PRICING EXCESS OF LOSS REINSURANCE WITH REINSTATEMENTS

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ABSTRACT

In this paper we discuss the concept of excess of loss reinsurance with reinstatements. The main objective is to provide a methodology to calculate the distribution of total aggregate losses for two or more consecutive layers when there is a limited number of reinstatements. We also compare different premium principles and their properties to price these treaties for any number of free or paid reinstatements.

KEYWORDS

Excess of loss reinsurance, reinstatements, multivariate recursions, premium principles, PH Transform.

1. INTRODUCTION

One of the common aspects of non-proportional reinsurance for some lines of business, such as catastrophe reinsurance, is the fact that the total number of losses to be paid by the reinsurer is limited. This concept is known in reinsurance jargon as *reinstatements*. Although it is a well known concept in practice it is rarely considered in the literature, and therefore the mathematical aspects of pricing such treaties have not been studied in detail. The problem was first studied by Sundt (1991) where he considered the concept of reinstatements and some mathematical aspects of this type of reinsurance. He also studied different premium principles such as pure premiums and the standard deviation principle to price a layer with any number free or paid reinstatements. Walhin and Paris (1999) studied the effect on the probability of ruin for the cedent when it buys excess of loss reinsurance with reinstatements and other aggregate conditions, and the reinsurance premium is calculated under different premium principles. Basically the premium principles studies in Walhin and Paris (1999) are pure premiums, the standard deviation principle and the Proportional Hazard Transform (PH-Transform) premium principle.

It is our objective in this paper to study the effect of having a limited number of reinstatements in each layer on the distribution of the sum of the total losses, and compare the results with the distributions of the losses if we consider the combined layer with the same number of reinstatements. We develop a methodology to calculate the distribution of total losses for two or more consecutive layers when there is a limited number of reinstatements. We also discuss some properties for different premium principles when pricing excess of loss reinsurance with free or paid reinstatements for two or more layers. The methodology to price excess of loss with reinstatements for pure premiums and the standard deviation principle was developed in Sundt (1991). In this paper we develop the methodology to calculate the initial premium under different risk adjusted premium principles for any number of free or paid reinstatements.

In Section 2 we introduce the notation and some basic concepts discussed in Sundt (1991) such as reinstatements and aggregate deductible. In Section 3 we study how to calculate the distribution of total aggregate losses for consecutive layers with reinstatements and compare it with the distribution of total losses if the combined layer is considered. Finally, in Section 4 we study the properties for the initial premiums for free and paid reinstatements under different premium principles such as pure premiums, standard deviation principle and risk adjusted premiums, and we prove analytically that some desirable properties do not hold for this particular type of reinsurance.

2. Excess of loss reinsurance in practice

We consider an insurance portfolio during a year; for convenience we will use the same notation as in Sundt (1991). We call N the number of claims in the year, and $Y_1, Y_2, \ldots$ the individual claim amounts to the portfolio. The $Y_i s$ are iid non-negative random variables with common distribution function $F_Y(t)$. An excess of loss reinsurance for this portfolio for the layer $m\, xs\, l$ would provide the following cover for each individual claim

$$Z_i = \min(\max(0, Y_i - l), m),$$

and therefore the aggregate claim amount for the reinsurer in a fixed period of time is

$$X = \sum_{i=1}^{N} Z_i \tag{1}$$

The aggregate amount X takes into account that the reinsurer would pay all the claims that hit the layer during the period under consideration, which is the usual assumption in the classical literature. In practice there are more complicated assumptions such as aggregate deductible and aggregate limit. If there is an aggregate deductible L, the reinsurer would pay the excess of L in the aggregate, i.e. $\max(0, X - L)$. Usually there is a limit in the number of losses covered by the reinsurer, where a loss is defined in the aggregate as a layer of the same size of the maximum amount of an individual claim to the reinsurer. This concept is known in practice as excess of loss reinsurance with reinstatements. The idea is that after each loss the layer must be reinstated. Reinsurance for a portfolio as described above for the layer $m\,xs\,l$, aggregate deductible L and K reinstatements provides total cover for the following amount

$$\min(\max(0, X - L), (K + 1)m),$$

where X is the aggregate claim amount defined in (1). A simple numerical example will make clear all these definitions.

Numerical illustration. Suppose we cover the layer 150 *xs* 100, with the amounts shown in Table 1 hitting the layer. In this case a loss will be completed once the aggregate amount has reached 150. If there is no aggregate deductible, the first three claims are considered the first loss, or *0th* reinstatement, the fourth claim goes beyond the limit and is considered the second loss or first reinstatement, and the fifth and sixth claim are the third loss or second reinstatement. If the aggregate deductible is $L = 150$, then the first three claims are paid by the ceding company and the reinsurer starts to pay from the fourth claim. If $L = 0$ and there is only 1 reinstatement available then the reinsurer would only pay the first and second losses which include four claims, with the other two claims going back to the ceding company.

TABLE 1

REINSURANCE FOR LAYER 150 *xs* 100

Claim number	1	2	3	4	5	6
Claim amount	175	150	125	300	220	130
Amount to layer	75	50	25	150	120	30

There are two kinds of reinstatements, free and paid reinstatements. For paid reinstatements, every time a claim hits the layer there is an extra premium charged at a pre-determined rate, pro rata to the claim size. If P is the initial premium, it is said that the reinstatement premium for the *nth* reinstatement is at $100c_n\%$. These rates are usually 100% or 50%, see Carter (1981). Returning to our numerical illustration, suppose there is only one reinstatement at 100%. After the first claim there is an extra premium equal to $\frac{75}{150}P = \frac{1}{2}P$, after the second claim the extra premium is $\frac{50}{150}P = \frac{1}{3}P$ and finally after the third claim the extra premium is $\frac{1}{6}P$. The sum of all these premiums form the reinstatement premium, which in this case adds up to P. The fourth claim is paid in full by the reinsurer and the contract finishes. At the end of the contract the reinsurer has paid four claims for an aggregate amount of 300, and received a total premium income of $2P$.

