CREDIBLE CLAIMS RESERVES: THE BENKTANDER METHOD

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ABSTRACT

A claims reserving method is reviewed which was introduced by Gunnar Benktander in 1976. It is a very intuitive credibility mixture of Bornhuetter/Ferguson and Chain Ladder. In this paper, the mean squared errors of all 3 methods are calculated and compared on the basis of a very simple stochastic model. The Benktander method is found to have almost always a smaller mean squared error than the other two methods and to be almost as precise as an exact Bayesian procedure.

KEYWORDS

Claims Reserves, Chain Ladder, Bornhuetter/Ferguson, Credibility, Standard Error

1. INTRODUCTION

This note on the occasion of the 80st anniversary of Gunnar Benktander focusses on a claims reserving method which was published by him in 1976 in "The Actuarial Review" of the Casualty Actuarial Society (CAS) under the title "An Approach to Credibility in Calculating IBNR for Casualty Excess Reinsurance". The Actuarial Review is the quarterly newsletter of the CAS and is normally not subscribed outside of North America. This might be the reason why Gunnar's article did not become known in Europe. Therefore, the method has been proposed a second time by the Finnish actuary Esa Hovinen in his paper "Additive and Continuous IBNR", submitted to the ASTIN Colloquium 1981 in Loen/Norway. During that colloquium, Gunnar Benktander referred to his former article and Hovinen's paper was not published further. Therefore it was not unlikely that the method was invented a third time. Indeed, Walter Neuhaus published it in 1992 in the Scandinavian Actuarial Journal under the title "Another Pragmatic Loss Reserving Method or Bornhuetter/Ferguson Revisited". He mentioned neither Benktander nor Horvinen because he did not know about their articles. In recent years, the method has been used occasionally in actuarial reports under the name "Iterated Bornhuetter/Ferguson Method". The present article gives a short review of the method and connects it with the name of its first publisher. Furthermore, evidence is given that the method is very useful which should already be clear from the fact that it has been invented so many times. Using a simple stochastic model it is shown that the Benktander method outperformes the Bornhuetter/Ferguson method and the chain ladder method in many situations. Moreover, simple formulae for the mean squared error of all three methods are derived. Finally, a numerical example is given and a comparison with a credibility model and a Bayesian model is made.

2. Review of the method

To keep notation simple we concentrate on one single accident year and on paid claims. Furthermore, we assume the payout pattern to be given, i.e. we denote with p_j, $0 < p_1 < p_2 < ... < p_n = 1$, the proportion of the ultimate claims amount which is expected to be paid after j years of development. After n years of development, all claims are assumed to be paid. Let U_0 be the estimated ultimate claims amount, as it is expected prior to taking the own claims experience into account. For instance, U_0 can be taken from premium calculation. Then, being at the end of a fixed development year $k < n$,

$$R_{BF} = q_k U_0 \quad \text{with} \quad q_k = 1 - p_k$$

is the well-known Bornhuetter/Ferguson (BF) reserve (Bornhuetter/Ferguson 1972). The claims amount C_k paid up to now does not enter the formula for R_{BF}, i.e. this reserving method ignores completely the current claims experience of the portfolio under consideration. Note that the axiomatic relationship between any reserve estimate $\hat{R}$ and the corresponding ultimate claims estimate $\hat{U}$ is always

$$\hat{U} = C_k + \hat{R} \quad \text{and} \quad \hat{R} = \hat{U} - C_k$$

because the same relationship also holds for the true reserve $R = C_n - C_k$ and the corresponding ultimate claims $U = C_n$, i.e. we have

$$U = C_k + R \quad \text{and} \quad R = U - C_k.$$

For the Bornhuetter/Ferguson method this implies that the final estimate of the ultimate claims is the posterior estimate

$$U_{BF} = C_k + R_{BF}$$

whereas the prior estimate U_0 is only used to arrive at an estimate of the reserve. Note further that the payout pattern $\{p_j\}$ is defined by $p_j = E(C_j)/E(U)$.

Another well-known claims reserving method is the chain ladder (CL) method. This method grosses up the current claims amount C_k, i.e. uses

$$U_{CL} = C_k / p_k$$

as estimated ultimate claims amount and

$$R_{CL} = U_{CL} - C_k$$

as claims reserve. Note that there

$$R_{CL} = q_k U_{CL}$$

holds. This reserving method considers the current claims amount C_k to be fully credibly predictive for the future claims and ignores the prior expectation U_0 completely. One advantage of CL over BF is the fact that with CL different actuaries come always to similar results which is not the case with BF because there may be some dissent regarding U_0.

