

RISK-MINIMIZING HEDGING STRATEGIES FOR UNIT-LINKED LIFE INSURANCE CONTRACTS

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ABSTRACT

A unit-linked life insurance contract is a contract where the insurance benefits depend on the price of some specific traded stocks. We consider a model describing the uncertainty of the financial market and a portfolio of insured individuals simultaneously. Due to incompleteness the insurance claims cannot be hedged completely by trading stocks and bonds only, leaving some risk to the insurer. The theory of risk-minimization is briefly reviewed and applied after a change of measure. Risk-minimizing trading strategies and the associated intrinsic risk processes are determined for different types of unit-linked contracts. By extending the model to the situation where certain reinsurance contracts on the insured lives are traded, the direct insurer can eliminate the risk completely. The corresponding self-financing strategies are determined.

KEYWORDS

Incomplete market, Martingale representation, Minimal martingale measure, Intrinsic risk, Reinsurance.

1 INTRODUCTION

Traditional actuarial analysis of life insurance contracts focuses on calculation of expected values of various discounted random cashflows; the fundamental principle of equivalence states that discounted premiums and benefits should balance on average for any contract. The corresponding premium is called the equivalence premium. Similarly, at any time during the insurance period, the prospective reserve is defined as the conditional expected value of all discounted future benefits less premiums, given the available information. The development of the reserve is described by

Thiele's differential equation, which originally dealt with constant deterministic interest and deterministic benefits, but has been widely generalized, see e.g. Norberg (1995) and Norberg and Møller (1996).

With a unit-linked life insurance contract, benefits depend explicitly on a specified stock index. Typically, the policyholder will receive the maximum of the stock price and some asset value guarantee stipulated in the contract, but other dependencies may be specified. These contracts have been analyzed by Brennan and Schwartz (1979), and more recently by e.g. Delbaen (1990), Bacinello and Ortu (1993), Aase and Persson (1994) and Nielsen and Sandmann (1995). The last of these authors allow the risk-free interest rate to be stochastic. Various exotic types of contract functions are considered in Ekern and Persson (1996). Aase and Persson (1994) derive a partial differential equation for the value of the reserve of a unit-linked life insurance, which is compared with Thiele's differential equation. They also present duplicating strategies that minimize the risk of the insurance company in a sense.

All the papers mentioned consider mortality risk as diversifiable or assume that the insurer is "risk neutral with respect to mortality" and replace the uncertain courses of the insured lives by the expected. In this way, the actual insurance claims, depending on uncertainty within the portfolio of insured lives and the financial markets, are replaced by similar claims which only include the financial uncertainty. These claims are then priced using standard no-arbitrage pricing theory. In the present paper we provide and examine a model where the uncertainty of a portfolio of lives to be insured and a certain financial market are described simultaneously, and consider the problem of hedging the actual claims which depend on both sources of uncertainty.

The insurance company issues life insurance contracts with insurance benefits linked to the price of a specified stock. This stock and one risk-free asset are traded freely on the financial market without transaction costs. We then consider the problem of defining optimal investment strategies. This situation differs from the case of standard life insurance, where the insurance company should try to maximize trading gains in order to compete with other companies on redistributions of bonus. With unit-linked contracts, benefits are already linked explicitly to the development of the market, and hence are not influenced by the factual gains generated by the investment strategies of the insurance company. However, by issuing these contracts, the insurer is exposed to a financial risk, and our objective here will be to minimize this risk. In this paper we will measure the risk associated with the contracts using the expected value (under an adjusted measure) of the square of the difference between the insurance benefits to be paid and the gains obtained from investments.

The insurance contracts are characterized as contingent claims in an incomplete model, such that the insurance claims cannot be perfectly duplicated by means of self-financing strategies. The theory of risk-minimization for incomplete markets introduced by Föllmer and Sonder-

mann (1986) and developed further by Föllmer and Schweizer (1988) and Schweizer (1991, 1994 and 1995) is reviewed and then applied after a change of measure. With its present formulation, this theory deals with the problem of hedging contingent claims that are payable at a fixed time only. The analysis of more general claims with intermediate payment times would require an extension of the original theory of Föllmer and Sondermann (1986), a problem which will be addressed in a forthcoming paper by Møller (1998). Thus, insurance contracts with payments occurring only at fixed times are analyzed within the original setup of Föllmer and Sondermann (1986), whereas some modifications are needed in order to deal with contracts where the sum insured falls due immediately upon the death of the insured. In the present paper, we assume that premiums are paid as single premiums and that all benefits are deferred to the term of the contract. In this way optimal investment strategies minimizing the risk (under the minimal martingale measure) associated with the assigned contracts are determined. Since the model is incomplete, risk cannot be eliminated completely by applying these strategies, leaving some minimum obtainable risk (called the intrinsic risk) to the insurer. This minimum risk process is determined for different types of standard contracts and is taken as a measure of the non-hedgeable risk inherent in the contracts.

In Section 2 we present the combined model and briefly mention some basic results from the theory of mathematical finance. We also introduce the basic types of insurance claims to be analyzed in the paper. Section 3 is devoted to a review of the most important concepts of risk-minimization. Unit-linked life insurance contracts by single premium are analyzed in Section 4. Section 5 deals with the situation where reinsurance contracts are traded freely on the market. Finally, some numerical results are presented in Section 6.

2 THE MODEL

In this section the two basic elements of the model, the financial market and a portfolio of individuals to be insured, are introduced. We set out by presenting the financial market and reviewing some well-known results from the theory of mathematical finance for complete markets. When extending the model by also including a portfolio of individuals to be insured, the market is no longer complete.

Throughout, we let T denote a fixed, finite time horizon and consider a given probability space (Ω, \mathcal{F}, P) .

2.1. The financial market

We consider a market consisting of only two traded assets: a stock with prices process S and a bond with price process B . At any time t these assets

are traded freely at prices S_t and B_t , respectively. The price processes are defined on a probability space (Ω, \mathcal{F}, P) and are given by the P -dynamics

$$dS_t = \alpha(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \quad (2.1)$$

$$dB_t = r(t, S_t)B_t dt, \quad (2.2)$$

$S_0 > 0$, $B_0 = 1$, where $W = (W_t)_{0 \leq t \leq T}$ is a standard Brownian motion on the time interval $[0, T]$. The filtration $\mathbf{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ generated by this economy is given by

$$\mathcal{G}_t = \sigma\{(S_u, B_u), u \leq t\} = \sigma\{S_u, u \leq t\}.$$

A solution to the equation (2.1) exists provided that the functions α and σ satisfy certain regularity conditions, see e.g. Duffie (1996, Appendix E). These conditions are assumed to be fulfilled henceforth. Furthermore, we assume that $\int_0^T r_t dt$ exists and is finite almost surely.