In the rest of this paper we will assume that there is no aggregate deductible, but it is not difficult to extend the results for any aggregate deductible. For paid reinstatements the initial premium P covers the $0th$ reinstatement which is

$$r_0 = \min(X, m).$$

The total reinstatement premium for the first reinstatement is $P_1 = c_1 P \frac{r_0}{m}$. In general, the nth reinstatement covers

$$r_n = \min(\max(0, X - nm), m).$$

For K reinstatements the total gross losses for the reinsurer are:

$$R_K = \sum_{n=0}^{K} r_n = \min(X, (K+1)m),$$

and therefore the total premium income at the end of the period of cover would be:

$$T_K = P\left(1 + \frac{1}{m}\sum_{n=1}^{K} c_n r_{n-1}\right).$$

Note that the random variables r_n are correlated, in the following sense: if there exists $0 \leq j \leq K$ such that $r_j = 0$, then $r_i = 0$ for all $i > j$, and if there is an $r_j > 0$ for $j > 0$, then $r_i = m$ for all $i < j$. In other words, these variables are greater than zero, only when all the previous reinstatements are equal to the maximum limit m.

3. Total losses for consecutive layers

It is common to see in practice that the reinsurance covers consecutive layers of a risk, but they are priced and treated separately because each layer could be subject to different conditions in the contract. For example, for catastrophe excess of loss there is usually one reinstatement at 100% for each layer. For other classes of reinsurance there might be one reinstatement at 100% and one reinstatement at 50%. In this section we study the distribution of total losses for consecutive layers with and without reinstatements, and compare these distributions when we consider the combined layer as a whole layer.

We will follow the same assumptions about the original portfolio as in Section 2, and consider an excess of loss reinsurance for the layers (m_1, m_2) and (m_2, m_3). We are going to study two consecutive layers but the results can be applied to any number of layers consecutive or non-consecutive. The reinsurer will cover for each layer the following amounts:

$$Z_{i1} = \min(\max(0, Y_i - m_1), m_2 - m_1) \qquad Z_{i2} = \min(\max(0, Y_i - m_2), m_3 - m_2).$$

If $F_Y(t)$ is the cumulative distribution of the Y_is, then the distribution functions for the Z_{i1}s and Z_{i2}s are

$$F_{Z_1}(t) = \begin{cases} F_Y(t + m_1) & 0 \leq t < m_2 - m_1 \\ 1 & t \geq m_2 - m_1 \end{cases}$$
$$F_{Z_2}(t) = \begin{cases} F_Y(t + m_2) & 0 \leq t < m_3 - m_2 \\ 1 & t \geq m_3 - m_2 \end{cases}$$

The aggregate claim amounts are respectively

$$S_1 = \sum_{i=1}^{N} Z_{i1} \quad \text{and} \quad S_2 = \sum_{i=1}^{N} Z_{i2}.$$

If we consider the combined layer (m_1, m_3) the individual claim amount for this layer would be $U_i = \min(\max(0, Y_i - m_1), m_3 - m_1) = Z_{i1} + Z_{i2}$, and therefore the aggregate claim amount for the combined layer is $S_c = S_1 + S_2$. Note that S_1, S_2 and S_c are compound random variables whose distributions can be calculated using Panjer's recursion.

In the next section we calculate the distribution of the total losses for two consecutive layers with reinstatements, and we will need the joint distribution of (S_1, S_2).

3.1. Joint distribution of aggregate losses for consecutive layers

In a recent paper, Sundt (1999) presents a multivariate version of Panjer's recursive formula to evaluate the joint distribution of two or more compound distributions, where all the aggregate variables are subject to the same events. This algorithm fits our particular case of aggregate losses for two consecutive layers. Although it can be used for any number of portfolios, for our purposes we are going to assume only two aggregate claim amounts. The assumptions of Sundt's algorithms are:

1. N, the number of claims, satisfies the recursion

$$P(N = n) = \left(a + \frac{b}{n}\right) P(N = n - 1), \qquad n = 1,\ 2,\ \ldots$$

for some constants a, b.

2. The individual claim amounts are integer-valued random variables, and the joint distribution of the individual claim amounts is known. The joint probability function is given by $f(z_1, z_2)$, for $z_1 = 0,\ 1,\ 2,\ \ldots, z_2 = 0,\ 1,\ 2,\ \ldots$.

The aggregate claim amounts are $S_1 = \sum_{i=1}^{N} Z_{i1}$ and $S_2 = \sum_{i=1}^{N} Z_{i2}$, and the recursion for the joint distribution of (S_1, S_2) is as follows:

$$g(s_1,s_2) = \sum_{u=0}^{s_1} \left(a + \frac{bu}{s_1}\right) \sum_{v=0}^{s_2} f(u,v) g(s_1 - u, s_2 - v), \tag{2}$$

for $s_1 = 1,\ 2,\ \ldots,\ s_2 = 0,\ 1,\ 2,\ \ldots$

$$g(s_1,s_2) = \sum_{v=0}^{s_2} \left(a + \frac{bv}{s_2}\right) \sum_{u=0}^{s_1} f(u,v) g(s_1 - u, s_2 - v), \tag{3}$$

for $s_1 = 0,\ 1,\ 2,\ \ldots,\ s_2 = 1,\ 2,\ \ldots$

We can use (2) to evaluate $g(s_1, s_2)$ for all (s_1, s_2) with $s_1 > 0$ and then evaluate $g(0, s_2)$ using (3).