BF and CL represent extreme positions. Therefore Benktander (1976) proposed to replace the prior U_0 with a credibility mixture

$$U_c = cU_{CL} + (1 - c)U_0.$$

As the credibility factor c should increase similarly as the claims C_k develop, he proposed to take $c = p_k$ and to estimate the claims reserve by

$$R_{GB} = R_{BF} \cdot \frac{U_{p_k}}{U_0}.$$

This is the method as proposed by Gunnar Benktander (GB). Observe that we have

$$R_{GB} = q_k U_{p_k}$$

and

$$U_{p_k} = p_k U_{CL} + q_k U_0 = C_k + R_{BF} = U_{BF},$$

i.e.

$$R_{GB} = q_k U_{BF}.$$

This last equation means that the Benktander reserve R_{GB} is obtained by applying the BF procedure in an additional step to the posterior ultimate claims amount U_{BF} which was arrived at by the normal BF procedure. This way has been taken in some recent actuarial reports and has there been called "iterated Bornhuetter/Ferguson method".

Note again that the resulting posterior estimate

$$U_{GB} = C_k + R_{GB} = (1 - q_k^2)U_{CL} + q_k^2 U_0 = U_{1-q_k^2}$$

for the ultimate claims is different from U_{p_k} which was used as prior.

Esa Hovinen (1981) applied the credibility mixture directly to the reserves instead of the ultimates, i.e. proposed the reserve estimate

$$R_{EH} = cR_{CL} + (1 - c)R_{BF},$$

again with $c = p_k$. But the Hovinen reserve

$$R_{EH} = p_k q_k U_{CL} + (1 - p_k) q_k U_0 = q_k U_{p_k} = R_{GB}$$

is identical to the Benktander reserve.

We have already seen that the functions $R(U) = q_k U$ and $U(R) = C_k + R$ are not inverse to each other except for $U = U_{CL}$. In addition, Table 1 shows that the further iteration of the methods of BF and GB for an arbitrary starting point U_0 finally leads to the chain ladder method.

We want to state this as a theorem:

Theorem 1. For an arbitrary starting point $U^{(0)} = U_0$, the iteration rule

$$R^{(m)} = q_k U^{(m)} \quad \text{and} \quad U^{(m+1)} = C_k + R^{(m)}, \quad m = 0,\ 1,\ 2,\ \dots,$$

gives credibility mixtures

$$U^{(m)} = (1 - q_k^m) U_{CL} + q_k^m U_0,$$
$$R^{(m)} = (1 - q_k^m) R_{CL} + q_k^m R_{BF}$$

between BF and CL which start at BF and lead via GB finally to CL for $m = \infty$.

TABLE 1

ITERATION OF BORNHUETTER/FERGUSON

Ultimate $U(R) = C_k + R$	***Connection***	***Reserve*** $R(U) = q_k U$
U_0		
	$\searrow$	
		$R_{BF} = q_k U_0$
$U^{(1)} = U_{BF} = C_k + R_{BF}$	$\swarrow$	
$= (1 - q_k) U_{CL} + q_k U_0$	$\searrow$	$R^{(1)} = R_{GB} = q_k U_{BF}$
$U^{(2)} = U_{GB} = C_k + R_{GB}$	$\swarrow$	$= (1 - q_k) R_{CL} + q_k R_{BF}$
$= (1 - q_k^2) U_{CL} + q_k^2 U_0$		
.....	$\searrow$	
		.....
$U^{(m)} = (1 - q_k^m) U_{CL} + q_k^m U_0$		
	$\searrow$	$R^{(m)} = q_k U^{(m)}$
$U^{(m+1)} = C_k + R^{(m)}$	$\swarrow$	$= (1 - q_k^m) R_{CL} + q_k^m R_{BF}$
$= (1 - q_k^{m+1}) U_{CL} + q_k^{m+1} U_0$		
.....	$\searrow$	.....
$U^{(\infty)} = U_{CL}$	$\longleftrightarrow$	$R^{(\infty)} = R_{CL}$

Walter Neuhaus (1992) analyzed the situation in a full Bühlmann/Straub credibility framework (see section 6 for details) and compared the size of the mean squared error $mse(R_c) = E(R_c - R)^2$ of

$$R_c = cR_{CL} + (1-c)R_{BF}$$

and the true reserve $R = U - C_k = C_n - C_k$ especially for

$c = 0$ (BF)
$c = p_k$ (GB, called PC-predictor by Neuhaus)
$c = c^*$ (optimal credibility reserve),

where $c^* \in [0;\ 1]$ can be defined to be that c which minimizes $mse(R_c)$. Neuhaus did not include $c = 1$ (CL) explicitely into his analysis.

Neuhaus showed that the mean squared error of the Benktander reserve R_{GB} is almost as small as of the optimal credibility reserve R_{c^*} except if p_k is small and c^* is large at the same time (cf. Figures 1 and 2 in Neuhaus (1992)). Moreover, he showed that the Benktander reserve R_{GB} has a smaller mean squared error than R_{BF} whenever $c^* > p_k/2$ holds. This result is very plausible because then c^* is closer to $c = p_k$ than to $c = 0$.

In the following we include the CL into the analysis and consider the case where U_0 is not necessarily equal to $E(U)$, i.e. consider the estimation error, too. This seems to be more realistic as in Neuhaus (1992) where $U_0 = E(U)$ was assumed. Instead of the credibility model used by Neuhaus, we introduce a less demanding stochastic model in order to compare the precision of R_{BF}, R_{CL} and R_{GB}. We derive a formula for the standard error of R_{BF} and R_{GB} (and R_{CL}) and show how the parameters required can be estimated. A numerical example is given in section 4. Moreover, there is a close connection to a paper by Gogol (1993) which will be dealt with in section 5. Finally, the connection to the credibility model is analyzed in section 6.