The process α is interpreted as the mean rate of return of S , and σ as the standard deviation of the rate of return. Similarly r is called the short rate of interest and denotes the rate of return of the risk-free asset. The process ν defined by $\nu_t = (\alpha_t - r_t)/\sigma_t$ is known as the market price of risk process associated with S . In addition to the assumptions above, we assume that ν satisfies the integrability conditions from Duffie (1996, Chapter 6). With constant coefficients α , σ and r , all conditions are satisfied, and we have the celebrated Black-Scholes model where S and B are given by

$$\begin{aligned} S_t &= S_0 \exp\left((\alpha - \frac{1}{2}\sigma^2)t + \sigma W_t\right), \\ B_t &= \exp(rt). \end{aligned}$$

The model above has been thoroughly investigated in the literature of mathematical finance, see e.g. Duffie (1996), Björk (1996) and Lamberton and Lapeyre (1996). Thus some concepts and results from the theory of finance, needed repeatedly in the sequel, will be quoted without explicit reference. Also Aase and Persson (1994) give a brief survey of this theory.

Recall that two measures P and P^* are said to be equivalent if, for each set $A \in \mathcal{F}$, we have that $P(A) = 0$ if and only if $P^*(A) = 0$. By definition, the probability measure P^* defined by

$$\frac{dP^*}{dP} = \exp\left(-\int_0^T \left(\frac{\alpha_u - r_u}{\sigma_u}\right) dW_u - \frac{1}{2}\int_0^T \left(\frac{\alpha_u - r_u}{\sigma_u}\right)^2 du\right) \equiv U_T \quad (2.3)$$

is equivalent to P . It can be verified that the discounted price process S^* , defined by

$$S_t^* = S_t/B_t = S_0 \exp\left(\int_0^t (\alpha_u - r_u) du + \int_0^t \sigma_u dW_u\right), \quad (2.4)$$

is a P^* -martingale. Thus P^* is called an *equivalent martingale measure*. In the above model, the martingale measure is unique.

A *trading strategy* or *portfolio strategy* is an adapted process $\varphi = (\xi, \eta)$ satisfying some integrability conditions (a precise definition will be given in Section 3). At any time $t \in [0, T]$, ξ_t and η_t represent, respectively, the number of shares and the number of bonds held in the portfolio. The *value process* \hat{V}^φ associated with φ is defined by

$$\hat{V}_t^\varphi = \xi_t S_t + \eta_t B_t, \tag{2.5}$$

and the strategy is said to be *self-financing* if

$$\hat{V}_t^\varphi = \hat{V}_0^\varphi + \int_0^t \xi_u dS_u + \int_0^t \eta_u dB_u, \tag{2.6}$$

for all $0 \leq t \leq T$. According to (2.6), any change in the value of the portfolio is generated by changes in the underlying price processes S and B . A *contingent claim* with maturity T is a random variable X that is \mathcal{G}_T -measurable and P^* -square integrable. In particular, X is called a *simple claim* whenever $X = g(S_T)$, for some function $g: \mathbf{R}_+ \rightarrow \mathbf{R}$. We say that a contingent claim X can be perfectly duplicated if there exists a self-financing portfolio φ such that $\hat{V}_T^\varphi = X$ P -a.s. In this case the claim is called *attainable*. If all contingent claims are attainable, then the market is said to be *complete*; otherwise the market is referred to as *incomplete*. A self-financing strategy φ is an *arbitrage* if $\hat{V}_0^\varphi < 0$ and $\hat{V}_T^\varphi \geq 0$ or if $\hat{V}_0^\varphi \leq 0$, $\hat{V}_T^\varphi \geq 0$ P -a.s. and $\hat{V}_T^\varphi > 0$ with positive probability. It is well-known that the market defined by (2.1)-(2.2) and filtration \mathbf{G} is complete and free of arbitrage under the above mentioned assumptions.

Note that if $\varphi = (\xi, \eta)$ is self-financing and duplicates the claim X , then we have the following representation from (2.5) and (2.6):

$$X = \xi_0 S_0 + \eta_0 B_0 + \int_0^T \xi_u dS_u + \int_0^T \eta_u dB_u \tag{2.7}$$

The arbitrage-free price process $(F(t, S_t))_{0 \leq t \leq T}$ associated with a simple claim specifying the payment $g(S_T)$ at time T can now be characterized by the partial differential equation (PDE)

$$-r(t, s)F(t, s) + F_t(t, s) + r(t, s)sF_s(t, s) + \frac{1}{2}\sigma(t, s)^2 s^2 F_{ss}(t, s) = 0, \tag{2.8}$$

with boundary value $F(T, s) = g(s)$. Here, exemplifying a general notational convention adopted throughout, $F_s(t, s)$ denotes the partial derivative of $F(t, s)$ with respect to s , $F_{ss}(t, s)$ denotes the second order partial derivative with respect to s , and so on.

The arbitrage-free price process associated with the claim $g(S_T)$ is also given in terms of the unique equivalent martingale measure by

$$F(t, S_t) = E^* \left[\exp \left(- \int_t^T r_u du \right) g(S_T) \middle| \mathcal{G}_t \right]. \quad (2.9)$$

(Throughout E^* denotes expectation with respect to P^*). Thus, the price is determined by discounting the T -payment with the asset B and then calculating the conditional expectation under the martingale measure P^* .