It is clear from the specification of the recursion that the aggregate claim amounts for two layers of the same risk satisfy all the above assumptions. To apply the bivariate recursion in order to calculate the joint distribution of (S_1, S_2) we require that S_1 and S_2 are integer-valued random variables. We therefore assume that the individual claim amounts Y_i are integer-valued random variables with probability function $F_Y(y) = P(Y_i = y)$ for $y = 0,\ 1,\ 2,\ \ldots$; if the $Y_i s$ are continuous random variables we can use the corresponding discretised distribution. Having the distribution of the $Y_i s$ we are able to calculate the joint distribution of the individual claim amounts for each layer, i.e. (Z_{i1}, Z_{i2}). We define $f(z_1, z_2) = P(Z_{i1} = z_1, Z_{i2} = z_2)$, and this joint probability function can be calculated as follows:

$$\begin{aligned}
&f(0,0) = P(Y_i \le m_1) && \\
&f(z_1,0) = P(Y_i = m_1 + z_1) && 0 < z_1 < m_2 - m_1 \\
&f(m_2 - m_1, z_2) = P(Y_i = m_2 + z_2) && 0 < z_2 < m_3 - m_2 \\
&f(m_2 - m_1, m_3 - m_2) = P(Y_i \ge m_3) &&
\end{aligned}$$

All the other possible combinations have probability zero. Hence, for consecutive layers we only need the recursion in (2) since $S_2 > 0$ only if $S_1 > 0$, and $g(0,0) = P(S_1 = 0) = pgf_N(f(0,0))$, where pgf_N is the probability generating function of N. In this case the numerical evaluation of the joint distribution of (S_1, S_2) can be simplified and this point is discussed in detail in Sundt (1999), Section 4.

3.2. Total losses for consecutive layers with reinstatements

In practice each layer is subject to different conditions, different numbers of reinstatements, aggregate deductible and reinstatement rates. For some classes of business the standard assumption is one reinstatement for each layer at 100%, see Carter (1981). For simplicity we are going to assume that there is no aggregate deductible and one reinstatement for each layer, but the results can be extended to include an aggregate deductible and any number of reinstatements.

Under these assumptions, and with the notation used in Section 2, the total losses in the aggregate for each layer are respectively:

$$S_1^* = \min(S_1, 2(m_2 - m_1)) \quad \text{and} \quad S_2^* = \min(S_2, 2(m_3 - m_2)),$$

while for the combined layer $S_c^* = \min(S_1 + S_2, 2(m_3 - m_1))$. In the previous section we discussed how to calculate the distribution of S_1, S_2, $S_1 + S_2$ and the joint distribution of (S_1, S_2). We can write the distribution function of $S_1^* + S_2^*$ as follows:

$$P(S_1^* + S_2^* = x) = \sum_{i=0}^{x} P(S_1^* = x - i, S_2^* = i),$$

where (S_1^*, S_2^*) is a function of (S_1, S_2) whose distribution was calculated in the previous section.

Example 1. Consider an insurance portfolio where the number of claims N follows a Poisson distribution with parameter λ. The individual claim amounts, Y_i, have a Pareto distribution with parameters $\alpha = 3$ and $\beta = 10$, and probability density function

$$f(y) = \frac{\alpha\beta^\alpha}{(y+\beta)^{\alpha+1}} \qquad y > 0.$$

The insurance company arranges an excess of loss contract for two consecutive layers, 10 *xs* 10 and 10 *xs* 20, and one reinstatement is available for each layer. Following the same notation as in Section 2, the aggregate claim amounts for each layer are

$$S_1 = \sum_{i=1}^{N} \min(\max(0, Y_i - 10), 10) \quad \text{and} \quad S_2 = \sum_{i=1}^{N} \min(\max(0, Y_i - 20), 10),$$

and therefore with one reinstatement the total *gross* losses for each layer are:

$$S_1^* = \min(S_1, 20) \quad \text{and} \quad S_2^* = \min(S_2, 20),$$

while for the combined layer of 20 *xs* 10 the total gross losses with one reinstatement are

$$S_c^* = \min(S_1 + S_2, 40).$$

To calculate the distributions of $S_1^* + S_2^*$ and S_c^* using the algorithms described above, we discretised the Pareto distribution on $1/50th$ of the mean using the discretisation method given by De Vylder and Goovaerts (1988). Hence, the probability mass is concentrated in the points $0, h, 2h, 3h, \ldots$, for $h = 0.1$, then we used the multivariate Panjer recursion or univariate Panjer recursion accordingly.