3. Calculation of the optimal credibility factor c^* and of the mean squared error of R_c

In order to compare R_{BF}, R_{CL} and R_{GB}, we use the mean squared error

$$mse(R_c) = E(R_c - R)^2$$

as criterion for the precision of the reserve estimate R_c (for a discussion see section 5). Because

$$R_c = cR_{CL} + (1-c)R_{BF} = c(R_{CL} - R_{BF}) + R_{BF}$$

is linear in c, the mean squared error $mse(R_c)$ is a quadratic function of c and will therefore have a minimum.

In the following, we consider U_0 to be an estimation function which is independent from C_k, R, U and has expectation $E(U_0) = E(U)$ and variance $Var(U_0)$. Then we have

Theorem 2. The optimal credibility factor c^* which minimizes the mean squared error $mse(R_c) = E(R_c - R)^2$ is given by

$$c^* = \frac{p_k}{q_k} \cdot \frac{Cov(C_k, R) + p_k q_k Var(U_0)}{Var(C_k) + p_k^2 Var(U_0)}. \tag{1}$$

Proof

$$\begin{aligned} E(R_c - R)^2 &= E[c(R_{CL} - R_{BF}) + R_{BF} - R]^2 \\ &= c^2 E(R_{CL} - R_{BF})^2 - 2cE[(R_{CL} - R_{BF})(R - R_{BF})] + E(R_{BF} - R)^2. \end{aligned}$$

$$0 = \frac{\partial}{\partial c} E(R_c - R)^2 = 2cE(R_{CL} - R_{BF})^2 - 2E[(R_{CL} - R_{BF})(R - R_{BF})]$$

yields

$$\begin{aligned} c^* &= \frac{E[(R_{CL} - R_{BF})(R - R_{BF})]}{E(R_{CL} - R_{BF})^2} = \frac{p_k}{q_k} \cdot \frac{E[(C_k - p_k U_0)(R - q_k U_0)]}{E(C_k - p_k U_0)^2} \\ &= \frac{p_k}{q_k} \cdot \frac{Cov(C_k - p_k U_0,\ R - q_k U_0)}{Var(C_k - p_k U_0)} = \frac{p_k}{q_k} \cdot \frac{Cov(C_k, R) + p_k q_k Var(U_0)}{Var(C_k) + p_k^2 Var(U_0)}. \end{aligned}$$

Here, we have used that $E(C_k) = p_k E(U_0)$ according to the definition of the payout pattern (and therefore $E(R) = q_k E(U_0)$). Q.E.D.

In order to estimate c^*, we need a model for $Var(C_k)$ and $Cov(C_k, R)$. The following model is not more than a slightly refined definition of the payout pattern:

$$E(C_k/U|U) = p_k, \tag{2}$$

$$Var(C_k/U|U) = p_k q_k \beta^2(U). \tag{3}$$

The factor q_k in (3) is necessary in order to secure that $Var(C_k|U) \to 0$ as k approaches n. A similar argument holds for p_k in case of very small values. A parametric example is obtained if the ratio C_k/U, given U, has a Beta(ap_k, aq_k)-distribution with $a > 0$; in this case $\beta^2(U) = (a+1)^{-1}$. Thus, in the simple cases, $\beta^2(U)$ depends neither on U nor on k. If the variability of C_k/U for high values of U is higher, then $\beta^2(U) = (U/U_0) \cdot \beta^2$ is a reasonable assumption.

From assumptions (2) and (3) and with $\alpha^2(U) := U^2\beta^2(U)$ we gather

$$\begin{aligned} &E(C_k|U) = p_k U, \\ &Var(C_k|U) = p_k q_k \alpha^2(U), \\ &E(C_k) = p_k E(U), \\ &Var(C_k) = p_k q_k E(\alpha^2(U)) + p_k^2 Var(U) \\ &\qquad\quad = p_k E(\alpha^2(U)) + p_k^2 (Var(U) - E(\alpha^2(U))), \end{aligned} \tag{4}$$

$$
\begin{aligned}
Cov(C_k, U) &= Cov(E(C_k|U), U) = p_k Var(U), \\
Cov(C_k, R) &= Cov(C_k, U) - Var(C_k) = p_k q_k \big(Var(U) - E(\alpha^2(U))\big), \qquad (5)\\
E(R) &= E(U) - E(C_k) = q_k E(U), \\
Var(R) &= Var(U) - 2Cov(C_k, U) + Var(C_k) \\
&= Var(U)(1 - 2p_k + p_k^2) + p_k q_k E(\alpha^2(U)) \\
&= q_k^2 Var(U) + p_k q_k E(\alpha^2(U)) \\
&= q_k E(\alpha^2(U)) + q_k^2 \big(Var(U) - E(\alpha^2(U))\big).
\end{aligned}
$$

