation principle (Hürlimann 1998). The simplest approach is the expected value principle, which sets the premium equal to the expected claim size multiplied by a loading factor. However, this principle fails to take account of the variability of the underlying risk. Consequently, many premium principles have been proposed which use higher order moments of the claims distribution or which use utility theory. Much of the research involves

the development of premium principles which satisfy certain desirable properties such as scale invariance, translation invariance and stochastic dominance (Wang 1996).

However, all these principles fail to account for the competitive nature of insurance pricing and it is this problem that we address here. The demand for an insurer's policies is in part determined by their price relative to other insurers. A lower relative price generates exposure within the insurance market at the expense of lower profits to the insurer. We suppose that the premium is set in order to optimise the wealth generated by selling insurance under a demand law. Specifically we apply both deterministic and stochastic optimal control theory (Gelfand & Fomin 2000; Sethi & Thompson 2000; Fleming & Rishel 1975) to find the optimal premium which maximises the (expected) terminal wealth of the insurer. Implicit in our formulation is that the insurance market ignores the strategy adopted by the insurer under consideration. This is reasonable as long as the insurer's exposure is small relative to the rest of the market.

Taylor (1986) was the first to consider how competition might affect an insurer's premium strategy. He observed violent changes in the premium rates offered by insurers in the Australian insurance market. Pricing insurance at an overall loss was often followed by a period of higher premium rates where considerable profits were taken. Moreover, an individual insurer appeared to follow the market rather than price its insurance based on its predicted claims distribution. These ideas lead to the formulation of a model based on a demand law as well as the distribution of claims. Taylor (1986) used a simple discrete time deterministic model. We have generalised his approach and used a stochastic model in continuous time, which we analyse using optimal control theory.

Optimal control theory has found widespread application in insurance. Such theory has been used for the determination of the optimal investment for an insurer (Hipp & Plum 2003), for optimal proportional reinsurance (Højgaard & Taksar 1997), and for the optimal choice of dividend barrier (Paulsen & Gjessing 1997). General reviews of the application of control theory to insurance can be found in Rantala (1988) and Brockett & Xia (1996).

In the following, we consider the continuous form of Taylor's model. We start in Section 2 by considering a deterministic premium strategy. By extending one strategy adopted by Emms, Haberman & Savoulli (2004) we show that the optimal premium strategy for this model is bang-bang. Consequently we extend the model in Section 3 by considering the range of premium rates accrued by the insurer that arise in a continuous model if the premium rate is fixed at the start of each policy. Again we consider a deterministic strategy and find its optimal form in Sections 3.1 & 3.2 for two parameterisations of new business generation. For both forms we examine the sensitivity of the model's predictions to the size of the parameters. In Section 4 we generalise further by considering a dynamic premium strategy, that is, a strategy which takes account of information available up to time t . Finally in Section 5 we compare the qualitative form of the deterministic and dynamic premium strategies.

2. EXTENDED DETERMINISTIC STRATEGY

Suppose at time t the insurer’s exposure is \mathbf{q} , the insurer’s premium (per unit exposure) is \mathbf{p} , the market average premium (per unit exposure) is $\bar{\mathbf{p}}$, the wealth process is \mathbf{w} and the mean claim size (per unit exposure) is $\boldsymbol{\pi}$. Consider the continuous form of the Taylor model (1986) studied by Emms, Haberman & Savoulli (2004):

$$d\mathbf{q} = \mathbf{q}g(\mathbf{p}/\bar{\mathbf{p}})dt, \tag{1}$$

$$d\mathbf{w} = -\alpha\mathbf{w}dt + \mathbf{q}(\mathbf{p} - \boldsymbol{\pi})dt, \tag{2}$$

where the demand function is $\exp(g)$ and α represents the loss of wealth due to returns paid to shareholders. Taylor (1986) determined the optimal control if $\bar{\mathbf{p}}$ is deterministic while Emms, Haberman & Savoulli (2004) specified a log-normal process for $\bar{\mathbf{p}}$. For the moment we assume that $\bar{\mathbf{p}}$ is a positive random process with finite mean at time t . We shall also leave the distribution for the mean claim size process $\boldsymbol{\pi}$ unspecified. Notice that we specify a premium rate at time t so that all premiums have units per unit time per unit exposure. With this formulation \mathbf{w} is an accurate reflection of the wealth of the company at time t since each policyholder pays a premium $p dt$ per unit exposure for each dt of cover. Consequently there are no outstanding liabilities at the end of the planning horizon T .

The principal assumption of this model is that all new and existing policyholders are *required* to pay the current premium rate \mathbf{p} . Figure 1 shows how the writing of policies affects the premium income of the insurer in discrete

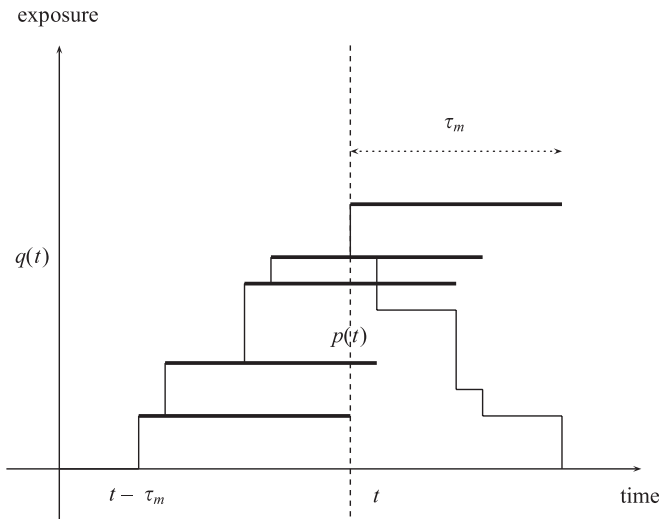


FIGURE 1: The insurer’s exposure as a function of time with policyholders paying the premium $p(t)$ currently in force for policies of mean length τ_m . Thick lines denote the duration of policies with the same start date. The accrued premium income rate at time t from all policies in force is $p(t)q(t)$.

terms. The change in wealth at time t due to premium income is denoted by the term $\mathbf{p}qdt$ in the wealth equation. Such an assumption is attractive since it means that all the random processes are Markov. It is reasonable if the premium rate does not change substantially over the course of a policy.

The demand parameterisation $g(\mathbf{p}/\bar{\mathbf{p}})$ in Taylor (1986) took two forms:

$$g = a(1 - \mathbf{p}/\bar{\mathbf{p}}), \quad (3)$$

$$g = -a \log(\mathbf{p}/\bar{\mathbf{p}}), \quad (4)$$

corresponding to an exponential or constant elasticity demand function. Here a is a constant which determines how much exposure is generated by a change in relative premium. The first of these forms was discussed solely in Taylor (1986), whilst the second has found widespread use in the financial literature (Lilien & Kotler 1983). Emms, Haberman & Savoulli (2004) considered two premium strategies and maximised the expected total utility of wealth. A similar analysis can be performed by specifying the simpler objective function

$$V = \max_{\mathbf{p}} \{ \mathbb{E}[\mathbf{w}(T) | S(0)] \},$$

that is maximising the expected wealth at the end of the planning horizon T given information on the state S at time $t = 0$. The adoption of a fixed form for the strategy effectively places a constraint on the variation of the premium.

We can generalise one of the premium strategies adopted in Emms, Haberman & Savoulli (2004) by considering the premium strategy

$$\mathbf{p} = k(t) \bar{\mathbf{p}}. \quad (5)$$

Emms, Haberman & Savoulli fixed k as a constant while here we maximise the objective over the functional $k(t)$. The deterministic function $k(t)$ need not be smooth and so it is useful for the analysis of models such as (1)-(2). Given the optimal relative premium $k(t)$, the corresponding optimal premium is stochastic and for the premium to be non-negative we require $k \geq 0$.

The demand functions and the form of strategy ensure that the exposure $q(t)$ is deterministic. This considerably simplifies the model and is one reason for adopting a strategy such as (5). If we adopt the terminology of optimal control theory then the control variable is k and the state variable is the exposure q which is governed by

$$\dot{q} = qg(k).$$

For both the constant elasticity and exponential demand functions g is a decreasing function of k . Note the forthcoming arguments still apply if we split up the exposure equation into new business generation and negative drift (representing policy termination) as long as the parameterisation of new business is of a similar form to the demand functions (3) and (4).