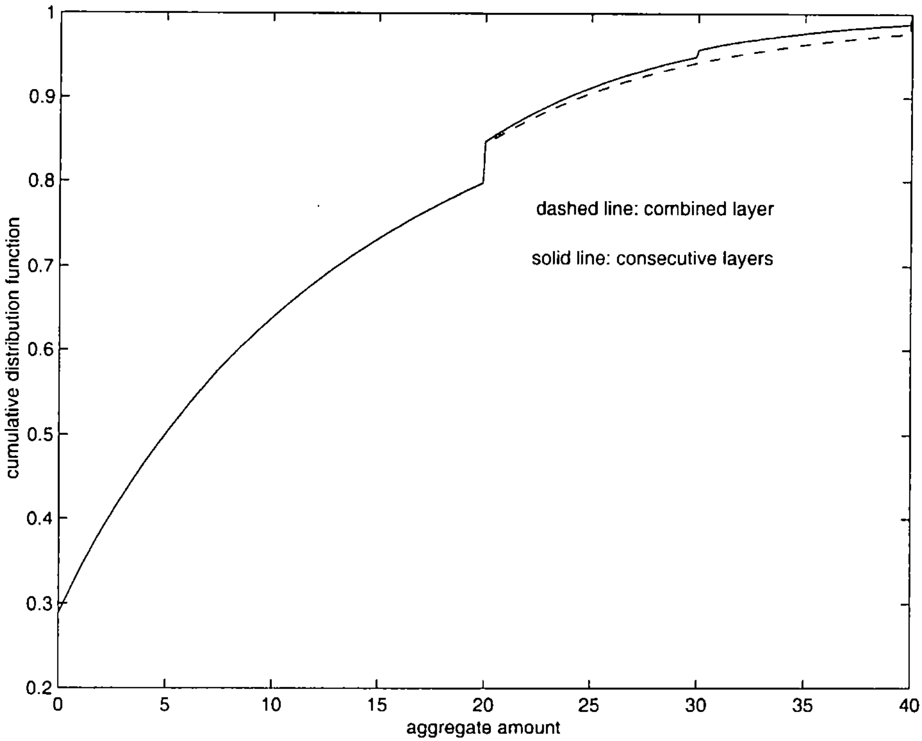

FIGURE 1: Total losses for consecutive layers $\lambda = 10$.

Figures 1 and 2 show the cumulative distribution function of the gross losses for the sum $S_1^* + S_2^*$ and for the combined layer S_c^* for different values of λ. We notice that for $\lambda = 10$ the distributions are the same up to 20, and then the distribution for the combined layer is lower than the distribution of the sum, due to the fact that there will be more claims hitting the first layer, and therefore the combined, than the second layer. Hence the combined layer provides more cover.

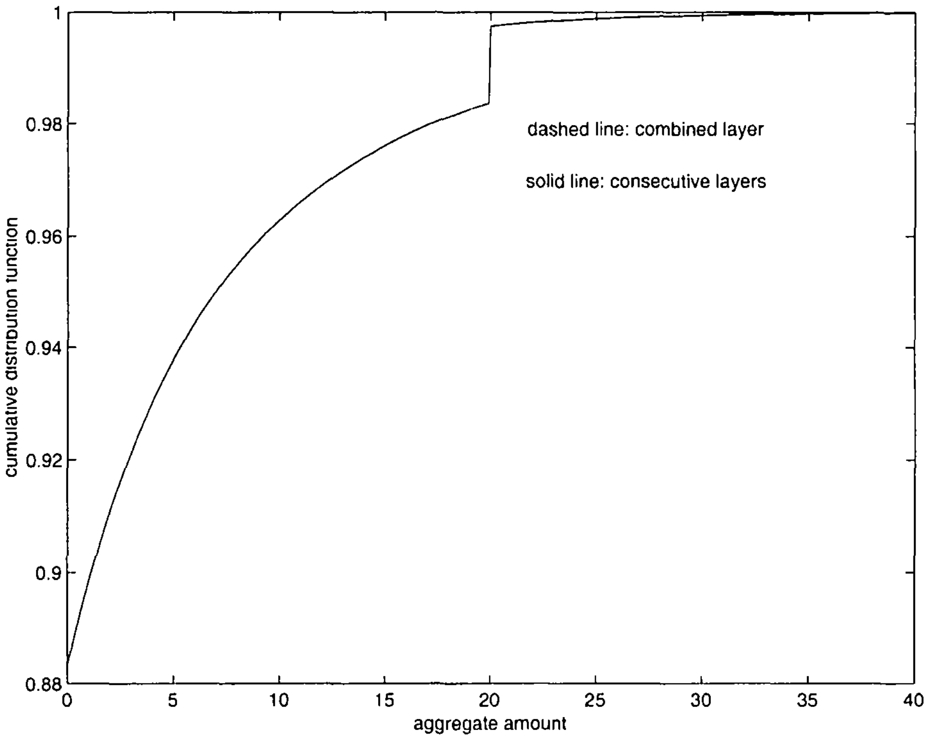

FIGURE 2: Total losses for consecutive layers $\lambda = 1$.

For $\lambda = 1$ we observe that the distributions get closer for all values of the aggregate amount, in fact in Figure 2 we do not see any difference between these distributions. The reason for this is that for small values of λ the losses for the second layer have a high probability of being zero and hence its effect in the sum $S_1^* + S_2^*$ is very small. In other words, for λ small the expected number of claims that hit the second layer is small, and if there is any claim affecting the reinsurer it would affect the first layer and therefore the combined. Hence the distributions of $S_1^* + S_2^*$ and S_c^* converge to the distribution of $S_1 + S_2$ which is the case of gross losses for unlimited reinstatements.

4. PREMIUM PRINCIPLES

In Section 2 we provided a brief summary of the concept of free and paid reinstatements. In this section we study different premium principles to calculate the initial premium P for any number of free and paid reinstatements. We compare the pure premium principle and the standard deviation principle developed in Sundt (1991) with different risk adjusted premium principles described in Wang (1996) and Silva and Centeno (1998). We follow the same notation and assumptions about the portfolio as in Section 2.

A premium principle is a rule π that assigns a non-negative number to a risk defined by its loss distribution function. There are some desirable properties that a premium principle should satisfy, these properties are well described by Wang (1996) and Silva and Centeno (1998). The basic properties we are going to study under the effect of a limited number of reinstatements are positive loadings, linearity and sub-additivity.

When the reinstatements are paid the total premium income becomes a random variable correlated to the total losses, therefore it is not obvious how to calculate the initial premium. In the next sections we describe how to use different premium principles to calculate the initial premium P for any number of free or paid reinstatements, and we study which properties the initial premiums satisfy.