By inserting (4) and (5) into (1), we immediately obtain

Theorem 3. Under the assumptions of model (2)-(3), the optimal credibility factor c^* which minimizes $mse(R_c)$ is given by

$$
c^* = \frac{p_k}{p_k + t} \quad \text{with} \quad t = \frac{E(\alpha^2(U))}{Var(U_0) + Var(U) - E(\alpha^2(U))}. \qquad (6)
$$

Some further straightforward calculations lead to

Theorem 4. Under the assumptions of model (2)-(3), we have the following formulae for the mean squared error:

$$
\begin{aligned}
mse(R_{BF}) &= E(\alpha^2(U)) q_k (1 + q_k/t), \\
mse(R_{CL}) &= E(\alpha^2(U)) q_k / p_k, \\
mse(R_c) &= E(\alpha^2(U)) \left(\frac{c^2}{p_k} + \frac{1}{q_k} + \frac{(1-c)^2}{t} \right) q_k^2.
\end{aligned}
$$

Proof

$$
\begin{aligned}
mse(R_{BF}) &= E(R_{BF} - R)^2 = Var(R_{BF} - R) = Var(R_{BF}) + Var(R) \\
&= q_k^2 Var(U_0) + q_k^2 \big(Var(U) - E(\alpha^2(U))\big) + q_k E(\alpha^2(U)) \\
&= E(\alpha^2(U)) (q_k + q_k^2/t), \\
mse(R_{CL}) &= E(R_{CL} - R)^2 = Var(R_{CL} - R) \\
&= Var(R_{CL}) - 2Cov(R_{CL}, R) + Var(R) \\
&= q_k^2 Var(C_k)/p_k^2 - 2 q_k Cov(C_k, R)/p_k + Var(R) \\
&= E(\alpha^2(U)) q_k / p_k, \\
mse(R_c) &= E(cR_{CL} + (1-c)R_{BF} - R)^2 \\
&= E[c(R_{CL} - R) + (1-c)(R_{BF} - R)]^2 \\
&= c^2 mse(R_{CL}) + 2c(1-c)E[(R_{CL} - R)(R_{BF} - R)] + (1-c)^2 mse(R_{BF}),
\end{aligned}
$$

$$
\begin{aligned}
E[(R_{CL}-R)(R_{BF}-R)] &= Cov(R_{CL}-R,\ R_{BF}-R)\\
&= -Cov(R_{CL},R)+Var(R)\\
&= Var(R)-q_k Cov(C_k,R)/p_k\\
&= q_k E\big(\alpha^2(U)\big).
\end{aligned}
$$

and putting all pieces together leads to the formula stated. Q.E.D.

An actuary who is able to assess $p_k = E(C_k/U|U)$ and U_0 (i.e. $E(U_0)$) should also be able to estimate $Var(U_0)$ and $Var(C_k/U|U)$ or $E(Var(C_k|U))$ as well as $Var(U)$. Therefrom, he can deduce $E\big(\alpha^2(U)\big) = E(Var(C_k|U))/(p_k q_k)$ – or $E\big(\alpha^2(U)\big) = Var(C_k/U|U)E(U^2)/(p_k q_k)$ if $Var(C_k/U|U)$ does not depend on U – and finally the parameter t. Then he has now a formula for the mean squared error of the BF method and a very simple formula for the CL method (where t is not needed) and can calculate the best estimate R_{c^*} including its mean squared error as well as the one of R_{GB}.

Regarding the very simple formula for $mse(R_{CL})$ we should note that this formula deviates from the corresponding one (i.e. for the unconditional mean squared error with known payout pattern) of the distribution-free chain ladder model of Mack (1993). The reason is that the models underlying are slightly different: Here we have

$$E\left(\frac{C_k}{U}|U\right) = p_k$$

and the model of Mack (1993) can be written as

$$E\left(\frac{U}{C_k}|C_k\right) = \frac{1}{p_k}.$$

Using theorem 4, we now compare the mean squared errors of the different methods in terms of p_k and t. First, we have

$$mse(R_{BF}) < mse(R_{CL}) \iff p_k < t,$$

i.e. we should use BF for the green years ($p_k < t$) and CL for the rather mature years ($p_k > t$). This is very plausible and the author is aware that some companies use this rule with $t = 0.5$. But the volatility measure t varies from one business to the other and therefore the actuary should try to estimate t in every single case as is shown in the next section.

Furthermore, we have

$$
\begin{aligned}
mse(R_{GB}) < mse(R_{BF}) &\iff t < 2 - p_k,\\
mse(R_{GB}) < mse(R_{CL}) &\iff t > p_k q_k/(1+p_k),
\end{aligned}
$$

i.e. GB is better than BF except t is very large and is better than CL except t is very small, see Figure 1 where for each of the three areas it is indicated which of BF, GB, CL is best. In the numerical example below, it will become clear that t is almost always in the GB area.