2.2. The insurance portfolio

In this paragraph we will introduce a model to describe the lifetimes in a group of individuals. For simplicity, we assume that the lifetimes are mutually independent and identically distributed. The i.i.d. assumption implies that the individuals are selected from a cohort of equal age x , say, and we denote by l_x the number of persons in the group. Mathematically, this is described by representing the individual remaining lifetimes as a sequence T_1, \dots, T_{l_x} of i.i.d. non-negative random variables defined on (Ω, \mathcal{F}, P) . Assuming that the distribution of T_i is absolutely continuous with hazard rate function μ_{x+t} , the survival function is

$${}_t p_x = P(T_i > t) = \exp \left(- \int_0^t \mu_{x+\tau} d\tau \right).$$

Now define a univariate process $N = (N_t)_{0 \leq t \leq T}$ counting the number of deaths in the group;

$$N_t = \sum_{i=1}^{l_x} I(T_i \leq t),$$

and denote by $\mathbf{H} = (\mathcal{H}_t)_{0 \leq t \leq T}$ the natural filtration generated by N , i.e. $\mathcal{H}_t = \sigma\{N_u, u \leq t\}$. By definition, N is cadlag (right-continuous with left-limits) and, since the lifetimes T_i are i.i.d., the counting process N is an \mathbf{H} -Markov process. The (stochastic) intensity process λ of the counting process N can be informally defined by

$$E[dN_t | \mathcal{H}_{t-}] = (l_x - N_{t-}) \mu_{x+t} dt \equiv \lambda_t dt,$$

the hazard rate function μ_{x+t} times the number of individuals under exposure just before time t . The *compensated counting process* M defined by

$$M_t = N_t - \int_0^t \lambda_u du \quad (2.10)$$

is an \mathbf{H} -martingale

2.3. The combined model

Now introduce the filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ generated by the economy and the insurance portfolio, that is

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t.$$

We assume throughout that \mathcal{G}_T and \mathcal{H}_T are independent and take

$$\mathcal{F} = \mathcal{G}_T \vee \sigma\{I(T_i \leq u), 0 \leq u \leq T, i = 1, \dots, l_\lambda\}.$$

At time 0 the insurance company issues an insurance contract for each of the l_λ individuals. These contracts specify payments of benefits and premiums that are contingent on the remaining lifetime of the policyholder, and are linked to the development on the financial market. During the period $[0, T]$ the company is allowed to trade the assets B and S freely (without transaction costs, taxes and short sales restrictions) based on the complete information \mathbf{F} . Furthermore, we allow for continuous rebalancing of the portfolio of stocks and bonds in order to hedge against the insurance claims.

In the following, we present the two basic forms of insurance contracts to be analyzed in this paper: the *pure endowment* and the *term insurance*. With a *pure endowment* contract, the sum insured is to be paid at the term T if the insured is then still alive. The sum is of the form $g(S_T)$ for some continuous function g stipulated in the contract, thus depending on the price of the risky asset at time T . Some specific functions will be considered as examples, e.g. $g(s) = s$ and $g(s) = \max(s, K)$ which are known from the literature as *pure unit-linked* and *unit-linked with guarantee* insurance policies, see Aase and Persson (1994). For each insured person the obligation of the insurance company is given by the *present value*

$$H_i = I(T_i > T)g(S_T)B_T^{-1} = I(T_i > T)g(S_T)e^{-\int_0^T r_u du} \quad (2.11)$$

Here we have adopted widely accepted actuarial usage of the term *present value*, it is taken to be the payments discounted using the bond price process described by (2.2). Thus, the present value is an \mathcal{F}_T -measurable random variable. This usage may be at variance with the economical one, where present value typically refers to an \mathcal{F}_0 -measurable value. The entire portfolio generates the discounted claim

$$H = g(S_T)B_T^{-1} \sum_{i=1}^{l_\lambda} I(T_i > T) = g(S_T)B_T^{-1}(l_\lambda - N_T), \quad (2.12)$$

where $(l_\lambda - N_T)$ is the number of survivors at the end of the insurance period. It should be noted that the undiscounted insurance claim HB_T taken from (2.12) is a function of S_T and N_T only. Insurance claims that are payable at time T and are functions of S_T and N_T only will be called *simple T-claims*, whereas more general insurance claims payable at time T are denoted (*general*) *T-claims*.

The *term insurance* states that the sum insured is due immediately upon death before time T . In this case, we consider a time dependent contract function $g_t = g(t, S_t)$. By the definition of the contract, payments can occur at any time during $[0, T]$ and obligations generated by such contracts do not form T -claims without introducing special assumptions. A simple way of transforming the obligations into a (general) T -claim is to assume that all payments are deferred to the term of the contract and are accumulated with the risk-free rate of interest r . With this specific construction, the heirs of a policyholder who died at time t would receive the benefit $g(t, S_t)B_T B_t^{-1}$ at time T . The deferred payments could as well be accumulated differently, for example by using some deterministic first order interest rate δ or by investing $g(t, S_t)$ according to a predefined strategy. These ways of modifying the contracts by deferring the benefits might seem most reasonable for contracts with short time horizons, say one year. Although time horizons associated with traditional life insurance contracts are typically much longer, we will assume that the benefits are actually deferred to the end of the insurance period. The insurer's liabilities in respect of a portfolio of term insurance contracts with payments that are deferred and accumulated using the riskless asset B are now described by the discounted general T -claim

$$H_T = B_T^{-1} \sum_{i=1}^{I_t} g(T_i, S_{T_i}) B_{T_i}^{-1} B_T I(T_i \leq T) = \sum_{i=1}^{I_t} \int_0^T g(u, S_u) B_u^{-1} dI(T_i \leq u),$$

which can be rewritten as an integral with respect to the counting process N :

$$H_T = \int_0^T g(u, S_u) B_u^{-1} dN_u \quad (2.13)$$

Various other insurance contracts can be obtained as combinations of the pure endowment and the term insurance. For example, with the *endowment insurance*, the sum insured is payable at the time of death of the insured persons or maturity, whichever comes first. The present value of this claim is a sum of (2.12) and (2.13). Throughout, we assume that premiums are paid as single premiums at time 0. Thus, the present value of all premiums is simply $\pi^s = I_s \cdot \pi_1$, where π_1 is the single premium paid by the insured.