Taking the expectation of the wealth equation given information up until $t = 0$ we obtain

$$\mathbb{E}[\mathbf{w}(T)] = e^{-\alpha T} \left[w(0) + \int_0^T F dt \right],$$

where

$$F = F(q, k, t) = e^{\alpha t} q(km_{\bar{p}} - m_{\pi}). \tag{6}$$

Here we adopt the notation

$$m_X(t; s) = \mathbb{E}[\mathbf{X}(t) | S(0) = s], \tag{7}$$

where S is the state of the system. Consequently the problem can be written in one of the standard forms for control theory (Sethi & Thompson 2000). We wish to determine the value function

$$V = \max_{k \geq 0} \left\{ J = \int_0^T F(q, k, t) dt \right\}, \tag{8}$$

where the optimal control k_* is denoted with an asterisk. The value of this objective function determines the maximum value of the expected terminal wealth.

The necessary conditions for an optimal control are determined by the Maximum Principle which can be stated in terms of the Hamiltonian defined by

$$H(q, k, \lambda, t) = F(q, k, t) + \lambda qg(k), \tag{9}$$

where λ is a Lagrange multiplier. The Maximum Principle states that

$$H(q_*, k_*, \lambda, t) \geq H(q, k, \lambda, t),$$

for all $k \geq 0$. For the exponential demand function $g = a(1 - k)$ and therefore the Hamiltonian is linear in the control. The optimal control is bang-bang: for $\lambda \leq 0$, $k_* = \infty$ while for $\lambda > 0$, $k_* = 0$ or ∞ depending on the parameters of the model. If $\lambda > 0$, $k_* = 0$ then this is the ultimate loss-leader: an insurer gives away insurance in order to capture the whole market and then charges those customers an infinite premium at $t = T$ in order to generate infinite wealth. If $k_* = \infty$ for $t \in [0, T]$ then it is optimal not to sell insurance.

For the constant elasticity demand function $g = -a \log k$ and the Hamiltonian is

$$H(q, k, \lambda, t) = e^{\alpha t} q(km_{\bar{p}} - m_{\pi}) - \lambda a q \log k. \tag{10}$$

If we suppose that the maximum of H over $k \in (0, \infty)$ is given by $H_k = 0$ then using (10) yields

$$e^{\alpha t} m_{\bar{p}} - \lambda a / k = 0. \tag{11}$$

Since k must be positive this requires λ to be positive also. This equation when coupled to the adjoint equation

$$\dot{\lambda} = -H_q = -e^{\alpha t}(km_{\bar{p}} - m_{\pi}) + \lambda a \log k \quad (12)$$

determines an optimal control and ultimately leads to the Euler-Lagrange equation of the Calculus of Variations (Gelfand & Fomin 2000). However, for a maximum of H the second-order condition is $H_{kk} \leq 0$ at $k = k_*$. From (10) we find

$$H_{kk} = \frac{\lambda a q}{k^2} > 0.$$

for $k > 0$, so that the turning point is a *minimum*. Looking at the form of the Hamiltonian it is clear that the optimal control is degenerate and not unique: if $\lambda \leq 0$ then $k_* = \infty$, while if $\lambda > 0$ then $k_* = 0$ or ∞ .

We have shown that for both demand functions (3) and (4) the optimal control is degenerate. Although for some demand functions it is simple to write down an equation that an extremal must satisfy, it is important to ensure that we actually have a maximal extremal. This applies to both deterministic and stochastic dynamic strategies, the latter of which are determined by a Bellman equation. Since the deterministic strategy generates infinite terminal wealth the solution to the Bellman equation for the (unconstrained) problem is also bang-bang.

A bang-bang strategy is optimal because of the principal assumption of the continuous model (1)-(2), that is, an insurer requires all existing customers to pay the current premium rate. However, there is a restriction on just how big an increase existing policyholders will be prepared to pay for insurance before terminating their cover. The optimal strategy is dependent on the value of this increase. One solution to the problem is to place a constraint on the premium strategy so that premium rates cannot change substantially over the policy (see Emms, Haberman & Savoulli 2004). An alternative approach is to change the assumptions of the model and this is the approach we adopt next.

3. ACCUMULATED PREMIUM INCOME

We describe a simplified continuous version of the model proposed by Gerrard & Glass (2004). We split up the change in exposure into that lost due to policy termination and that gained due to new business (or renewals). To do this we need a parameterisation for the rate of generation of new business \mathbf{n} . Motivated by the previous demand functions (3) and (4) we adopt a relationship of the form

$$\mathbf{n} = \mathbf{q}G(\mathbf{p}/\bar{\mathbf{p}}), \quad (13)$$

where G is a *non-negative* demand function. This parameterisation reflects the idea that the reputation of a company is proportional to its exposure in the market and that it is in part the reputation of an insurer which increases its likelihood to generate new business. New business generation is also determined by the premium that the insurer sets relative to the market, which is

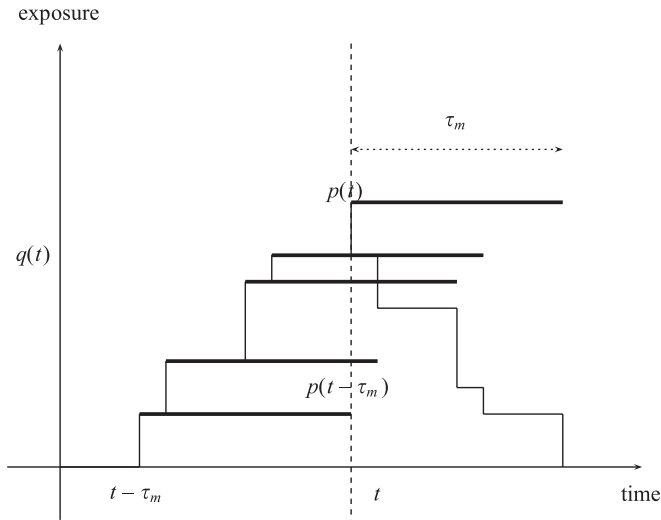


FIGURE 2: The insurer’s exposure as a function of time with the premium set at the start of the policy and held constant over its duration. The accrued premium income is $Q = \int_{t-\tau_m}^t p(s) n(s) ds$.

represented by the demand function G . We shall consider two forms of this demand function in the work to follow.

Let us denote the mean length of an insurance policy (including renewals) by τ_m . The change in exposure at time t is dependent on the generation of new business from $t - \tau_m$ up to t (see Figure 2). Therefore the process \mathbf{q} is non-Markov, which considerably complicates the modelling. In order to keep the model Markov and use conventional control theory we parameterise the loss of exposure due to policy termination: we suppose that this loss is proportional to $\kappa \mathbf{q}$ where $\kappa = \tau_m^{-1}$.

We cannot require that all existing customers pay the current premium rate, so instead we suppose that clients pay the premium rate in effect at the start of their policies. This means that at time t there is a range of premium rates being paid to the insurer and so we introduce the accumulated premium income rate \mathbf{Q} . Figure 2 shows that the change in the accumulated income depends on new business generation and premium rates between $t - \tau_m$ and t , so that \mathbf{Q} is also non-Markov. For a Markov model we suppose that the loss in accumulated premium income due to policy termination is proportional to the current accumulated premium income i.e. $\kappa \mathbf{Q}$.

Finally, Emms, Haberman & Savoulli (2004) suppose that the mean claim size $\boldsymbol{\pi}$ is constant and that the market average price is lognormally distributed. This is unrealistic in that it assumes that the market average price and the mean claim size are independent. If the market uses the expected value principle then $\bar{\mathbf{p}}$ should be proportional to $\boldsymbol{\pi}$. Thus we shall assume

$$\boldsymbol{\pi} = \gamma \bar{\mathbf{p}}, \tag{14}$$

where the constant γ is a measure of the market loading factor. It is expected that $\gamma \lesssim 1$ so that the market, on average, makes money selling insurance. We may consider the case that γ is a deterministic or stochastic function of time in subsequent work.