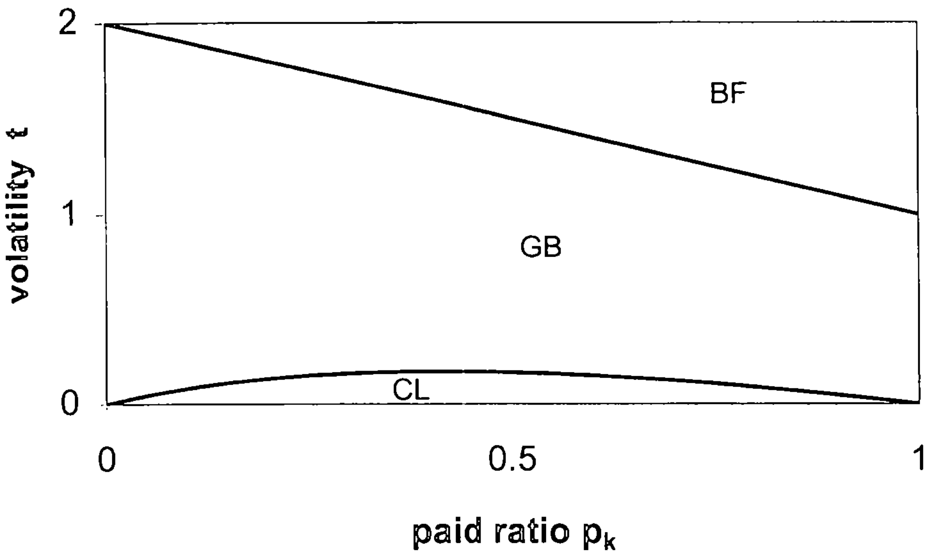

FIGURE 1: Areas of smallest mean squared error.

4. NUMERICAL EXAMPLE

Assume that the a priori expected ultimate claims ratio is 90% of the premium, i.e. $U_0 = 90\%$. Assuming further $p_k = 0.50$ for $k = 3$, we have $R_{BF} = 45\%$ (all % ages relate to the premium). Let the paid claims ratio be $C_k = 55\%$, then $U_{CL} = 110\%$ and $R_{CL} = 55\%$. Taken together, we have $R_{GB} = 50\%$.

In order to calculate the standard errors, we have to assess $Var(U)$, $Var(U_0)$ and $E(\alpha^2(U))$. For $Var(U)$, we can use a consideration of the following type: We assume that the ultimate claims ratio will never be below 60% and only once every 20 years above 150%. Then, assuming a shifted lognormal distribution with expectation 90%, we get $Var(U) = (35\%)^2$. This rather high variance is typical for a reinsurance business or a small direct portfolio.

Regarding $E(\alpha^2(U))$, we consider here the special case where $\beta^2(U) = \beta^2$ does not depend on U (e.g. using a Beta distribution), i.e. $E(\alpha^2(U)) = E(U^2)\beta^2 = E(U^2)Var(C_k/U|U)/(p_k q_k)$. Therefore, we have to assess $Var(C_k/U|U)$, i.e. the variability of the payment ratio C_k/U around its mean p_k. If we assume – e.g. by looking at the ratios C_k/U of past accident years – that C_k/U will be almost always between 0.30 and 0.70, then – using the two-sigma rule from the normal distribution – we have a standard deviation of 0.10, i.e. $Var(C_k/U|U) = 0.10^2$, which leads to $\beta^2 = Var(C_k/U|U)/(p_k q_k) = 0.20^2$ and $E(\alpha^2(U)) = E(U^2)\beta^2 = 0.193^2$.

Finally, the most difficult task is to assess $Var(U_0)$ but this has much less influence on t than $Var(U)$ (which is always larger) and $E(\alpha^2(U))$. Moreover, an actuary who is able to establish a point estimate U_0 should also be able to estimate the uncertainty $Var(U_0)$ of his point estimate. Thus, there will be a certain interval or range of values where the actuary takes his choice of U_0 from. Then he can take this interval and use the two-sigma rule to produce the standard deviation $\sqrt{Var(U_0)}$. Let us assume that in our example $Var(U_0) = (15\%)^2$.

Now we can calculate $t = 0.346$ and all standard errors (= square root of the estimated mean squared error) as well as the optimal credibility reserve R_{c^*}:

$$
\begin{aligned}
R_{BF} &= 45\% \pm 21.3\% \\
R_{CL} &= 55\% \pm 19.3\% \\
R_{GB} &= 50\% \pm 17.3\% \\
R_{c^*} &= 50.9\% \pm 17.2\% \qquad \text{with} \quad c^* = 0.591.
\end{aligned}
$$

Note that these standard errors are based on the *un*conditional mean squared error (cf. discussion in the next section) and on a known pattern $\{p_j\}$. Including the uncertainty of the p_j will increase the standard error.