In Section 2.1 it was pointed out that in the complete market every contingent claim can be represented as an integral with respect to the price processes S and B , see (2.7). As we will show later, this property is not preserved when the model consists of the assets (B, S) and filtration \mathbf{F} . Intuitively, this follows from the fact that the claims (2.12)-(2.13) are not generated by the price processes (B, S) alone since the uncertainty concerning the insured lives contributes essentially to the final outcome of the claims

We end this section by discussing choice of martingale measure in the combined model. For any \mathbf{H} -predictable process h , such that $h > -1$, define a likelihood process L by

$$dL_t = L_{t-}h_t dM_t, \tag{2.14}$$

and initial conditional $L_0 = 1$ Provided that $E^P[L_T]$, a new probability measure \hat{P} can be defined by

$$\frac{d\hat{P}}{dP} = U_T \cdot L_T, \tag{2.15}$$

where U_T is given by (2.3). Using the definition of the measure \hat{P} and the independence between N and (B, S) under P we see that S^* defined by (2.4) is also a \hat{P} -martingale: for $u < t$ we have

$$\hat{E}[S_t^* | \mathcal{F}_u] = \frac{E[S_t^* U_T L_T | \mathcal{F}_u]}{E[U_T L_T | \mathcal{F}_u]} = \frac{E[S_t^* U_T | \mathcal{F}_u]}{E[U_T | \mathcal{F}_u]} \frac{E[L_T | \mathcal{F}_u]}{E[L_T | \mathcal{F}_u]} = E^*[S_t^* | \mathcal{F}_u] = S_u^*,$$

using that S^* is a P^* -martingale, and so each \hat{P} is an equivalent martingale measure. Due to this non-uniqueness of the equivalent martingale measure, contracts cannot in general be priced uniquely by no-arbitrage pricing theory alone. Actually, all prices

$$\pi(\hat{P}) = E^{\hat{P}}[H]$$

for the claims (2.12)-(2.13) obtained by admissible choices of h are consistent with absence of arbitrage. Furthermore, (B, S) and N are independent under \hat{P} and, by the Girsanov theorem, the process M^h defined by

$$M_t^h = N_t - \int_0^t \lambda_u(1 + h_u) du$$

is an (\mathbf{F}, \hat{P}) -martingale. The term L_T in (2.15) essentially changes the hazard rate in the model to $\mu_{\cdot+t}(1 + h_t)$. In particular, the measure P^* defined by (2.3) can be obtained from (2.15) with $h \equiv 0$. Note that the change of measure from P to P^* does not affect the distribution of N and that M is an (\mathbf{F}, P^*) -martingale.

Throughout this paper we will apply the specific martingale measure P^* defined by (2.3) which is also known as the *minimal martingale measure*, cf Schweizer (1991, 1995). This particular measure is normally applied to pricing of unit-linked contracts, the motivation being the insurer's risk neutrality with respect to mortality, see e.g. Aase and Persson (1994). Thus, we consider the probability space $(\Omega, \mathcal{F}, P^*)$ endowed with the filtration \mathbf{F} . Note that \mathbf{F} is equivalently generated by the P^* -martingales S^* and M :

$$\mathcal{F}_t = \sigma\{S_u^*, M_u, 0 \leq u \leq t\}.$$

In the analysis below, we could equally well apply any of the martingale measures \hat{P} defined by (2.15) for admissible choices of h . In this case we would obtain similar results with the hazard rate function μ replaced by $(1+h)\mu$ and M replaced by M^h . However, there do exist martingale measures which do not preserve independence between (B, S) and N , and such choices of martingale measures would certainly complicate calculations in Section 4 greatly.

3. A REVIEW OF RISK-MINIMIZATION

In the previous section, a model describing a financial market and an insurance portfolio was introduced. It was pointed out that this market is incomplete in the sense that contingent claims cannot in general be perfectly duplicated by means of self-financing strategies. In this section, we briefly review some results on the theory of risk-minimization, dealing with incomplete as well as complete markets.

Föllmer and Sondermann (1986) extended the established theory for complete markets to the case of an incomplete market. By introducing the concept of *mean-self-financing* strategies they obtained optimal strategies in the sense of minimization of a certain squared error process. In Föllmer and Schweizer (1988) a discrete time multiperiod model was examined within this set-up, and they obtained recursion formulas describing the optimal strategies. The theory has been further developed by Schweizer (1991, 1994). Föllmer and Sondermann (1986) originally considered the case where the original probability measure P is in fact a martingale measure. Schweizer (1991) introduced the concept of *local risk-minimization* for price processes which are only semimartingales and this criterion was similar to performing risk-minimization using the minimal martingale measure P^* .

Recall the space $(\Omega, \mathcal{F}, P^*)$, filtration \mathbf{F} and the (\mathbf{F}, P^*) -martingales S^* and M . The deflated value process V^φ is defined by

$$V_t^\varphi = \hat{V}_t^\varphi B_t^{-1} = \xi_t S_t^* + \eta_t, \quad (3.1)$$

where \hat{V}^φ is given by (2.5). From Föllmer and Sondermann (1986) and Schweizer (1994) we have a slightly modified definition of strategies and the value process. Introducing the space $\mathcal{L}^2(P_S^*)$ of \mathbf{F} -predictable square-integrable processes ξ satisfying

$$\mathbf{E}^* \left[\int_0^T \xi_u^2 d\langle S^* \rangle_u \right] < \infty,$$

they state:

Definition 3.1 An \mathbf{F} -strategy is any process $\varphi = (\xi, \eta)$ with $\xi \in \mathcal{L}^2(P_S^*)$ and η \mathbf{F} -adapted such that the (deflated) value process V^φ is cadlag and $V_t^\varphi \in \mathcal{L}^2(P^*)$ for all t .

The cost process C^φ associated with the strategy φ is defined by

$$C_t^\varphi = V_t^\varphi - \int_0^t \xi_u dS_u^*, \quad (3.2)$$

and the risk process R^φ of φ is defined by

$$R_t^\varphi = \mathbb{E}^* \left[(C_T^\varphi - C_t^\varphi)^2 | \mathcal{F}_t \right]. \quad (3.3)$$

In this definition, the notion *risk process* is attached to the conditioned expected squared value of future costs. This usage differs from the traditional actuarial one, where “risk process” would typically denote the cash flow of premiums and benefits

The cost C^φ is the value of the portfolio less the accumulated income from the asset S . The total costs C_t^φ incurred in $[0, t]$ decompose into the costs incurred during $(0, t]$ and an initial cost $C_0^\varphi = V_0^\varphi$, which typically is greater than zero. A strategy is said to be *mean-self-financing* if the cost process $C^\varphi = (C_t^\varphi)_{0 \leq t \leq T}$ is an (\mathbf{F}, P^*) -martingale. Furthermore, it should be noted that the strategy $\varphi = (\xi, \eta)$ is *self-financing* if and only if

$$V_t^\varphi = V_0^\varphi + \int_0^t \xi_u dS_u^*,$$

that is, if and only if $C_t^\varphi = C_0^\varphi = V_0^\varphi$ P^* -a.s.

Let us now turn to the problem of characterizing the optimal strategies. We consider a general contingent claim specifying the \mathcal{F}_T -payment H at time T and focus on *admissible* strategies φ satisfying

$$V_T^\varphi = H \text{ a.s.}$$

By means of admissible strategies, the hedger is able to generate the contingent claim, but only at some cost defined by C_T^φ . In particular, for attainable claims, $C_T^\varphi = C_0^\varphi = V_0^\varphi$ is known at time 0.