With these assumptions the modified model is

$$d\mathbf{q} = (\mathbf{n} - \kappa\mathbf{q})dt, \tag{15}$$

$$d\mathbf{Q} = (\mathbf{p}\mathbf{n} - \kappa\mathbf{Q})dt, \tag{16}$$

$$d\mathbf{w} = -\alpha\mathbf{w}dt + \mathbf{Q}dt - \gamma\bar{\mathbf{p}}\mathbf{q}dt. \tag{17}$$

Again, we adopt the deterministic strategy (5) so that the exposure q and the rate of generation of new business n are deterministic. We assume that there is an explicit expression for $m_{\bar{p}}$ independent of the other state variables so that there are now two unknown state variables: q and m_Q . For example, if $\bar{\mathbf{p}}$ is log-normally distributed with drift μ then $m_{\bar{p}} = \bar{p}(0)e^{\mu t}$ using the notation defined in (7).

The first state equation is (15) whilst the second comes from taking the expectation of (16), which yields

$$\frac{dm_Q}{dt} = qkG(k)m_{\bar{p}} - \kappa m_Q,$$

using (5) and (13). On integrating this equation we obtain

$$\begin{aligned} m_Q(t) &= m_Q(0) + \int_0^t e^{\kappa(s-t)} m_p(s) n(s) ds \\ &= m_Q(0) + \int_0^{B(t)} e^{\kappa(s(b)-t)} m_p(s(b)) db, \end{aligned} \tag{18}$$

where $B(t) = \int_0^t n(s)ds$ is the total amount of business generated over time t . Here we have supposed that B is a strictly increasing function of t so that its inverse $t = t(B)$ is well-defined.

Figure 3 shows the profile of premium rates that this model might deliver at time t . The correct premium profile depends on the history of the premium rates between $t - \tau_m$ and t . By making the model Markov, we have from (18) that this profile is a weighted function of the entire premium history from $[0, t]$ albeit in such a way that only those premium values in the range $[t - \tau_m, t]$ are important. However, we note that this does impose artificially a premium structure which differs from that actually received except in the case that the premium rate is constant. If m_p is constant taking the expectation of (15) and (16) and integrating yields $m_Q = qm_p$ using $m_Q(0) = q(0)m_p$, which is the accumulated premium income an insurer obtains from exposure q .

Taking the expectation of the wealth equation (17) and integrating gives

$$m_w(t) = e^{-\alpha t} \left[m_w(0) + \int_0^t e^{\alpha s} (m_Q(s) - \gamma q m_{\bar{p}}(s)) ds \right].$$

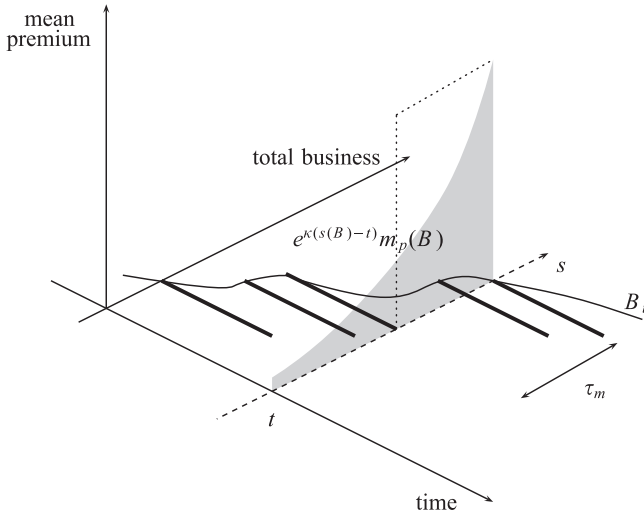


FIGURE 3: The mean premium rate profile received at time t where the premium rate is set at the start of the policy and held constant over its duration. Individual policies are represented by the solid lines. If the premium rate were constant then the accrued premium $Q(t)$ is given by the shaded region. The actual accrued premium income in a continuous model is that enclosed by the dotted lines.

We define the value function in Lagrange form as

$$V = \max_k \left\{ J = \int_0^T F dt \right\},$$

where

$$F(x, t) = e^{\alpha t} (m_Q - \gamma q m_{\bar{p}}), \tag{19}$$

and we write the state vector as

$$x = (q, m_Q)^T.$$

Alternatively we could define a value function in linear Mayer form at the expense of an extra state equation for m_w . For this system we write the state equations as

$$\dot{x} = f(x, k, t) = \begin{pmatrix} q(G(k) - \kappa) \\ qkG(k)m_{\bar{p}} - \kappa m_Q \end{pmatrix}. \tag{20}$$

The Hamiltonian is defined by

$$H(x, k, \lambda, t) = F + \lambda f = e^{\alpha t} (m_Q - \gamma q m_{\bar{p}}) + \lambda_1 q(G(k) - \kappa) + \lambda_2 (qkG(k)m_{\bar{p}} - \kappa m_Q),$$

where the Lagrange multiplier vector is $\lambda = (\lambda_1, \lambda_2)$.

We suppose the optimal control k_* is given by the first order condition for a maximum of H , the state equations and the adjoint equations:

$$H_k = 0, \quad (21)$$

$$\dot{x} = H_x, \quad (22)$$

$$\dot{\lambda} = -H_\lambda. \quad (23)$$

We must also ensure the second order condition holds: $H_{kk} \leq 0$ at $k = k_*$. The last two equations are the canonical Euler equations (Gelfand & Fomin 2000) and reduce to the Euler-Lagrange equation if the control is sufficiently smooth. The boundary conditions for this system are

$$x(0) = (q(0), m_Q(0))^T, \quad \lambda(T) = 0,$$

the last of which is the transversality condition. In general, for two state variables, this is a fourth-order boundary value problem. However, H is linear in the state variables and $H_k = 0$ only depends on m_p so that the system decouples independently of the particular parameterisation for G . In order to determine the optimal control k_* we need only solve the initial value problem consisting of the adjoint equations (23) and the transversality condition.

The second of the adjoint equations is independent of the choice of demand function G . From (19) and (20) we have

$$\dot{\lambda}_2 = -H_{m_Q} = \kappa \lambda_2 - e^{\alpha t}, \quad (24)$$

with boundary condition $\lambda_2(T) = 0$. This can be integrated immediately to obtain

$$\lambda_2(t) = \frac{e^{\alpha t}}{\kappa - \alpha} (1 - e^{(\kappa - \alpha)(t - T)}). \quad (25)$$

Consequently for $0 \leq t \leq T$ we have $\lambda_2 \geq 0$.

3.1. Power law demand function

The demand function G must be a non-negative decreasing function of the relative premium price. Therefore a suitable parameterisation, which is defined for all positive premiums, takes the form of a power law:

$$G = b_1 k^{-a_1}, \quad (26)$$

where $a_1, b_1 > 0$: a_1 is dimensionless while b_1 has units per unit time. Although G is defined for all $k > 0$, (26) is an unrealistic parameterisation as k becomes large. If the optimal strategy depends upon new business generation for large relative premium rates then this is not a good model for the demand function.

The Hamiltonian for this demand function is

$$H = e^{\alpha t}(m_Q - \gamma q m_{\bar{p}}) + \lambda_1 q(b_1 k^{-a_1} - \kappa) + \lambda_2(q b_1 k^{-a_1+1} m_{\bar{p}} - \kappa m_Q),$$

which has derivatives

$$\begin{aligned} H_k &= q b_1 k^{-a_1-1}((1-a_1)\lambda_2 m_{\bar{p}} k - a_1 \lambda_1), \\ H_{kk} &= a_1 b_1 q k^{-a_1-2}((a_1+1)\lambda_1 - (1-a_1)\lambda_2 k m_{\bar{p}}). \end{aligned}$$

Suppose the extremum of H is at an interior point k_{i^*} , that is $k_{i^*} \in (0, \infty)$. Therefore the optimal strategy is given by $H_k = 0$:

$$k_{i^*} = \frac{a_1}{1-a_1} \left(\frac{\lambda_1}{m_{\bar{p}} \lambda_2} \right), \tag{27}$$

which yields a maximum providing that

$$H_{kk} = a_1 b_1 q k_{i^*}^{-a_1-2} \lambda_1 < 0.$$

Consequently there is an interior maximum of H if $\lambda_1 < 0$ and $k_{i^*} \neq 0$. Further from (25) and (27) we must have $a_1 > 1$. If there is no interior maximum the optimal strategy is at either end-point of k . The optimal behaviour can be summarised by the following table for $a_1 > 1$:

	$\lambda_2 = 0$	$\lambda_2 > 0$
$\lambda_1 < 0$	$k_* = \infty$	interior
$\lambda_1 = 0$	undefined	$k_* = 0$
$\lambda_1 > 0$	$k_* = 0$	$k_* = 0$

The remaining Lagrange multiplier is determined by

$$\dot{\lambda}_1 = \gamma m_{\bar{p}} e^{\alpha t} + \lambda_1 \kappa + \lambda_1 b_1 \left(\frac{1}{a_1 - 1} \right) \left(\frac{(1-a_1)\lambda_2 m_{\bar{p}}}{a_1 \lambda_1} \right)^{a_1}, \tag{28}$$

using (25) with boundary condition $\lambda_1(T) = 0$. This is similar to a Bernoulli equation but it is non-homogeneous and so in general it does not have an analytical solution.