For the purpose of comparison, we look at a more stable business, too: Assume that $Var(U) = (10\%)^2$, $Var(U_0) = (5\%)^2$ and $Var(C_k/U|U) = (0.03)^2$. Then, everything else being equal, we obtain $\beta^2 = 0.06^2$, $E(\alpha^2(U)) = 0.054^2$, $t = 0.309$ and

$$
\begin{aligned}
R_{BF} &= 45\% \pm 6.2\% \\
R_{CL} &= 55\% \pm 5.4\% \\
R_{GB} &= 50\% \pm 4.9\% \\
R_{c^*} &= 51.2\% \pm 4.9\% \qquad \text{with} \quad c^* = 0.618.
\end{aligned}
$$

In both cases, GB has a smaller mean squared error than BF and CL, and the size of t has not changed much, because the relative sizes of the three variances $Var(U)$, $Var(U_0)$, $Var(C_k/U|U)$ are similar. A closer look at formula (6) shows that the size of t is changed more if $E(\alpha^2(U))$ (i.e. $Var(C_k/U|U)$) is changed than if $Var(U)$ or $Var(U_0)$ are changed. In the first example, for instance, we had $Var(C_k/U|U) = 0.10^2$ and GB was better than CL and BF. If we change the variability of the paid ratio to $Var(C_k/U|U) \geq 0.153^2$, then $t \geq 1.51$ and BF is better than GB and CL. If we change it to $Var(C_k/U|U) \leq 0.074^2$, then $t \leq 0.164$ and CL is better than GB and BF, see Figure 1. But in the large range of normal values of $Var(C_k/U|U)$, GB is better than CL and BF. Because $Var(U_0)$ is always smaller than $Var(U)$, the size of t is essentially determined by the ratio $Var(C_k/U|U)/Var(U)$.

5. Application of an exact Bayesian model to the numerical example

If we make distributional assumptions for U and $C_k|U$, we can determine the exact distribution of $U|C_k$ according to Bayes' theorem. This was done by Gogol (1993) who assumed that U and $C_k|U$ have lognormal distributions because then $U|C_k$ has a lognormal distribution, too.

Applied to our first numerical example, this model is:

$$U \sim \text{Lognormal } (\mu, \sigma^2) \quad \text{with} \quad E(U) = 90\%, \ Var(U) = (35\%)^2,$$
$$C_k|U \sim \text{Lognormal } (\nu, \tau^2) \quad \text{with} \quad E(C_k|U) = p_k U, \ Var(C_k|U) = p_k q_k \beta^2 U^2$$

where $\beta^2 = 0.20^2$ is as before, i.e. such that $Var(C_k/U|U) = 0.10^2$.
This yields

$$\sigma^2 = \ln(1 + Var(U)/(E(U))^2) = 0.375^2,$$
$$\mu = \ln(E(U)) - \sigma^2/2 = -0.176,$$
$$\tau^2 = \ln\big(1 + \beta^2 q_k/p_k\big) = 0.198^2.$$

Then (see Gogol (1993)),

$$U|C_k \sim \text{Lognormal } (\mu_1, \sigma_1^2)$$

with

$$\mu_1 = z\big(\tau^2 + \ln(C_k/p_k)\big) + (1 - z)\mu = 0.067,$$
$$\sigma_1^2 = z\tau^2 = 0.175^2,$$
$$z = \sigma^2/(\sigma^2 + \tau^2) = 0.782.$$

This yields (at $C_k = 55\%$)

$$E(U|C_k) = \exp(\mu_1 + \sigma_1^2/2) = 108.6\%,$$
$$E(R|C_k) = E(U|C_k) - C_k = 53.6\%,$$
$$Var(R|C_k) = Var(U|C_k) = (E(U|C_k))^2\big(\exp(\sigma_1^2) - 1\big) = (19.2\%)^2.$$

If we compare this last result with the mean squared errors obtained in section 4, we should recall that $E(R|C_k)$ minimizes the *conditional* mean squared error

$$E\Big((\hat{R} - R)^2|C_k\Big) = Var(R|C_k) + \big(\hat{R} - E(R|C_k)\big)^2$$

among all estimators $\hat{R}$ which are a square integrable function of C_k as well as it minimizes the *un*conditional mean squared error

$$E\big(\hat{R} - R\big)^2 = E(Var(R|C_k)) + E\big(\hat{R} - E(R|C_k)\big)^2$$

because the first term of the r.h.s. does not depend on $\hat{R}$. But the resulting minimum values $Var(R|C_k)$ and $E(Var(R|C_k))$ are different.

Basically, in claims reserving we should minimize the *conditional* mean squared error, given C_k, because we are only interested in the future variability and because C_k remains a fixed part of the ultimate claims U. But if $E(R|C_k)$ is a linear function of C_k (like R_c), this function can be found by minimizing the unconditional (average) mean squared error. Moreover, the latter can often be calculated easier than the conditional mean squared error as it is the case in model (2)-(3). The unconditional mean squared error is the appropriate measure to compare the precision of different reserving methods.

Altogether, it is clear that the mean squared errors calculated in section 4 are average (unconditional) mean squared errors, averaged over all possible values of C_k. Therefore, in order to make a fair comparison of the various methods in our numerical example, we must calculate the unconditional mean squared error $E(Var(R|C_k))$ in the Bayesian model, too.