As a first result, admissible strategies minimizing the mean squared error R_0^φ defined by (3.3) are determined. For any admissible φ we have

$$C_T^\varphi = V_T^\varphi - \int_0^T \xi_u dS_u^* = H - \int_0^T \xi_u dS_u^*, \quad (3.4)$$

hence

$$R_0^\varphi = \mathbb{E}^* \left[(C_T^\varphi - C_0^\varphi)^2 \right] = \mathbb{E}^* \left[\left(H - \int_0^T \xi_u dS_u^* - C_0^\varphi \right)^2 \right], \quad (3.5)$$

and so R_0^φ is minimized for $C_0^\varphi = E^*[H]$ ($= E^*[C_T^\varphi]$). Thus, we should choose ξ so as to minimize the variance

$$E^* \left[(C_T^\varphi - E^*[C_T^\varphi])^2 \right] \quad (3.6)$$

This criterion does not yield a unique strategy, but it characterizes an entire class of strategies all minimizing the mean squared error (3.5). The non-uniqueness of the optimal admissible strategy is a natural consequence of the simple criterion of minimizing (3.5), which involves only the value of the cost process C^φ at time T , given by (3.4). Furthermore, note that $H = \xi_T S_T^* + \eta_T$, which does not depend on $(\eta_t)_{0 \leq t < T}$. Thus, we should not expect the minimization criterion associated with the squared error (3.5) to impose any constraints on the number of bonds held in the time interval $(0, T)$.

The construction of the strategies is based on an application of the Galtchouk-Kunita-Watanabe decomposition, see Föllmer and Sondermann (1986). Defining the *intrinsic value process* V^* by

$$V_t^* = E^*[H | \mathcal{F}_t],$$

and noting that V^* is an (\mathbf{F}, P^*) -martingale, the Galtchouk-Kunita-Watanabe decomposition theorem allows us to write V_t^* uniquely in the form

$$V_t^* = E^*[H] + \int_0^t \xi_u^H dS_u^* + L_t^H, \quad (3.7)$$

where $L^H = (L_t^H)_{0 \leq t \leq T}$ is a zero-mean (\mathbf{F}, P^*) -martingale, L^H and S^* are orthogonal, and ξ^H is a predictable process in $\mathcal{L}^2(P_S^*)$. By applying the orthogonality of the martingales L^H and S^* , and using $V_T^* = H$, Föllmer and Sondermann (1986, Theorem 1) prove.

Theorem 3.2 (Föllmer and Sondermann) *An admissible strategy $\varphi = (\xi, \eta)$ has minimal variance*

$$E^* \left[(C_T^\varphi - E^*[C_T^\varphi])^2 \right] = E^* \left[(L_T^H)^2 \right]$$

if and only if $\xi = \xi^H$.

Note that if, furthermore, the number of bonds held at time 0 is determined such that the initial value of the portfolio equals $E^*[H]$, i.e.

$$\eta_0 = E^*[H] - \xi_0 S_0^*,$$

then $R_0^\varphi = E^* \left[(C_T^\varphi - E^*[C_T^\varphi])^2 \right]$. Thus, the variance is interpreted as the minimal obtainable risk.

A more precise result is obtained by looking for admissible strategies, that is $V_T^\varphi = H$, minimizing the *remaining risk*, defined by R_t^φ at any time t . Such strategies are said to be *risk-minimizing*. Now fix some admissible strategy φ . When considering the remaining risk R_t^φ at some point in time t , only admissible strategies $\tilde{\varphi}$ coinciding with φ in the interval $[0, t)$ should be compared. This condition ensures, that the cost processes are given by the same value $C_t^\varphi = C_t^{\tilde{\varphi}}$ at the time of consideration. In this case the strategy $\tilde{\varphi}$ is said to be an *admissible continuation* of φ at time t , see Föllmer and Sondermann (1986) for more details. The risk-minimizing strategy, minimizing the risk process $(R_t^\varphi)_{0 \leq t \leq T}$ is determined by Föllmer and Sondermann (1986, Theorem 2).

Theorem 3.3 (Föllmer and Sondermann) *There exists a unique admissible risk-minimizing strategy $\varphi = (\xi, \eta)$ given by*

$$(\xi_t, \eta_t) = (\xi_t^H, V_t^* - \xi_t^H S_t^*), \quad 0 \leq t \leq T$$

The associated risk process is given by $R_t^\varphi = \mathbb{E}^ \left[(L_T^H - L_t^H)^2 | \mathcal{F}_t \right]$*

The risk process associated with the risk-minimizing strategy is also called the *intrinsic risk process*

4. UNIT-LINKED CONTRACTS WITH SINGLE PREMIUM

In this section, we apply the technique of risk-minimization in the investigation of the insurance contracts introduced in Section 2. An important step will be the construction of the decomposition (3.7) of the present values (2.12)-(2.13). Having determined this, risk-minimizing strategies and the intrinsic risk process associated with the pure endowment and the deferred term insurance contract can be determined by Theorems 3.2 and 3.3.

From the classical actuarial theory it is known that in the case of fixed premiums and sum insured, the “relative risk” associated with the portfolio decreases as the size l_λ of the portfolio increases. More precisely, this means that the ratio between the standard deviation of the present value of all payments and the size of the portfolio l_λ will converge to 0 as l_λ is increased. In the present set-up, we cannot expect such results since the payments associated with different insurance contracts are now linked to the same asset and hence are no longer stochastically independent. However the initial intrinsic risk R_0 can be taken as a measure of the risk associated with the non-hedgeable part of the claims, and we will accordingly examine the ratio $\sqrt{R_0}/l_\lambda$.