We nondimensionalise the market average premium using its initial value. The remaining scales are taken as

$$\begin{aligned} b_1 \lambda_2 &= \omega, & b_1 \lambda_1 &= \bar{p}(0)\lambda, & t &= T(1-s), & \hat{\alpha} &= \alpha T, \\ \kappa &= K b_1, & \varepsilon &= (b_1 T)^{-1}, & m_{\bar{p}} &= \bar{p}(0)M(s). \end{aligned}$$

Substituting these scales into (28) we obtain the non-dimensional adjoint equation:

$$\varepsilon \frac{d\lambda}{ds} = -\gamma M(s) e^{\hat{\alpha}(1-s)} - K\lambda - \left(\frac{1}{a_1 - 1}\right) \left(\frac{(1 - a_1) M(s) \omega}{a_1 \lambda}\right)^{a_1}, \tag{29}$$

where from (25)

$$\omega(s) = \frac{e^{\hat{\alpha}(1-s)}}{K - \hat{\alpha}\varepsilon} \left(1 - e^{(\hat{\alpha} - \frac{K}{\varepsilon})s}\right),$$

and the optimal strategy is

$$k_{i^*} = \left(\frac{1}{1 - a_1}\right) \frac{\lambda}{M\omega}. \tag{30}$$

At $s = 0$, $\lambda = \omega = 0$ so that k_{i^*} is undefined. We can only find the limiting behaviour numerically since substituting Taylor series expansions for λ and ω about $s = 0$ leads to the algebraic equation:

$$\varepsilon \lambda'(0) = -\gamma M(0) e^{\hat{\alpha}} - \left(\frac{1}{a_1 - 1}\right) \left(\frac{(1 - a_1) M(0) e^{\hat{\alpha}}}{\varepsilon a_1 \lambda'(0)}\right)^{a_1}.$$

Given the numerical root of this equation we can integrate (29) numerically with initial value $\lambda(\delta s) = \lambda'(0)\delta s$.

If we suppose $\varepsilon \ll 1$ then to leading-order we have $\omega \sim e^{\hat{\alpha}(1-s)}K$ and an algebraic equation for λ :

$$\gamma M(s) e^{\hat{\alpha}(1-s)} - K(-\lambda) + \left(\frac{1}{a_1 - 1}\right) \left(\frac{(a_1 - 1) M(s) e^{\hat{\alpha}(1-s)}}{K(-\lambda)}\right)^{a_1} = 0. \tag{31}$$

In general the solution to this equation can only be determined numerically.

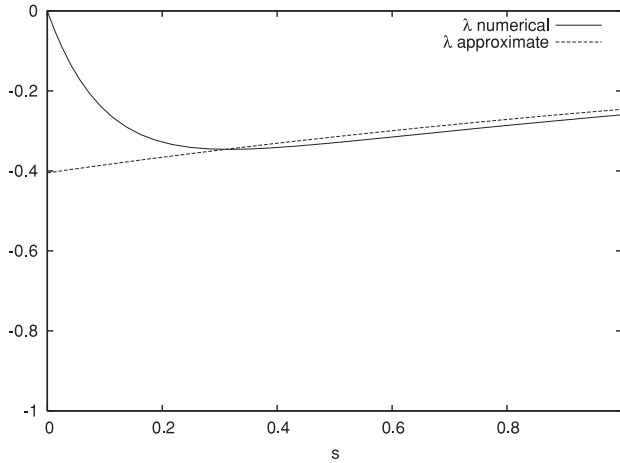
For simplicity, we suppose $\bar{\mathbf{p}}$ is constant so that $M \equiv 1$. In order to generate a parameter set we shall suppose that if the insurer sets its premium at 80% of the market value then this leads to a 40% increase in the insurer's exposure after one year. Thus we choose $a_1 = 2$ and obtain $b_1 = 0.256$ p.a. from (26). The mean policy length is set at one year and the planning horizon is 10 years. Depreciation of wealth is taken as 6% and the premium ratio $\gamma = 0.67$. Figure 4 shows the numerical solution to (29) and (31) for comparison using the sample data in Table 1. The strategy proceeds from $s = 1$ corresponding to $t = 0$ to $s = 0$ at time $t = T$. Notice that there is a region of thickness ε where the algebraic equation does not give a good approximation to the adjoint differential equation. From (15) the state equation for the exposure is

$$dq = qb_1(k^{-a_1} - K)dt. \tag{32}$$

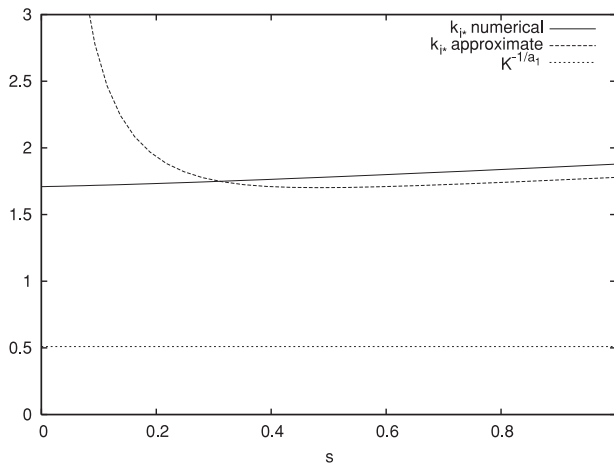
It can be seen in Figure 4(ii) that the optimal strategy k_{i^*} is always above K^{-1/a_1} so that exposure is decreasing exponentially with increasing time t . Thus for this parameter set the optimal strategy represents a withdrawal from the market.

Time horizon T	10 yr
Demand growth a_1	2
Demand parameterisation b_1	0.256 p.a.
Demand growth a_2	2 p.a.
Demand Parameterisation b_2	1
Depreciation of wealth α	0.06 p.a.
Ratio of break-even to initial market average premium γ	0.67
Mean length of policy $\tau = \kappa^{-1}$	1 p.a.

TABLE 1: Typical parameter values.

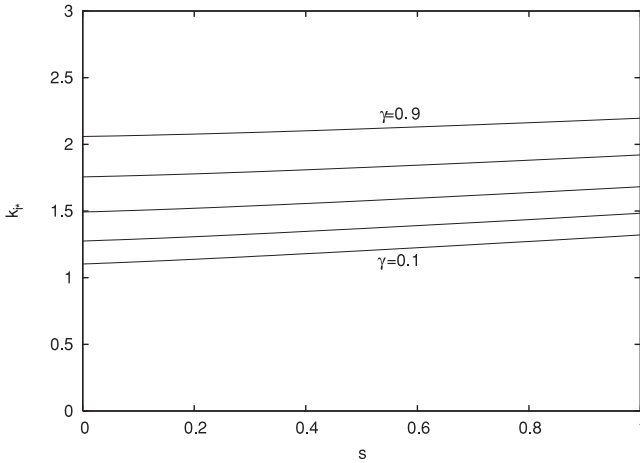


(i)

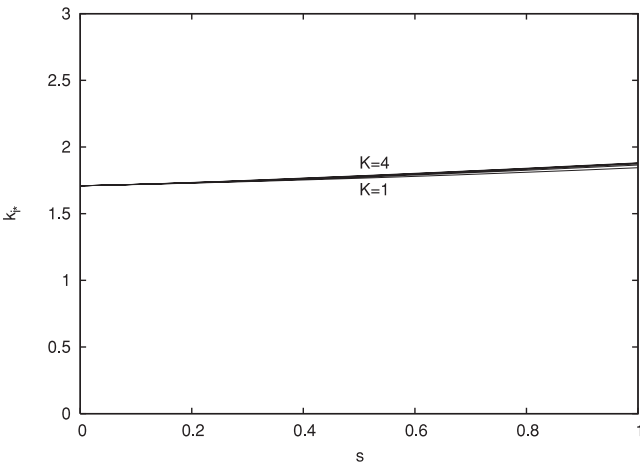


(ii)

FIGURE 4: The optimal strategy for a power law demand function G with parameters as in Table 1. The first graph (i) shows the numerical solution for the adjoint variable λ_1 and its approximate value, whilst graph (ii) shows the optimal interior relative premium $k_{i,s}$.