4.1. Pure premiums

Sundt (1991) presented the methodology to calculate pure premiums for any number of free or paid reinstatements. In this section we summarise briefly some of his results. Following the same notation as in Section 2, under the pure premium principle the initial premium should be such that the following equality is satisfied

$$E[T_K] = E[R_K],$$

therefore the initial premium is given by

$$P = \frac{E\left[\sum_{n=0}^{K} r_n\right]}{\left(1 + \frac{1}{m} E\left[\sum_{n=1}^{K} c_n r_{n-1}\right]\right)}. \tag{4}$$

Note that the initial premium P can be calculated uniquely from (4). When all the rates are the same, i.e. $c_n = c$ for all $n = 1, \ldots, K$, the initial premium is

$$P = \frac{E\left[\sum_{n=0}^{K} r_n\right]}{\left(1 + \frac{c}{m} E\left[\sum_{n=1}^{K} r_{n-1}\right]\right)} = \frac{E[R_K]}{\left(1 + \frac{c}{m} E[R_{K-1}]\right)}.$$

When we apply pure premiums for paid reinstatements, there are some properties that do not hold. With paid reinstatements the initial premiums are not linear. Suppose we define a new risk whose aggregate losses are $Y = aX + b$, where X is the aggregate claim amount defined in (1) and $a, b > 0$. We notice that if the original risk covers a layer of size m then the size of the layer for the risk Y is am. Suppose that for the new risk Y there are K reinstatements available, then each reinstatement provides cover for a maximum amount of am. Using the same notation as in Section 2, the $0th$ reinstatement for the risk Y is given by

$$u_0 = \min(Y, am) = \min(aX + b, am) = ar_0 + ab_1,$$

where r_0 represents the losses for the $0th$ reinstatement for the risk X, as defined in Section 2, and $b_1 = \min(b/a, m)$. In general, the nth reinstatement for Y is given by

$$u_n = \min(Y - nam, am) = \min(aX + b - nam, am) = ar_n + ab_1.$$

Therefore, if P_Y is the initial premium for the new risk Y and all the reinstatement rates are the same, the total premium income from the risk Y with K reinstatements is given by

$$T_K = P_Y\left(1 + \frac{c}{am}\sum_{n=1}^{K} u_{n-1}\right) = P_Y\left(1 + \frac{ac}{am}(R_{K-1} + Kb_1)\right),$$

where R_{K-1} represents the losses for $K - 1$ reinstatements for the original risk X. Therefore, applying the pure premium principle the initial premium for the risk Y is given by

$$P_Y = \frac{aE[R_K] + a(K+1)b_1}{\left(1 + \frac{ac}{am}(E[R_{K-1}] + Kb_1)\right)} \neq aP_X + b$$

where P_X is the initial pure premium for the original risk.

Sub-additivity is also a desirable property for a premium principle, otherwise any risk could be split in two small risks to reduce the premium cost. However, in the case of excess of loss with reinstatements care must be taken on how to define the sum of two risks. For K free reinstatements the initial pure premium is given by $P = E[R_K]$, and there is no extra premium, therefore in this case for any two layers with free reinstatements the initial pure premiums are additive. For paid reinstatements the sum of the risk could be defined in terms of the total net losses. Under the assumptions about the original portfolio as in Section 2 the total net losses for a risk with initial premium P can be defined as

$$W(P) = \sum_{n=0}^{K} r_n - \frac{P}{m}\sum_{n=1}^{K} c_n r_{n-1}, \tag{5}$$

and it can be seen that the initial pure premium given by formula (4) satisfies the equation $P = E[W(P)]$. If we want to prove that for paid reinstatements the initial pure premiums are sub-additive we have to define the premium for the sum of the risks as the expected value of the sum of the total net losses for each risk. Although sub-additivity is a desirable property in theoretical terms, in the case of excess of loss with paid reinstatements the interpretation of the sum of the net losses of two risks does not make practical sense since the resulting risk is no longer a layer and therefore the definition of reinstatement premiums and other definitions given in Section 2 lose their natural interpretation in practice. For excess of loss with paid reinstatements it is more reasonable to compare the sum of the premiums for two layers with the corresponding premium if the combined layer is considered.

Example 2. Under the same distributional assumptions given in Example 1 we calculate the initial pure premium for each layer and for the combined layer for one free or one reinstatement at 100%, and for unlimited reinstatements. Note that for unlimited reinstatements it is not necessary to use a discretised distribution for the claim amounts. It is only necessary to calculate the expected value of the total number of claims and the expected value of a single claim to the layer which can be calculated analytically. However, for limited reinstatements we require to calculate the compound distribution of the aggregate claim amount X in order to calculate the expected value of the losses with limited reinstatements.

Tables 2 and 3 show the pure premiums calculated using formula (4) for different values of λ. We observe that the premiums for the combined layer in each case are greater than or equal to the sum of the premiums when each layer is priced separately with the same number of reinstatements. However, for small values of λ, see Table 3, the premium for the combined layer is very close to the sum of the premiums.

TABLE 2

PURE PREMIUMS $\lambda = 10$

Reinstatements	$K = 1$		$K = \infty$	
Layer	$c = 0$	$c = 1$	$c = 0$	$c = 1$
10 *xs* 10	6.6128	4.3709	6.9444	4.0983
10 *xs* 20	2.4103	1.9806	2.4305	1.9553
SUM	9.0231	6.3515	9.3749	6.0536
20 *xs* 10	9.2173	6.6181	9.3749	6.3829

As we discussed in Example 1, the distribution of the sum of the gross losses is very similar to the distribution of the gross losses for the combined layer for small values of λ, and hence the pure premiums are also very similar. Particularly for free reinstatements we notice from Table 3 that the pure premium for one free reinstatement converges to the pure premium for unlimited free reinstatements. In other words, for small values of λ the probability of the reinsurer being liable for more than one claim is very small, and therefore a limited number of reinstatements does not make much difference to the premiums to be charged.