For this purpose, we have to integrate $Var(R|C_k)$ over C_k and therefore need the distribution of C_k which we obtain by mixing the distributions of $C_k|U$ and U:

$$C_k/p_k \sim \text{Lognormal } (\mu - \tau^2/2,\ \sigma^2 + \tau^2),$$
$$\exp(2z\ \ln(C_k/p_k)) \sim \text{Lognormal } \left(2z\mu - z\tau^2,\ 4z^2(\sigma^2 + \tau^2)\right).$$

This yields

$$\begin{aligned} E(Var(R|C_k)) &= E\left(\exp\left(2\mu_1 + \sigma_1^2\right)\left(\exp(\sigma_1^2) - 1\right)\right) \\ &= E(\exp(2z\ \ln(C_k/p_k)))\exp\left(3z\tau^2 + 2(1-z)\mu\right)\left(\exp(z\tau^2) - 1\right) \\ &= \exp\left(2\mu + 2\sigma^2\right)\left(\exp(z\tau^2) - 1\right) \\ &= (17.0\%)^2. \end{aligned}$$

This shows finally, that the exact Bayesian model on average has only a slightly smaller mean squared error than the optimal credibility reserve R_{c^*} and the Benktander reserve R_{GB}. But if we recall that, with the exact Bayesian procedure, we assume to exactly know the distributional laws without any estimation error, then the slight improvement in the mean squared error does not pay for the strong assumptions made.

6. CONNECTION TO THE CREDIBILITY MODEL

Finally, we establish an interesting connection between the model (2)-(3) and the credibility model used in Neuhaus (1992). There, the Bühlmann/Straub credibility model was applied to the incremental losses and payouts: For $j = 1, \ldots, n$ (where n is such that $p_n = 1$) let

$$m_j = p_j - p_{j-1}$$

be the incremental payout pattern and

$$S_j = C_j - C_{j-1}$$

be the incremental claims (with the convention $p_0 = 0$ and $C_0 = 0$). Then the Bühlmann/Straub credibility model makes the following assumptions:

$$S_1|\Theta, \ldots, S_n|\Theta \quad \text{are independent,} \tag{7}$$

$$E(S_j/m_j|\Theta) = \mu(\Theta), \qquad 1 \leq j \leq n, \tag{8}$$

$$Var(S_j/m_j|\Theta) = \sigma^2(\Theta)/m_j \qquad 1 \leq j \leq n, \tag{9}$$

where Θ is the unknown distribution quality of the accident year. Assumption (7) can be crucial in practise. Model (7)-(9) can be set up without refering to p_j by just requiring $m_j > 0$ and $m_1 + \ldots + m_n = 1$. Then the following formulae still hold using $p_k := m_1 + \ldots + m_k$.

From (7)-(9) we obtain

$$E(C_k|\Theta) = p_k\mu(\Theta),$$
$$Var(C_k|\Theta) = p_k\sigma^2(\Theta).$$

The latter formula shows, that the credibility model is different from model (2)-(3) where we have $Var(C_k|U) = p_k q_k \alpha^2(U)$, i.e. we do not have $\Theta = U$.

In the credibility model (7)-(9) we obtain further

$$\begin{aligned}
E(C_k) &= p_k E(\mu(\Theta)) = p_k E(C_n) = p_k E(U),\\
Var(C_k) &= p_k E\big(\sigma^2(\Theta)\big) + p_k^2 Var(\mu(\Theta)),\\
Cov(C_k, U) &= E(Cov(C_k, C_k|\Theta)) + Cov(p_k\mu(\Theta), \mu(\Theta))\\
&= p_k\big(E\big(\sigma^2(\Theta)\big) + Var(\mu(\Theta))\big),\\
Cov(C_k, R) &= p_k q_k Var(\mu(\Theta)),\\
E(R) &= q_k E(\mu(\Theta)) = q_k E(U),\\
Var(R) &= q_k E\big(\sigma^2(\Theta)\big) + q_k^2 Var(\mu(\Theta)).
\end{aligned} \tag{10}$$

If we compare these formulae with the corresponding formulae of model (2)-(3) and take into account that here

$$Var(\mu(\Theta)) = Var(U) - E\big(\sigma^2(\Theta)\big)$$

holds (from (10) with $k = n$), then we see that these formulae are completely identical if $E\big(\alpha^2(U)\big) = E\big(\sigma^2(\Theta)\big)$. This leads immediately to

Theorem 5. The formulae of theorems 3 and 4 hold for model (7)-(9), too, after having replaced $E\big(\alpha^2(U)\big)$ with $E\big(\sigma^2(\Theta)\big)$.

In the credibility model, a natural estimate of $E(\sigma^2(\Theta))$ can be established: From

$$Var(S_j/m_j|\Theta) = \sigma^2(\Theta)/m_j$$

and

$$\sum_{j=1}^{k} m_j \frac{S_j}{m_j} \Big/ \sum_{j=1}^{k} m_j = C_k/p_k = U_{CL}$$

it follows that

$$\sigma^2 = \frac{1}{k-1} \sum_{j=1}^{k} m_j \left(\frac{S_j}{m_j} - U_{CL}\right)^2$$

is an unbiased estimator of $E(\sigma^2(\Theta))$. We can write

$$\sigma^2 = p_k s^2/(k-1)$$

where

$$s^2 = \sum_{j=1}^{k} m_j \left(\frac{S_j}{m_j} - U_{CL}\right)^2 \Big/ \sum_{j=1}^{k} m_j$$

can be calculated easily as the m_j-weighted average of the squared deviations of the observed ratios S_j/m_j from their weighted mean U_{CL}. Note that each S_j/m_j is an unbiased estimate of the expected ultimate claims $E(U)$.