4.1. The pure endowment

Consider the claim with present value H in (2.12);

$$H = g(S_T)B_T^{-1}(l_\lambda - N_T), \quad (4.1)$$

and define the (deflated) intrinsic value process $V^* = (V_t^*)_{0 \leq t \leq T}$ by

$$V_t^* = E^*[H|\mathcal{F}_t],$$

for all $t \in [0, T]$. Due to the stochastic independence between N and (B, S) under P^* , we get

$$V_t^* = E^*[(l_\lambda - N_T)|\mathcal{F}_t]B_t^{-1}E^*[g(S_T)B_tB_T^{-1}|\mathcal{F}_t]. \quad (4.2)$$

Here, the first factor is easily determined as

$$\begin{aligned} E^*[(l_\lambda - N_T)|\mathcal{F}_t] &= E^*\left[\sum_{i=1}^{l_\lambda} I(T_i > T) \middle| \mathcal{F}_t\right] = \sum_{i: T_i > t} E^*[I(T_i > T)|T_i > t] \\ &= \sum_{i: T_i > t} {}_{T-t}p_{\lambda+i} = (l_\lambda - N_t)_{T-t}p_{\lambda+i}, \end{aligned}$$

that is, at any time t the expected number of individuals alive at the time of maturity T is simply the number of survivors at time t multiplied by the probability ${}_{T-t}p_{\lambda+i}$ of survival to T for an individual, conditional on his/her survival to t . The second factor in (4.2) corresponds to the representation (2.9) of the unique arbitrage-free price process associated with the simple T -claim $g(S_T)$ in the complete model with filtration \mathbf{G} . In the present model, the insured lives are included in the filtration \mathbf{F} , and arbitrage-free prices are in general not unique. However, as N and (B, S) are stochastically independent, the conditional distribution of (B, S) given \mathcal{F}_t does not depend on information concerning the insured lives \mathcal{H}_t and thus

$$E^*[g(S_T)B_tB_T^{-1}|\mathcal{F}_t] = E^*[g(S_T)B_tB_T^{-1}|\mathcal{G}_t] = F^g(t, S_t),$$

where the function $F^g(t, s)$ satisfies the same second order PDE as in the complete case (2.8). Consequently, we arrive at the expression

$$V_t^* = (l_\lambda - N_t)_{T-t}p_{\lambda+i}B_t^{-1}F^g(t, S_t). \quad (4.3)$$

The process V^* can be interpreted as the market value process associated with the entire portfolio of pure endowment contracts, using the pricing rule P^* . In particular, the initial value $V_0^* = l_{\lambda T}p_{\lambda}F^g(0, S_0)$ is a natural candidate for the single premium for the entire portfolio. This specific choice of single premium would be in accordance with the well established actuarial principle of equivalence (stating that premiums and benefits should balance on average), but exercised under the martingale measure P^*

Applying the Itô formula to (4.3), we get

$$V_t^* = V_0^* + \int_0^t (l_\lambda - N_{u-}) B_u^{-1} F^g(u, S_u)_{T-u} p_{\lambda+u} l_{\lambda+u} du \\ + \int_0^t (l_\lambda - N_{u-})_{T-u} p_{\lambda+u} d(B_u^{-1} F^g(u, S_u)) + \sum_{0 < u \leq t} (V_u^* - V_{u-}^*).$$

To determine the integral involving $d(B_t^{-1} F^g(t, S_t))$, recall the definition of the deflated price process $S_t^* = S_t B_t^{-1}$, implying that

$$dS_t = S_t^* dB_t + B_t dS_t^* = S_t r_t dt + B_t dS_t^*.$$

Using the Itô-formula and the PDE (2.8), it is seen that

$$d(B_t^{-1} F^g(t, S_t)) = -r(t, S_t) B_t^{-1} F^g(t, S_t) dt \\ + B_t^{-1} \left(F_t^g(t, S_t) dt + F_{S_t}^g(t, S_t) dS_t + \frac{1}{2} F_{S_t S_t}^g(t, S_t) \sigma(t, S_t)^2 S_t^2 dt \right) \\ = F_{S_t}^g(t, S_t) dS_t^*$$

Also, since

$$\sum_{0 < u \leq t} (V_u^* - V_{u-}^*) = - \int_0^t B_u^{-1} F^g(u, S_u)_{T-u} p_{\lambda+u} dN_u,$$

we obtain.

Lemma 4.1 *For the contingent claim H in (4.1) the process V^* defined by $V_t^* = \mathbb{E}^*[H | \mathcal{F}_t]$ has the decomposition*

$$V_t^* = V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t \nu_u^H dM_u,$$

where (ξ^H, ν^H) are given by

$$\xi_t^H = (l_\lambda - N_{t-})_{T-t} p_{\lambda+t} F_{S_t}^g(t, S_t), \quad (4.4)$$

$$\nu_t^H = -B_t^{-1} F^g(t, S_t)_{T-t} p_{\lambda+t}, \quad 0 \leq t \leq T. \quad (4.5)$$

Admissible strategies minimizing the variance

$$\mathbb{E}^* \left[(C_T^\varphi - \mathbb{E}^*[C_T^\varphi])^2 \right] \quad (4.6)$$

can now be characterized by applying Theorem 3.2 and Lemma 4.1. By use of the Fubini theorem, the associated minimum obtainable variance is rewritten as

$$\begin{aligned}
 \mathbb{E}^* \left[\left(\int_0^T \nu_u^H dM_u \right)^2 \right] &= \mathbb{E}^* \left[\int_0^T (\nu_u^H)^2 d\langle M \rangle_u \right] \\
 &= \mathbb{E}^* \left[\int_0^T (B_u^{-1} F^g(u, S_u))_{T-u}^2 \lambda_u du \right] \\
 &= \int_0^T \mathbb{E}^* \left[(B_u^{-1} F^g(u, S_u))^2 \right]_{T-u} p_{\lambda+u}^2 \mathbb{E}^* [(I_\lambda - N_u) \mu_{\lambda+u}] du \\
 &= \int_0^T \mathbb{E}^* \left[(B_u^{-1} F^g(u, S_u))^2 \right]_{T-u} p_{\lambda+u}^2 l_{\lambda+u} \mu_{\lambda+u} du \\
 &= l_{\lambda+T} p_{\lambda} \int_0^T \mathbb{E}^* \left[(B_u^{-1} F^g(u, S_u))^2 \right]_{T-u} p_{\lambda+u} \mu_{\lambda+u} du \quad (4.7)
 \end{aligned}$$

Thus we have obtained

Theorem 4.2 Consider the pure endowment given by the contingent claim H in (4.1). Admissible strategies φ^* minimizing the variance (4.6) are determined by

$$\begin{aligned}
 \xi_t^* &= (I_\lambda - N_{t-})_{T-t} p_{\lambda+t} F_\lambda^g(t, S_t), \quad 0 \leq t \leq T, \\
 \eta_T^* &= H - \xi_T^* S_T^*.
 \end{aligned}$$

The minimal variance is given by (4.7)

The insurance company is able to reduce the total risk associated with the portfolio of unit-linked insurance contracts to the “intrinsic risk” R_0^φ , by following a strategy according to Theorem 4.2 which also satisfies $C_0^\varphi = \mathbb{E}^*[H]$. In particular, it is seen that R_0^φ is proportional to l_λ , implying that the ratio between $\sqrt{R_0^\varphi}$ and l_λ converges to 0 as l_λ converges to infinity.