(i)



(ii)

The optimal control is a function of the parameters γ , K , a_1 , ε , and $\hat{\alpha}$. Numerical experiments and previous sensitivity analyses (Emms, Savoulli & Haberman 2004) suggest that we should expect a reasonable optimal strategy for $\varepsilon \ll 1$ and $\hat{\alpha} \sim 1$. Figure 5 shows how the optimal control varies if we vary γ , K and a_1 in turn. As we increase γ the optimal relative premium increases and the insurer withdraws from the market. The optimal strategy is insensitive to K because from (31), λ/ω is independent of $K = \kappa/b_1$ at leading order. If K is sufficiently small (corresponding to long policies) then (32) shows that exposure can grow exponentially but that the optimal relative premium is still larger than the market average premium. Of course when the mean policy length is longer than the planning horizon T then very few policyholders leave the insurer no matter what premium is set: they continue to pay the premium rate set at

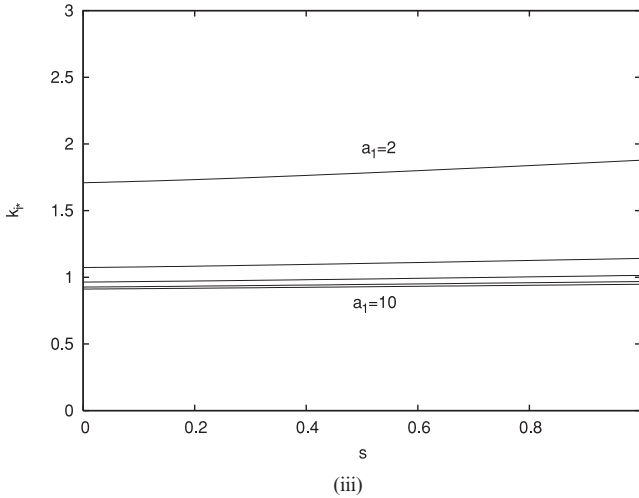


FIGURE 5: Sensitivity of the optimal strategy to model parameters for a power law demand function G . One parameter is varied while the remaining are taken from Table 1. The optimal control is shown for for $\gamma = 0.1-0.9$ in steps of 0.2 in (i), $K = 1-4$ in steps of 1 in (ii), and for $a_1 = 2.0-10.0$ in steps of 2.0 in (iii).

the start of their policies. This highlights a limitation of the model: the mean policy length is independent of the premium rate. It is simple to add a parameterisation for $\tau_m(k)$ to the model at the expense of increasing the parameter space.

From Figure 5(iii) we can see that as the demand parameter a_1 is increased the optimal premium decreases below the market average premium. However, for the parameter range examined the optimal relative premium is above $K^{-1/a_1} \approx 0.5$ so that this premium strategy leads to market withdrawal. It is clear that the optimal strategy is strongly dependent on the demand parameterisation. Notice also that for the chosen parameter set the optimal relative premium $k_{i^*} > \gamma$, which implies that $\mathbf{p} > \boldsymbol{\pi}$, so that none of these strategies are loss-leading.

3.2. Linear demand function

We simplify the parameterisation for new business by taking

$$G = \begin{cases} a_2(b_2 - k) & \text{if } k \leq b_2, \\ 0 & \text{if } k > b_2, \end{cases} \tag{33}$$

where $a_2 > 0$ has dimension per unit time and b_2 is dimensionless. Linear demand functions are often used in the economics literature (Lilien & Kotler 1983). Note that this is a different parameterisation to g in (3) since G is a non-negative function characterising the demand for new business rather than the fractional change in exposure.

The Hamiltonian in this case is

$$H = e^{\alpha t}(m_Q - \gamma q m_{\bar{p}}) + \lambda_1 q(a_2(b_2 - k) - \kappa) + \lambda_2(qka_2(b_2 - k)m_{\bar{p}} - \kappa m_Q),$$

which has derivatives

$$\begin{aligned} H_k &= -a_2 q \lambda_1 + \lambda_2 q m_{\bar{p}} a_2 (b_2 - 2k), \\ H_{kk} &= -2\lambda_2 q m_{\bar{p}} a_2, \end{aligned}$$

if $k \leq b_2$. Therefore we have a maximum for H at

$$k_{i^*} = \frac{1}{2} \left(b_2 - \frac{\lambda_1}{m_{\bar{p}} \lambda_2} \right), \tag{34}$$

which gives the optimal interior strategy providing that $0 \leq k_{i^*} \leq b_2$ and $\lambda_2 > 0$.

The remaining adjoint equation determines the optimal interior control:

$$\dot{\lambda}_1 = \gamma m_{\bar{p}} e^{\alpha t} - \lambda_1(a_2(b_2 - k_{i^*}) - \kappa) - \lambda_2 a_2 k_{i^*} (b_2 - k_{i^*}) m_{\bar{p}}, \tag{35}$$

with boundary condition $\lambda_1(T) = 0$. The second Lagrange multiplier is just a function of t so we can substitute (34) into (35) to obtain

$$\dot{\lambda}_1 = \gamma m_{\bar{p}} e^{\alpha t} - \frac{a_2 b_2^2 \lambda_2 m_{\bar{p}}}{4} - \lambda_1 \left(\frac{a_2 b_2}{2} - \kappa \right) - \frac{a_2 \lambda_1^2}{4 m_{\bar{p}} \lambda_2}, \tag{36}$$

which is a Riccati equation.

Next we rescale using the following change of variables:

$$t = T(1 - s), \quad a_2 \lambda_1 = \bar{p}(0) \lambda, \quad a_2 \lambda_2 = \omega, \quad m_{\bar{p}} = \bar{p}(0) M(s) \tag{37}$$

and introduce the following nondimensional parameters:

$$\kappa = K a_2, \quad \varepsilon = \frac{1}{a_2 T}, \quad \hat{\alpha} = \alpha T.$$

Note that the definition of K and ε has changed from the previous section because of the change in demand function. From (15) the exposure is governed by

$$dq = \begin{cases} qa_2(b_2 - k - K) & \text{if } k \leq b_2, \\ -qa_2 K & \text{if } k > b_2. \end{cases} \tag{38}$$

If $K \geq b_2$ then policies expire at a rate greater than the rate at which new business is generated irrespective of the level of the relative premium. This is clearly unrealistic so we must have $K < b_2$.

The nondimensional Riccati equation is

$$\varepsilon \frac{d\lambda}{ds} = \frac{b_2^2 M(s) \omega(s)}{4} + \lambda \left(\frac{b_2}{2} - K \right) + \frac{\lambda^2}{4M(s)\omega(s)} - \gamma M(s) e^{\hat{a}(1-s)}, \quad (39)$$

with

$$\omega(s) = \frac{e^{\hat{a}(1-s)}}{K - \hat{a}\varepsilon} \left(1 - e^{(\hat{a} - \frac{K}{\varepsilon})s} \right), \quad (40)$$

and the boundary condition is $\lambda(0) = 0$. In terms of these new variables the optimal interior strategy is

$$k_{i*} = \frac{1}{2} \left(b_2 - \frac{\lambda}{M\omega} \right), \quad (41)$$

providing that $|\lambda| < M\omega$. Since both $\lambda = \omega = 0$ at $s = 0$, this expression is undefined at end of the time horizon. However, if we suppose that λ is sufficiently smooth near $s = 0$ then a Taylor series expansion substituted into (39) gives

$$k_{i*} = \frac{1}{2} (b_2 + \gamma), \quad \text{as } s \rightarrow 0+. \quad (42)$$

This expression gives the terminal optimal relative premium rate.