If in our numerical example in addition to $p_3 = 0.50$ and $C_3 = 55\%$ we have $p_1 = 0.10$, $p_2 = 0.30$, $C_1 = 15\%$, $C_2 = 27\%$, then $m_1 = 0.10$, $m_2 = 0.20$, $m_3 = 0.20$, $S_1 = 15\%$, $S_2 = 12\%$, $S_3 = 28\%$, and the ratios $S_1/m_1 = 1.5$, $S_2/m_2 = 0.6$, $S_3/m_3 = 1.4$ have a variance $s^2 = 0.41^2$. Then the estimate for $E(\sigma^2(\Theta))$ is $\sigma^2 = 0.205^2$. With $C_1 = 10\%$ and $C_2 = 30\%$ we would get $\sigma^2 = 0.061^2$ indicating a more stable case.

Note that for the estimation of $E(\alpha^2(U))$ the observation of several accident years is necessary. Anyhow, model (2)-(3) is less demanding than model (7)-(9).

7. Conclusion

In claims reserving, the actuary has usually two independent estimators R_{BF} and R_{CL}, at his disposal: One is based on prior knowledge (U_0), the other is based on the claims already paid (C_k). It is a well-known lemma of Statistics that from several independent and unbiased estimators one can form a better estimator (i.e. with smaller variance) by putting them together via a linear combination. From this general perspective, too, it is clear that the *GB* reserve should be superior to *BF* or *CL*.

More precisely, the foregoing analysis has shown that *GB* has a smaller mean squared error than *BF* and *CL* if the payout pattern is neither extremely volatile nor extremely stable. This conclusion is derived within a model whose assumptions are nothing more than a precise definition of the term 'payout pattern'. Therefore, actuaries should include the Benktander method in their standard reserving methods.

Finally, we want to emphasize that all formulae derived rely on the assumption that the prior estimate U_0 and the observed claims C_k are independent. This means that these formulae probably will not hold any more for a 'prior' U_0 which has been adjusted during the development period as it is often done in practise. Such an adjustment is like choosing an U_c with an unknown c and gives a procedure which is much less objective than the Benktander method.

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REFERENCES

BENKTANDER, G. (1976) An Approach to Credibility in Calculating IBNR for Casualty Excess Reinsurance. In *The Actuarial Review*, April 1976, p. 7.

BORNHUETTER, R.L. and FERGUSON, R.E. (1972) The Actuary and IBNR. In *Proceedings of the Casualty Actuarial Society*, Vol. LIX, 181-195.

GOGOL, D. (1993) Using Expected Loss Ratios in Reserving. In *Insurance: Mathematics and Economics* **12**, 297-299.

HOVINEN, E. (1981) Additive and Continuous IBNR. *ASTIN Colloquium Loen/Norway*.

MACK, Th. (1993) Distribution-free Calculation of the Standard Error of Chain Ladder Reserve Estimates. *ASTIN Bulletin* **23** (1993), 213-225.

NEUHAUS, W. (1992) Another Pragmatic Loss Reserving Method or Bornhuetter/Ferguson Revisited. In *Scand. Actuarial J.* 1992, 151-162.

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Correction Note

to the paper
„Credible Claims Reserves: The Benktander Method"
by Thomas Mack

In Chapter 5 („Application …"), there is a mistake.
The equation for μ_1 should be as follows:

$$\mu_1 = z\,(\,\tau^2/2 + \ln(C_k/p_k)\,) + (1 - z)\,\mu = 0.05155\ ,$$

i. e. $\tau^2/2$ instead of τ^2 and a slightly different numerical result. This mistake entails the following further alterations later on in the same chapter:

$E(U|C_k) = \ldots = 106.9\%$ (instead of 108.6%),

$E(R|C_k) = \ldots = 51.9\%$ (instead of 53.6%),

$Var(R|C_k) = \ldots = \ldots = (18.9\%)^2$ (instead of 19.2%).

Finally, the last equations of Chapter 5 change as follows:

$$\begin{aligned} E(Var(R|C_k)) &= E(\,\exp(2\mu_1 + \sigma_1^2)\,(\exp(\sigma_1^2) - 1)\,) \\ &= E(\exp(2z\ln(C_k/p_k)))\,\exp(2z\tau^2 + 2(1-z)\mu)\,(\exp(z\tau^2) - 1) \\ &= \exp(2\mu + (1+z)\sigma^2)\,(\exp(z\tau^2) - 1) \\ &= (16.8\%)^2\ . \end{aligned}$$

(i. e. $2z\tau^2$ instead of $3z\tau^2$ in the second line, $(1+z)\sigma^2$ instead of $2\sigma^2$ in the third line and 16.8% instead of 17.0% in the forth line.) This concludes the list of corrections.