Before determining the unique risk-minimizing strategy, we present one specific strategy from Theorem 4.2, see Föllmer and Sondermann (1986, Example 1).

Example 4.3 We shall present one strategy φ that does not require any extra investments during the time interval $(0, T)$. It is self-financing on $(0, T)$, followed by a possible extra payment at time T . Define the strategy by

$$\xi_t = \xi_t^H, \quad 0 \leq t \leq T, \quad (4.8)$$

$$\eta_t = \mathbb{E}^*[H] + \int_0^t \xi_u dS_u^* - \xi_t S_t^*, \quad 0 \leq t < T, \quad (4.9)$$

and $\eta_T = H - \xi_T S_T^*$. By definition, this strategy is self-financing on the interval $(0, T)$. Substituting the decomposition of H from Lemma 4.1 into the expression of η_T , we get

$$\eta_T = H - \xi_T S_T^* = E^*[H] + \int_0^T \xi_u^H dS_u^* + \int_0^T \nu_u^H dM_u - \xi_T S_T^*$$

Likewise we have from (4.9) that

$$\eta_{T-} = E^*[H] + \int_0^{T-} \xi_u dS_u^* - \xi_{T-} S_{T-}^* = E^*[H] + \int_0^T \xi_u dS_u^* - \xi_T S_T^*,$$

which proves that

$$\eta_T - \eta_{T-} = \int_0^T \nu_u^H dM_u = L_T^H$$

Thus, the loss L_T^H is an extra payment/investment to be made at time T in order to satisfy the condition of admissibility.

The variance-minimizing trading strategy in Example 4.3 represents a very simple dynamic portfolio strategy from the point of view of the insurer. According to this strategy he is to make an initial investment at time 0 in stocks and bonds. During the time interval $(0, T)$ this portfolio is then adjusted continuously without any additional inflow or outflow of capital as defined by the equations (4.8)-(4.9). At the term T the insurance company now provides the difference L_T^H between the claim H and the value V_{T-}^φ of the portfolio. However, there are reasons why this strategy should not be applied. Indeed, it does minimize the variance or the initial intrinsic risk, but at any time t during the insurance period the value V_t^φ of the portfolio will in general not equal the conditional expected present value of the claim V_t^* . Since this difference may be substantial due to adverse development within the insurance portfolio, one should at least require that the value of the portfolio equals V_t^* in order to enhance the solvency of the insurer. This additional requirement, in addition with the minimal variance criterion, is actually sufficient to determine the unique risk-minimizing strategy φ . The associated intrinsic risk process is described in Theorem 3.3, and we get

$$\begin{aligned} E^* \left[(L_T^H - L_t^H)^2 \mid \mathcal{F}_t \right] &= E^* \left[\left(\int_t^T \nu_u^H dM_u \right)^2 \mid \mathcal{F}_t \right] = E^* \left[\int_t^T (\nu_u^H)^2 \lambda_u du \mid \mathcal{F}_t \right] \\ &= \int_t^T E^* \left[(\nu_u^H)^2 \mid \mathcal{F}_t \right] E^* \left[(l_\lambda - N_u) \mu_{\lambda+u} \mid \mathcal{F}_t \right] du \\ &= (l_\lambda - N_t) \int_t^T E^* \left[(\nu_u^H)^2 \mid \mathcal{F}_t \right] {}_{u-t}p_{\lambda+t} \mu_{\lambda+u} du. \end{aligned} \quad (4.10)$$

From Theorem 3.3 we now have:

Theorem 4.4 *For the pure endowment given by the contingent claim (4.1) the unique admissible risk-minimizing strategy is given by*

$$\begin{aligned}\xi_t^* &= (I_\lambda - N_{t-}) {}_{T-t}p_{\lambda+t} F_t^g(t, S_t), \\ \eta_t^* &= (I_\lambda - N_t) {}_{T-t}p_{\lambda+t} B_t^{-1} F^g(t, S_t) - \xi_t^* S_t^*, \quad 0 \leq t \leq T\end{aligned}$$

The intrinsic risk process R^{φ^*} is given by (4.10)

In the model the insurance company is allowed to trade the assets S and B continuously, thus being able to hedge all contingent claims involving these assets only. This eliminates a part of the total uncertainty, leaving only the uncertainty of “not knowing how many of the insured persons will die in the insurance period”. The latter is described by the martingale M , which generates the insurer’s loss L^H .

$$dL_t^H = \nu_t^H dM_t = -B_t^{-1} F^g(t, S_t) {}_{T-t}p_{\lambda+t} (dN_t - \lambda_t dt). \quad (4.11)$$

The insurer adjusts his trading strategy according to the conditional expected number of insured persons surviving the insurance period. During the infinitesimal time interval $[t, t + dt)$ the insurer will experience the gain dM_t multiplied by the term $B_t^{-1} F^g(t, S_t) {}_{T-t}p_{\lambda+t}$, the latter denoting the price at time t of one security with payment $g(S_T)$ at time T contingent on the survival of some individual. That is, a death will produce an immediate gain for the insurer due to the downwards adjustment of the expected number of survivors, whereas no deaths will cause a small loss. The expression (4.11) for the loss is similar to the one obtained by Norberg (1992) for general payment streams, using a quite different approach. With this terminology, the term $(\nu_t^H B_t)$ is recognized as the *sum at risk* at time t .

We now turn to some examples in the case of constant deterministic short rate of interest, constant drift term α , and volatility parameter σ on S . We will investigate three different contract functions: pure unit-linked, where $g(s) = s$; unit-linked with guarantee, where $g(s) = \max(s, K)$; and the case of deterministic benefits, $g(s) = K$.