Let us suppose $\varepsilon \ll 1$. To leading-order $\omega \sim e^{\hat{a}(1-s)}/K$ and

$$\lambda \sim M \left(\frac{2K - b_2 - 2(K^2 - b_2K + \gamma K)^{1/2}}{K e^{\hat{a}(s-1)}} \right), \quad (43)$$

providing that

$$\gamma \geq \gamma_c = b_2 - K. \quad (44)$$

Again, we suppose that if the insurer sets its premium at 80% of the market premium then that leads to a gain in exposure of 40% over one year. Thus we choose $a_2 = 2$ p.a. and obtain $b_2 = 1$ from (33). The other parameters are taken from Table 1 and yield $\gamma_c = 1/2$. If $\gamma < 1/2$ then the optimal control does not equilibriate. In this case there is the possibility that the optimal control becomes negative or even that there is a spontaneous singularity for $s \in (0, 1]$. Using (43) the approximate optimal interior strategy is

$$k_{i*} = b_2 - K - (K^2 - b_2K + K\gamma)^{1/2}. \quad (45)$$

The leading-order value of k_{i*} is independent of both M and s . It should be remembered that these approximate expressions only remain valid if the solution to the Riccati equation does equilibriate. In the following numerical calculations we find that as the solution tends towards a bang-bang control then these approximations become invalid.

Figure 6 shows the approximate premium strategy along with the numerical solution of the Riccati equation (39) using the parameters in Table 1. For the moment we set $b_2 = 1$ so that there is no demand for insurance if the insurer's premium is above market average. We shall relax this assumption in the forthcoming sensitivity analysis. In addition, we set $M \equiv 1$ so that there is no drift in the market average premium. Figure 6 (i) reveals that there is an inner region where the approximate expression for λ does not satisfy the boundary condition however, this region appears unimportant, when we plot the approximate optimal premium ratio k_{i^*} . Note that the strategy proceeds over time from $s = 1$ to $s = 0$ and, the exposure is always decreasing as s decreases (since $k > b_2 - K$) so that this strategy effectively represents a gradual withdrawal from the market. In contrast to the power law demand function, the optimal control k_{i^*} for a linear demand function increases over time.

The optimal control is a function of six parameters: γ , K , b_2 , M , $\hat{\alpha}$ and ε . The last of these parameters, ε , is a measure of how fast the adjoint λ_1 reaches its equilibrium (should it do so). Figure 7(i)-(iv) show how the optimal control varies as we vary each of these parameters in isolation.

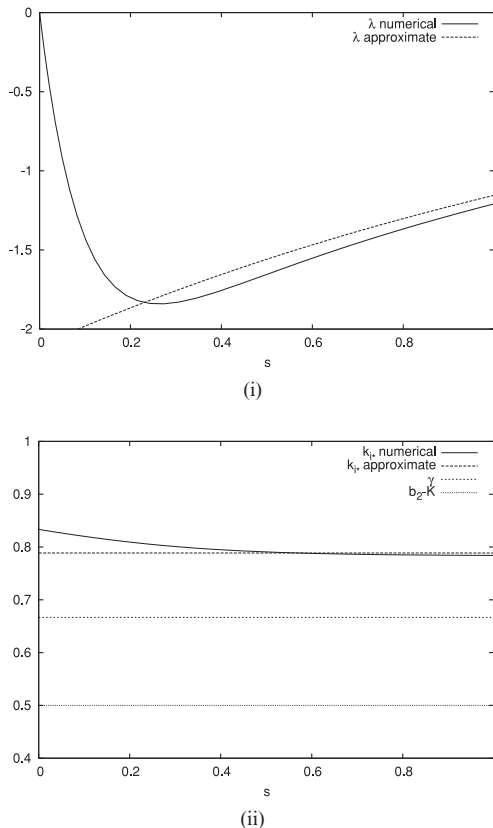
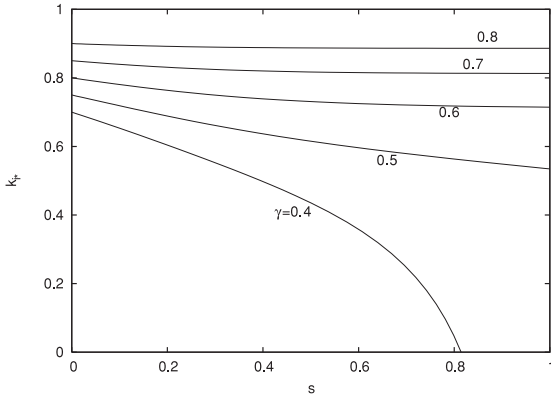
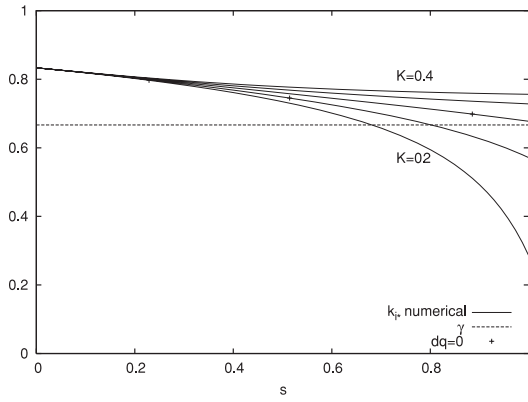


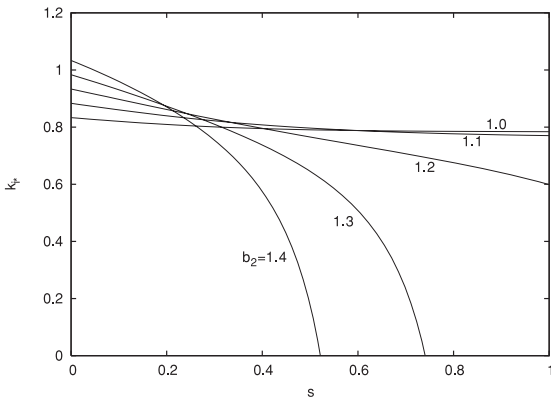
FIGURE 6: The optimal premium strategy for a linear demand function G with parameters taken from Table 1. Graph (i) shows the adjoint variable λ while (ii) shows the optimal relative premium k_{i^*} .



(i)



(ii)



(iii)

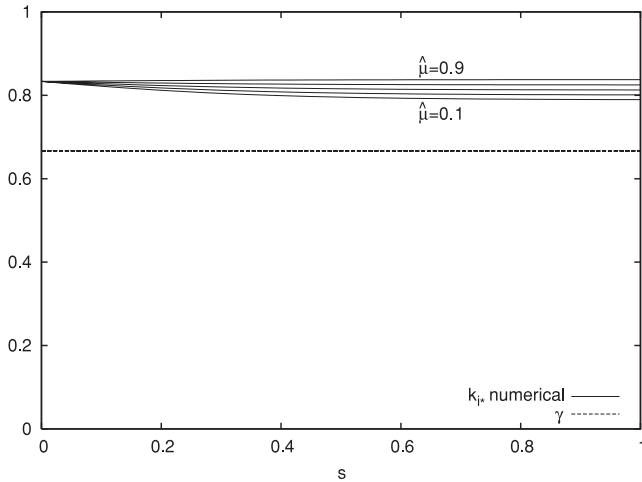


FIGURE 7: Sensitivity analysis of the optimal premium strategy for a linear demand function G . For each figure one parameter is varied with the others held constant and taken from Table 1. The optimal control is shown for $\gamma = 0.40-0.80$ in steps of 0.1 in (i), for $K = 0.2-0.4$ in steps of 0.05 in (ii), for $b_2 = 1.0-1.4$ in steps of 0.1 in (iii), and for $\hat{\mu} = 0.1-0.9$ in steps of 0.2 in (iv).