TABLE 3

PURE PREMIUMS $\lambda = 1$

Reinstatements	$K = 1$		$K = \infty$	
Layer	$c = 0$	$c = 1$	$c = 0$	$c = 1$
10 *xs* 10	0.6938	0.6501	0.6944	0.6493
10 *xs* 20	0.2428	0.2371	0.2431	0.2373
SUM	0.9366	0.8872	0.9375	0.8866
20 *xs* 10	0.9370	0.8957	0.9375	0.8955

4.2. The standard deviation principle

If Y is a risk, the standard deviation principle gives the premium as

$$P = E[Y] + \gamma\sqrt{Var[Y]},$$

where γ is a positive constant. For paid reinstatements the total premium income becomes a random variable, therefore the application of the standard deviation principle is not direct. Sundt (1991) proposes to solve the following equation for P

$$E[T_K] = E[R_K] + \gamma\sqrt{Var(R_K - T_K)}, \tag{6}$$

where T_K is the total premium income with K reinstatements and R_K is the total gross loss with K reinstatements as defined in Section 2. Using equation (6) Sundt (1991) presents a detailed development of the formula for the initial premium under this premium principle for any number of free or paid reinstatements. Under this premium principle there is not always a solution for the initial premium P, furthermore the solution might not be unique. It has been discussed that the standard deviation principle does not satisfy some desirable properties, and in the case of pricing excess of loss reinsurance with reinstatements it has the extra disadvantage that the solution for the initial premium does not necessarily exist. Furthermore, when pricing two consecutive layers of the same risk for some values of γ the combined layer attracts lower premiums than the sum of the premiums for each layer. We show this in the next example.

Example 3. Using the same distributional assumptions as in Example 1, Table 4 shows the initial premium under the standard deviation principle for $\gamma = 0.2$.

TABLE 4

INITIAL PREMIUMS FOR S.D.P. $\lambda = 10$, $\gamma = 0.2$

Reinstatements	$K = 1$		$K = \infty$	
Layer	$c = 0$	$c = 1$	$c = 0$	$c = 1$
10 *xs* 10	12.7648	6.6123	13.2320	5.1290
10 *xs* 20	5.6294	3.7600	5.6614	3.6148
SUM	18.3942	10.3723	18.8934	8.7438
20 *xs* 10	18.1891	10.3435	18.4033	6.4792

From Table 4 we observe that the combined layer attracts lower premiums than the sum of the premiums for each layer in all the cases. This is not desirable since we have discussed before that when there is a limited number of reinstatements the combined layer provides more cover than if the layers are considered separately. In the case of unlimited free reinstatements the combined layer provides the same cover as the sum of the two separate layers, therefore in this case it is reasonable that the premium for the combined layer is higher than the sum of the premiums to avoid splitting risks to reduce premium costs.

From the discussion above we conclude that in the case of excess of loss with limited reinstatements the standard deviation principle has some disadvantages that make its application in the pricing process very limited.

In the next section we discuss some advantages of a class of risk adjusted premium principles introduced by Wang (1995) that has some attractive properties that the standard deviation does not satisfy. Furthermore, we develop the method to use these premium principles when pricing excess of loss reinsurance with reinstatements.

4.3. Risk adjusted premiums

Wang (1995) first introduced the concept of risk adjusted premiums by means of the Proportional Hazard Transform as a loaded premium with desirable properties that the standard deviation principle and the variance principle do not satisfy. Wang (1996) studies the desirable properties of this premium principle and some applications to Utility Theory. Silva and Centeno (1998) compared different risk adjusted premiums with the expected value principle from the reinsurer's point of view and concluded that the PH Transform is very similar to the expected value principle for stop-loss and excess of loss as defined in Classical Risk Theory. Christofides (1998) also studied some advantages of the PH Transform as a premium principle over the standard deviation principle when pricing financial risks. Although the latter paper focuses on the use of this premium principle as a loaded premium in practice, in the case of reinsurance treaties it only studies the case of stop-loss contracts as defined in Classical Risk Theory. Walhin and Paris (1999) used the PH Transform to price excess of loss with reinstatements; however, they calculate the initial premium for paid reinstatements under the PH Transform using a numerical recursion.

It is our objective in this section to develop formulae that allow us to calculate the initial risk adjusted premium for any number of free or paid reinstatements.

Definition 1. *Let X be a positive risk, with cdf $F_X(t)$ and survival function $S_X(t)$. Define*

$$S_Y(t) = (S_X(t))^{1/\rho} \qquad \rho \geq 1,$$

so that $S_Y(t)$ defines another survival function for any $\rho \geq 1$. The mapping $\Pi_\rho(X) : S_X(t) \to S_Y(t)$ is called the Proportional Hazard (PH) Transform.

Definition 2. *The risk adjusted premium according to the PH Transform principle for a positive risk X is defined as*

$$\pi_\rho(X) = \int_0^\infty (1 - F_X(t))^{1/\rho} dt = \int_0^\infty (S_X(t))^{1/\rho} dt \qquad \text{for } \rho \geq 1,$$

where $F_X(t)$ is the cdf and $S_X(t)$ is the survival function of X.

Note that $\pi_\rho(X) = E_{\Pi_\rho}[X]$, i.e. the PH Transform premium is equivalent to calculating the expected value of the losses from the risk X with respect to the distorted survival function. The PH Transform premium principle satisfies all the desirable properties for a premium principle, see Wang (1996) and Silva and Centeno (1998). In particular, the PH Transform premium principle is additive for comonotonic risks, see Wang (1996).