Figure 7(i) and 7(ii) show how the optimal strategy varies with γ and K respectively. As γ decreases the initial loss increases until, for $\gamma \lesssim \gamma_c$ given by (44), the optimal strategy is determined by the constraint $k \geq 0$. Similar behaviour occurs as K is decreased, corresponding to increasing the mean length of the policies. Furthermore, for $K \lesssim 0.35$ the exposure increases over the initial course of the strategy, so that this condition might be taken as that required to enter the market. Initially the insurer makes a loss to build up exposure which increases the company’s reputation. Towards the end of the time horizon the premium is raised in order to make a profit. Notice also that for sufficiently small K the optimal strategy is loss-leading because $k_{i^*} < \gamma$.

Figure 7(iii) shows how the optimal control varies as we increase b_2 , which corresponds to increasing the cutoff relative premium: if $k > b_2$ there is no demand for insurance. We see that as b_2 increases so does the bound on the critical loss ratio γ_c given by (44). Therefore, if there is demand for a high relative premium, the optimal control is likely to be limited by the constraint that the premium must be non-negative. Moreover, if we set $\gamma > \gamma_c$ then the terminal optimal relative premium increases with b_2 . Clearly, more wealth is generated by selling insurance at these high relative premium rates even though they generate little exposure. Of course, it is implausible that new business would be generated at such high premium rates: the rate of generation of new business would be zero if the premium is set too much above the market average. We must ensure that the demand parameterisation reflects this feature if we wish to obtain a realistic optimal strategy.

Finally, Figure 7(iv) shows the variation of the optimal strategy if we suppose \mathbf{p} is lognormally distributed, which gives $M = e^{\hat{\mu}(1-s)}$, where $\hat{\mu} = \mu T$ is the nondimensional drift. As predicted from the leading-order outer solution, the

variation of this parameter has little effect on the optimal strategy. The mean of the market average premium $M(0)$ cancels in the inner region so that there is little variation in the optimal control over the entire time horizon. The optimal control shows little variation with $\hat{\alpha}$ so the results are not shown.

4. PURE STOCHASTIC STRATEGY

We have found smooth deterministic optimal strategies for an accrued premium model for certain parameter choices. Using the linear demand function we look next for the dynamic optimal premium and derive the correspondence between this premium and the deterministic premium strategy.

Let us suppose the process for the market average premium follows an Ito process

$$d\bar{p} = \mu_t dt + \sigma_t d\mathbf{Z}_t, \tag{46}$$

where \mathbf{Z}_t is a standard Brownian motion and the drift μ_t and the volatility σ_t depend on time and the current state but not the insurer's premium. We use the relative premium $\mathbf{k} = \mathbf{p}/\bar{p}$ as the control and define the value function to maximise the expected terminal wealth:

$$V(\bar{p}, q, Q, w, t) = \max_k \{ \mathbb{E}[\mathbf{w}(T) | \bar{p}(t) = \bar{p}, \mathbf{q}(t) = q, \mathbf{Q}(t) = Q, \mathbf{w}(t) = w] \}. \tag{47}$$

Bellman's principle of optimality states that

$$V(\bar{p}, q, Q, w, t) = \max_k \{ \mathbb{E}[V(\bar{p} + d\bar{p}, q + dq, Q + dQ, w + dw, t + dt) | \bar{p}(t) = \bar{p}, \mathbf{q}(t) = q, \mathbf{Q}(t) = Q, \mathbf{w}(t) = w] \}. \tag{48}$$

Therefore rearranging on the right-hand side we obtain the Bellman equation

$$\max_k \mathbb{E}_t \left[\frac{dV}{dt} \right] = 0.$$

For the insurance model (46) and (15)-(17) the Bellman equation is

$$V_t + \mu V_{\bar{p}} + \frac{1}{2} \sigma^2 V_{\bar{p}\bar{p}} + V_w (-\alpha w + Q - \gamma \bar{p} q) + \max_k \{ q(G - \kappa) V_q + (pqG - \kappa Q) V_Q \} = 0, \tag{49}$$

with boundary condition

$$V(t = T) = w, \tag{50}$$

providing the value function is sufficiently smooth.

The first order condition for a maximum gives

$$G'V_q + V_Q\bar{p}(G + kG') = 0.$$

For the demand function (33), $G' = -a_2$ providing $k \leq b_2$ so that this condition becomes

$$k = \frac{1}{2} \left(b_2 - \frac{V_q}{\bar{p}V_Q} \right), \tag{51}$$

which is similar in form to (34). If the value function is sufficiently smooth and we write the “shadow prices” $V_q = \lambda_1$ and $V_Q = \lambda_2$ then the optimal premiums have the same form (see p. 229, Yong & Zhu 1999). This demonstrates the wider correspondence between the Maximum Principle and Dynamic Programming. Substituting (51) into the Bellman equation removes the maximum operator:

$$V_t + \mu V_{\bar{p}} + \frac{1}{2} \sigma^2 V_{\bar{p}\bar{p}} + V_w(-\alpha w + Q - \gamma\bar{p}q) + q \left(\frac{a_2}{2} \left(b_2 + \frac{V_q}{\bar{p}V_Q} \right) - \kappa \right) V_q + \left(\frac{a_2 q \bar{p}}{4} \left(b_2^2 - \frac{V_q^2}{\bar{p}^2 V_Q^2} \right) - \kappa Q \right) V_Q = 0. \tag{52}$$

This is a quasi-linear partial differential equation with four space variables. Given its high dimension and nonlinearity this problem appears difficult to solve numerically. However, by assuming a particular form for the value function motivated by the deterministic optimisation problem, we can find a solution of (52) and apply a verification theorem (Fleming & Rishel 1975).

On the boundary $t = T$, $V = w$ and therefore $V_q = V_Q = 0$ so that the premium given by (51) is undefined in the same way as it was for the deterministic strategy. If we approximate V_t by a first-order difference then

$$V_t \approx \frac{V(T) - V(T - \delta t)}{\delta t},$$

where δt is a small time step. Substituting into the Bellman equation (52) yields

$$V(T - \delta t) = w + (-\alpha w + Q - \gamma\bar{p}q)\delta t, \tag{53}$$

so that $V_Q = \delta t$ and $V_q = -\gamma\bar{p}\delta t$ at $t = T - \delta t$ providing the spatial boundary conditions are consistent with the finite difference approximation (53). Consequently the terminal premium is well-defined as $t \rightarrow T^-$ and is given by

$$p_{T^-} = \frac{1}{2} \bar{p}(b_2 + \gamma). \tag{54}$$

This should be compared with the optimal deterministic strategy given by (42): near the boundary the terminal premium is identical to the deterministic optimal

premium strategy as long as the control is sufficiently smooth. The result holds irrespective of the distribution of $\bar{\mathbf{p}}$.

Given the structure of (53) we further restrict the process for $\bar{\mathbf{p}}$ by supposing $\mu = \mu(\bar{p}, t)$ and $\sigma^2 = \sigma^2(\bar{p}, t)$. Now we look for a value function of the form

$$V = e^{-\alpha T}(e^{\alpha t}w + \Lambda_1(\bar{p}, t)q + \Lambda_2(t)Q), \tag{55}$$

with $\Lambda_1(T) = \Lambda_2(T) = 0$. Consequently, from (51) a candidate for the dynamic optimal control is

$$k_c(\bar{p}, t) = \frac{1}{2} \left(b_2 - \frac{\Lambda_1(\bar{p}, t)}{\bar{p}\Lambda_2(t)} \right), \tag{56}$$

as long as the control is interior. Substituting (55) into (52) yields a PDE for Λ_1 and an ODE for Λ_2 :

$$\Lambda_{1t} + \mu(\bar{p}, t)\Lambda_{1\bar{p}} + \frac{1}{2}\sigma^2(\bar{p}, t)\Lambda_{1\bar{p}\bar{p}} + \left(\frac{1}{2}a_2b_2 - \kappa \right)\Lambda_1 + \frac{1}{4}a_2\bar{p}b_2^2\Lambda_2 + \frac{a_2\Lambda_1^2}{4\bar{p}\Lambda_2} = \gamma e^{\alpha t}\bar{p}. \tag{57}$$

$$\dot{\Lambda}_2 = \kappa\Lambda_2 - e^{\alpha t}. \tag{58}$$

The equation for Λ_1 is a semi-linear PDE and, in general, can only be solved numerically. The equation for Λ_2 is identical to (24) so that its solution is given by (25).