Clearly for paid reinstatements the PH Transform premium principle cannot be applied directly since the total premium income becomes a random variable correlated to the total losses. Walhin and Paris (1999) propose that the initial premium should be the solution of the following equation

$$P = \pi_\rho(W(P)) = E_{\Pi_\rho}[W(P)] , \tag{7}$$

where $W(P)$ represents the total net loss defined in (5). In the latter paper they solve equation (5) using a numerical recursion.

However, since the PH Transform premium principle is equivalent to calculating an expected value with respect to the distorted survival function we propose to apply the same scheme as for pure premiums. Hence, we equate the expected value of the total premium income to the expected value of the total losses with respect to the distorted survival function for the same value of ρ. In other words, the initial risk adjusted premium must be such that the following equality holds

$$E_{\Pi_\rho}[T_K] = E_{\Pi_\rho}[R_K]\,. \tag{8}$$

As discussed in Section 2 each reinstatement represents a layer of the aggregate claim amount X, and layers of the same risk are comonotonic random variables, see Wang (1995). Therefore, the variables $r_n s$ are comonotonic. Hence, using the properties of linearity and additivity for comonotonic risk of E_{Π_ρ}, we can rewrite (8) as follows:

$$P\left(1 + \frac{1}{m}\sum_{n=1}^{K} c_n E_{\Pi_\rho}(r_{n-1})\right) = E_{\Pi_\rho}(R_K).$$

Solving for P, the initial risk adjusted premium is given by

$$P = \frac{E_{\Pi_\rho}(R_K)}{\left(1 + \frac{1}{m}\sum_{n=1}^{K} c_n E_{\Pi_\rho}(r_{n-1})\right)}\,. \tag{9}$$

It can be easily verified that if we use the same properties of the PH Transform in equation (7) we obtain the same solution as in formula (9). Therefore, it is not necessary to solve equation (7) numerically, furthermore the initial premium under the PH Transform premium principle is uniquely determined by formula (9) for any number of free or paid reinstatements.

We notice that the initial risk adjusted premium P has the same form as the pure premiums defined in Section 4.1, except that the expected value is calculated according to the distorted survival function.

Under the PH Transform premium principle the initial premiums for free reinstatements are not additive, but they are still sub-additive due to the property of sub-additivity of the PH Transform premium principle. In general, the property of linearity does not hold under this premium principle. We proved in Section 4.1. that for pure premiums with paid reinstatements linearity does not hold, which is the case when $\rho = 1$ for the PH Transform premium principle.

In general the PH Transform premium principle provides a loaded premium with certain advantages over the classic loaded premium of the standard deviation principle and the variance principle. Moreover, in the case of excess of loss reinsurance with reinstatements the PH Transform premium principle has the extra advantage that the initial premium is uniquely determined by formula (9) and it is very simple to compute.

Wang (1996) proposes the use of other transforms similar to the PH Transform that also have desirable properties, except that he considers it desirable that if $g(x)$ is the transform, then $g'(0) = \infty$ which is only true for the PH Transform. For more details see Wang (1996) and Silva and Centeno (1998).

All the proposed transforms are increasing, continuous and concave functions for $0 \leq x \leq 1$, with $g(0) = 0$ and $g(1) = 1$. In all these cases the premium principle for a positive risk X is calculated as

$$\pi_g(X) = \int_0^\infty g(S_X(t))dt, \tag{10}$$

where $S_X(t)$ is the survival function of the random variable X. Note that in this case $\pi_g(X) = E_g[X]$.

If we want to use these premium principles for a limited number of reinstatements, using the same notation as for the PH Transform, the initial premium must satisfy the following equation

$$E_g[T_K] = E_g[R_K] \,. \tag{11}$$

All these transforms have the same desirable properties as the PH Transform, see, for example, Silva and Centeno (1998). Therefore, under any of the risk adjusted premium principles the initial premium can be calculated using formula (9) replacing E_{Π_ρ} by E_g where g is the corresponding transform.

Example 4. We assume the same portfolio as in Example 1, and we discretise the Pareto distribution in the same units as in Example 2. For free reinstatements we calculate the distributions of $S_1^* + S_2^*$, S_c^*, S_1^* and S_2^* as described in Section 3. Then we calculate the survival function, distort it and finally calculate the expected value according to the new survival function for each value of ρ.

Tables 5 and 6 show the risk adjusted premiums under the PH Transform premium principle for different values of ρ and different values of λ for one free reinstatement. From Table 5 we notice the following inequality

$$\pi_\rho(S_1^* + S_2^*) \leq \pi_\rho(S_1^*) + \pi_\rho(S_2^*) \leq \pi_\rho(S_c^*),$$

where the first inequality is the property of sub-additivity discussed above and the second inequality is due to the fact discussed above that for a limited number of reinstatements the combined layer provides more cover, and should be more expensive.

TABLE 5

RISK ADJUSTED PREMIUMS WITH PH TRANSFORMS $\lambda = 10$

ρ	$\pi_\rho(S_1^* + S_2^*)$	$\pi_\rho(S_1^*)$	$\pi_\rho(S_2^*)$	$\pi_\rho(S_1^*) + \pi_\rho(S_2^*)$	$\pi_\rho(S_c^*)$
1.0	9.0232	6.6128	2.4103	9.0232	9.2173
1.2	10.9094	7.7403	3.2344	10.9747	11.1852
1.4	12.6235	8.7116	4.0313	12.8183	12.9715
1.6	14.1775	9.5518	4.7875	14.3393	14.5856
1.8	15.5862	10.2828	5.4971	15.7799	16.0425
2.0	16.8645	10.9230	6.1590	17.0820	17.3585

However, for $\lambda = 1$ we observe

$$\pi_\rho(S_1^* + S_2^*) \leq \pi_\rho(S_c^*) \leq \pi_\rho(S_1^*) + \pi_\rho(S_2^*),$$

for $\rho \geq 1.2$, see Table 6. As discussed in Example 3 for the standard deviation principle it is not convenient that the combined layer attracts lower premiums than the sum of the premiums since it provides more cover. As we observe in Figure 2, for small values of λ the distribution of the losses for the combined layer and the distribution of the sum of the losses for each layer converge. Therefore, the premium for the combined layer is very close to the premium for the sum, i.e. $\pi_\rho(S_1^* + S_2^*) \approx \pi_\rho(S_c^*)$, while the sum of the premiums under the PH Transform principle is always greater than the premium for the sum. It might also occur that for $\lambda = 10$ for higher values of ρ we also lose this property which is not in favour of the PH Transform premium principle.

TABLE 6

RISK ADJUSTED PREMIUMS WITH PH TRANSFORMS $\lambda = 1$

ρ	$\pi_\rho(S_1^* + S_2^*)$	$\pi_\rho(S_1^*)$	$\pi_\rho(S_2^*)$	$\pi_\rho(S_1^*) + \pi_\rho(S_2^*)$	$\pi_\rho(S_c^*)$
1.0	0.9367	0.6938	0.2428	0.9367	0.9370
1.2	1.5534	1.1043	0.4558	1.5601	1.5543
1.4	2.2591	1.5566	0.7207	2.2773	2.2609
1.6	3.0215	2.0312	1.0236	3.0548	3.0248
1.8	3.8172	2.5147	1.3531	3.8678	3.8281
2.0	4.6291	2.9980	1.7000	4.6980	4.6361

For paid reinstatements, we assume one reinstatement at 100%, therefore using formula (9) we have the initial premium given by

$$P = \frac{\pi_\rho(R_1)}{1 + \frac{1}{m}\pi_\rho(R_0)},$$

where R_1 and R_0 are the total gross losses with 1 and 0 reinstatements respectively, and m is the size of the corresponding layer. We calculate the distribution of R_K for $K = 0,\ 1$ using Panjer's recursion with a discretised Pareto distribution as described in Example 1, then we distort the survival function for different values of ρ and finally we calculate the expected value according to the new survival function. We then input these values in equation (9).

TABLE 7

INITIAL RISK ADJUSTED PREMIUM $\lambda = 10$

ρ	*layer* 10 *xs* 10		*layer xs* 20		*layer xs* 10		*Sum*	
	$c = 0$	$c = 1$	$c = 0$	$c = 1$	$c = 0$	$c = 1$	$c = 0$	$c = 1$
1.0	6.6128	4.3709	2.4103	1.9806	9.2173	6.6181	9.0231	6.3515
1.2	7.7403	4.9245	3.2344	2.5287	11.1852	7.6885	10.9747	7.4608
1.4	8.7116	5.3831	4.0313	3.0214	12.9715	8.6124	12.7429	8.4118
1.6	9.5518	5.7683	4.7875	3.4620	14.5856	9.4156	14.3393	9.2372
1.8	10.2828	6.0958	5.4971	3.8559	16.0425	10.1188	15.7799	9.9596
2.0	10.9230	6.3773	6.1590	4.2090	17.3585	10.7384	17.0820	10.5950

Tables 7 and 8 show the initial risk adjusted premiums for different values of ρ and different values of λ. For paid reinstatements, for $\lambda = 10$ the combined layer attracts higher premiums than the sum for the premiums. For $\lambda = 1$ for $\rho \geq 1.6$ the sum of the premiums is greater than the premium for the combined layer for the same reasons we discussed for free reinstatements. Therefore, in the case of excess of loss with a limited number of reinstatements the PH Transform premium principle has a common disadvantage with the standard deviation premium principle which is not desirable in practice.

TABLE 8

INITIAL RISK ADJUSTED PREMIUM $\lambda = 1$

ρ	*layer* 10 *xs* 10		*layer xs* 20		*layer xs* 10		*Sum*	
	$c = 0$	$c = 1$	$c = 0$	$c = 1$	$c = 0$	$c = 1$	$c = 0$	$c = 1$
1.0	0.6938	0.6501	0.2428	0.2371	0.9370	0.8957	0.9366	0.8875
1.2	1.1043	0.9996	0.4558	0.4364	1.5543	1.4454	1.5601	1.4370
1.4	1.5566	1.3607	0.7207	0.6740	2.2609	2.0415	2.2773	2.0359
1.6	2.0312	1.7172	1.0236	0.9335	3.0848	2.6516	3.0548	2.6524
1.8	2.5147	2.0606	1.3531	1.2027	3.8221	3.2559	3.8678	3.2653
2.0	2.9980	2.3870	1.7000	1.4734	4.6361	3.8434	4.6980	3.8640

With the numerical results we provided in this paper we wanted to illustrate how to apply the methodology discussed in Sections 3 and 4 and how calculate premiums for excess of loss reinsurance for free and paid reinstatements using different premium principles. The numerical results showed the difference in the premiums if two consecutive layers are priced separately or if the combined layer is priced with the same number of reinstatements. When the expected number of claims that affect the layers is large we observed clear differences between the combined layer and the sum of the layers when they are considered separately. However, when the expected number of claims is small there is not a significant difference between the losses for the combined and the sum of the losses for separate layers with the same number of reinstatements. Furthermore, the numerical results showed that for loaded premiums there are cases when the combined layer attracts lower premiums than the sum of the premiums which is not a desirable property from the reinsurer's point of view.

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