We can make further progress by supposing $\bar{\mathbf{p}}$ is lognormally distributed so that its drift and volatility are linear in the state variable: $\mu(\bar{p}, t) = \mu\bar{p}$ and $\sigma^2(\bar{p}, t) = \sigma^2\bar{p}$. Consequently Λ_1 takes the form

$$\Lambda_1(\bar{p}, t) = m_{\bar{p}}\bar{p}\lambda_1(t), \tag{59}$$

where λ_1 satisfies

$$\dot{\lambda}_1 = \gamma m_{\bar{p}} e^{\alpha t} - \frac{1}{4}a_2b_2^2\Lambda_2m_{\bar{p}} - \lambda_1 \left(\frac{1}{2}a_2b_2 - \kappa \right) - \frac{a_2\lambda_1^2}{4m_{\bar{p}}\Lambda_2}. \tag{60}$$

Now the candidate dynamic premium strategy is of the form $\mathbf{p}_c = k(t)\bar{\mathbf{p}}$ from (56) and (60) is identical to (36).

The deterministic strategy (5) is of a similar form to the candidate dynamic premium in the case that the market average premium is lognormally distributed and the control is smooth. In (7) we take the expectation of the governing processes based on information up until $t = 0$. However, we could instead take conditional expectations using the information available up until time t . The analysis in Section 3.2 would be identical and the optimal premium strategy would be the same except that the state equations must be integrated using the current state rather than that at time $t = 0$. Consequently, if we evaluate the deterministic premium strategy then this yields the candidate optimal dynamic

premium in the form $\mathbf{p}_c = k_{i^*} \bar{\mathbf{p}}$ if we use the current value of the lognormally distributed market average premium. However, if $\bar{\mathbf{p}}$ has some other distribution then we must solve the PDE (57) numerically with boundary condition $\Lambda_1(\bar{\mathbf{p}}, T) = 0$.

It remains to ascertain under what conditions the candidate premium strategy (56) is optimal. If the market average premium is lognormally distributed then we can apply the verification Theorem 4.1 of Fleming & Rishel (1975) p. 159. Let us restrict the set of feedback controls to

$$U = \{k(t) \in C^1(t) : 0 \leq k(t) \leq b_2, 0 \leq t \leq T\}$$

so that the exposure equation (15) has smooth bounded coefficients. We assume that there exists a sufficiently smooth unique solution to (60) over $[0, T]$ which leads to a control $k(t) \in U$. The value function defined by (55) and (59) is twice continuously differentiable in the state variables because it is linear in those variables and $\Lambda_2(t)$ is a smooth function of time. The value function is also once continuously differentiable in time so $V \in C_p^{2,1}(Q)$ where the domain $Q = \mathbb{R}^4 \times (0, T)$ using the notation of Fleming & Rishel.

By construction V satisfies the Bellman equation (49) and boundary condition (50) providing the first order condition yields the maximum in the equation. The expression inside the maximum operator is a quadratic function of k where the coefficient of k^2 is

$$-e^{\alpha T} a_2 \bar{\mathbf{p}} q \Lambda_2(t) \leq 0$$

so that the maximum is given by the first order condition as long as $k_c \in U$. It remains to determine whether the feedback control $k_c = k_c(t)$ leads to a well-defined state trajectory, that is, we must verify that the control is *admissible*. If we substitute the control into the state equations (15)-(17) then we obtain a system of linear stochastic differential equations:

$$\begin{aligned} d\bar{\mathbf{p}} &= \mu \bar{\mathbf{p}} dt + \sigma \bar{\mathbf{p}} d\mathbf{Z}_t, \\ d\mathbf{Q} &= (k(t) q(t) G(k(t)) \bar{\mathbf{p}} - \kappa \mathbf{Q}) dt, \\ d\mathbf{w} &= -\alpha \mathbf{w} dt + \mathbf{Q} dt - \gamma q(t) \bar{\mathbf{p}} dt. \end{aligned}$$

The exposure, $q(t)$, is deterministic and governed by a linear equation with bounded smooth coefficients so that it is integrable over $[0, T]$. Therefore, by the uniqueness and existence Theorem 4.1 of Fleming & Rishel (1975) p. 118 there is a unique state trajectory corresponding to the optimal control. The linearity and the smoothness of the coefficients ensure that both the Lipschitz and linear growth conditions are satisfied over the domain Q . Consequently the conditions of Theorem 4.1 of Fleming & Rishel (1975) are satisfied and $k_c(t) \in U$ is the optimal control.

The verification of optimality reduces to the existence of a smooth solution of the Riccati equation (60) which yields a control in the set U . It is well known that Riccati equations may blow-up in finite time (see Bender & Orszag 1978). We can see this behaviour in Figure 7(i)-(iii) as γ, K are decreased or b is increased. The validity of the optimal control depends implicitly on the

parameters through the solution of (60): if for a given parameter set the computed control lies in U then it is the optimal control. This is the case for a number of the controls in Figure 7. In the absence of an analytical solution, the range of parameters which lead to either negative controls or blow-up must be determined numerically.

5. CONCLUSIONS

Emms, Haberman & Savoulli (2004) found the optimal insurance premium given that the relative insurer to market average premium was constant: $\mathbf{p}/\bar{\mathbf{p}} = k$. They found that the insurer should set k so that \mathbf{p} is just above the breakeven rate if there is no drift in the market average premium. We have modified their model in a number of ways.

First, we suppose that the loss ratio $\gamma = \pi/\bar{\mathbf{p}}$ is constant. Emms, Haberman & Savoulli (2004) supposed that the breakeven premium rate was constant, which complicates the behaviour of the optimal control. Specifically, if the market average premium drifts above breakeven the optimal control is necessarily a loss-leader. However, one would expect the main reason for greater premiums is that claims are higher so that there is a direct correlation between the market average premium and the expected mean claim size (or breakeven premium rate).

Second, we have generalised the deterministic premium strategy to be of the form $\mathbf{p}/\bar{\mathbf{p}} = k(t)$. In an unconstrained model we find that the optimal control $k(t)$ is bang-bang. This is a direct consequence of the assumption that the insurer can force existing customers to pay the current premium rate. The optimal control strongly depends on how much the insurer can raise the premium rate during the course of a policy. We are led to a modification of the model which fixes the premium rate at the start of a policyholder's contract. For two choices of the demand function a smooth optimal control was calculated. We find that withdrawal from the market, setting a premium above break-even or loss-leading can be optimal and that the qualitative form of optimal premium strategy is sensitive to the form of the demand function. A loss-leading premium strategy is optimal for a linear demand function when the loss ratio is sufficiently small or the mean contract length is sufficiently large. If we adopt a parameterisation which increases the demand for insurance with a high relative premium then this leads to an unsmooth optimal control with a high terminal premium rate.

The premium strategy of loss-leading followed by profit-taking is one possible cause of the observed actuarial cycle (Daykin *et al.* 1994). Many insurance companies prohibit loss-leading which imposes a restriction on the premium charged to policyholders. Taylor (1986) modelled this restriction by modifying the demand function. However, using optimal control theory the requirement becomes a constraint on the relative premium and may lead to a non-smooth control. Deterministic premium strategies can be investigated numerically for a variety of constraints including those which involve the state of the insurer. This is a further reason for the study of this class of control and forms the subject of ongoing research.

In the last section we compared the optimal deterministic strategies for a linear demand function with the dynamic premium strategy predicted by a Bellman equation. If the market average premium rate is modelled as a log-normal process we find that the deterministic premium strategy and dynamic premium are of the same form. In addition, since we have found a smooth optimal deterministic control under parametric restrictions, the deterministic control is the optimal dynamic control using a verification theorem (Fleming & Rishel 1975). A similar analysis can also be carried out if we consider the objective of maximising the expected total utility of wealth with a utility function which is linear in the wealth process. Further work is aimed at generalising the loss ratio γ (here assumed constant) to include both deterministic and stochastic models.

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