Example 4.5 Consider a standard Black-Scholes market, where all coefficients r , α and σ are constant. Let the contract function be of the simple form $g(s) = s$, i.e. the insured is to be paid the value of the stock at the maturity date. In this case, the process $(F^g(t, S_t))_{0 \leq t \leq T}$ is easily determined as

$$F^g(t, S_t) = E^* \left[e^{-r(T-t)} S_T | \mathcal{F}_t \right] = S_t,$$

implying that $F_s^g(t, S_t) = 1$. The intrinsic value process is

$$V_t^* = (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} e^{-rt} S_t = (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} S_t^*,$$

and in particular $V_0^* = l_\lambda {}_T p_\lambda S_0^*$. From Theorem 4.4 we have the unique risk-minimizing strategy

$$(\xi_t, \eta_t) = ((l_\lambda - N_{t-}) {}_{T-t}p_{\lambda+t}, -\Delta N_t {}_{T-t}p_{\lambda+t} S_t^*), \tag{4.12}$$

where $\Delta N_t = N_t - N_{t-}$. Finally, we have the aggregated loss

$$L_T^H = - \int_0^T S_u^* {}_{T-u}p_{\lambda+u} dM_u,$$

and the intrinsic risk process

$$\begin{aligned} R_t^\varphi &= (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} \int_t^T E^* \left[(S_u^*)^2 | \mathcal{F}_t \right] {}_{T-u}p_{\lambda+u} \mu_{\lambda+u} du \\ &= (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} (S_t^*)^2 \int_t^T e^{\sigma^2(u-t)} {}_{T-u}p_{\lambda+u} \mu_{\lambda+u} du \end{aligned}$$

The risk-minimizing strategy given by (4.12) is easy to interpret: at any time t the insurance company should hold a number of stocks, corresponding to the expected number of surviving individuals. Since the number of stocks is controlled by a predictable process ξ , some adjustments are made each time a death occur within the portfolio in order to ensure that $V_t^* = V_t^\varphi$ for all t . This is described by the adapted process η , which denotes the amount to be cashed by the insurance company in connection with the observed death.

Example 4.6 Now consider the contract function $g(s) = \max(s, K)$, where K is some non-negative constant. Note, that $K = 0$ is just the case treated above in Example 4.5. As in the previous example, prices are described by a standard Black-Scholes market.

Writing the contract function $\max(s, K)$ on the form $K + (s - K)^+$, the process $(F^g(t, S_t))_{0 \leq t \leq T}$ can be evaluated by means of the well-known Black-Scholes formula

$$\begin{aligned} F^g(t, S_t) &= E^* \left[e^{-r(T-t)} (K + (S_T - K)^+) | \mathcal{F}_t \right] \\ &= Ke^{-r(T-t)} + \left(S_t \Phi(z_t) - Ke^{-r(T-t)} \Phi \left(z_t - \sigma \sqrt{T-t} \right) \right) \\ &= Ke^{-r(T-t)} \Phi \left(-z_t + \sigma \sqrt{T-t} \right) + S_t \Phi(z_t), \end{aligned} \tag{4.13}$$

where Φ is the standard normal distribution function and

$$z_t = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}.$$

In particular, the first order partial derivative is $F_s^g(t, S_t) = \Phi(z_t)$. Thus, the risk-minimizing strategy is given by

$$\xi_t = (l_\lambda - N_{t-}) {}_{T-t}p_{\lambda+t} \Phi(z_t), \quad (4.14)$$

$$\begin{aligned} \eta_t &= (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} e^{-rt} F^g(t, S_t) - (l_\lambda - N_{t-}) {}_{T-t}p_{\lambda+t} \Phi(z_t) S_t^* \\ &= (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} K e^{-rT} \Phi\left(-z_t + \sigma\sqrt{T-t}\right) \\ &\quad - \Delta N_t {}_{T-t}p_{\lambda+t} \Phi(z_t) S_t^*, \end{aligned} \quad (4.15)$$

and the intrinsic risk process R^φ is now given by

$$R_t^\varphi = (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} \int_t^T \mathbb{E}^* \left[(e^{-ru} F^g(u, S_u))^2 | \mathcal{F}_t \right] {}_{T-u}p_{\lambda+u} \mu_{\lambda+u} du,$$

with F^g defined by (4.13).

Example 4.7 As a last example, consider the case of deterministic benefits, that is $g(S_T) = K$ for some non-negative K . Here, the risk-minimizing strategy is given by

$$(\xi_t, \eta_t) = (0, (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} K e^{-rT}), \quad (4.16)$$

and the intrinsic risk process is

$$\begin{aligned} R_t^\varphi &= (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} \int_t^T K^2 e^{-2rT} {}_{T-u}p_{\lambda+u} \mu_{\lambda+u} du \\ &= (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} (1 - {}_{T-t}p_{\lambda+t}) K^2 e^{-2rT}. \end{aligned}$$

In Example 4.5-4.7, we have determined risk-minimizing strategies for three different contract functions, in the setting of a standard Black-Scholes market. The strategies are associated with an entire portfolio l_λ ; single-life strategies are obtained by specializing to $l_\lambda = 1$. For example, the strategy (4.14)-(4.15) for a single life becomes

$$\xi_t = I(T_1 \geq t) {}_{T-t}p_{\lambda+t} \Phi(z_t), \quad (4.17)$$

$$\begin{aligned} \eta_t &= I(T_1 > t) {}_{T-t}p_{\lambda+t} K e^{-rT} \Phi\left(-z_t + \sigma\sqrt{T-t}\right) \\ &\quad - I(T_1 = t) {}_{T-t}p_{\lambda+t} \Phi(z_t) S_t^*, \end{aligned} \quad (4.18)$$

and the intrinsic value process is

$$V_t^* = I(T_1 > t) {}_{T-t}p_{\lambda+t} \left(K e^{-rT} \Phi\left(-z_t + \sigma\sqrt{T-t}\right) + S_t^* \Phi(z_t) \right)$$

The process V^* is in a sense similar to a traditional prospective reserve. First, an indicator function appears, which guarantees that the reserve is only different from zero as long as the policyholder is still alive. The rest of the

terms are interpreted as the conditional expected present value of the insurance benefit, given the policyholder is alive at t . Provided that the policyholder survives to the maturity date, that is $T_1 > T$, the risk-minimizing strategy (4.17)-(4.18) for a single life reduces to the strategy

$$(\xi_t^0, \eta_t^0) = \left({}_{T-t}p_{\lambda+t} \Phi(z_t), {}_{T-t}p_{\lambda+t} K e^{-tT} \Phi\left(-z_t + \sigma \sqrt{T-t}\right) \right),$$

which is exactly equal to the corresponding duplicating strategy obtained by Aase and Persson (1994). The result (4.17)-(4.18) is to be interpreted as follows: As long as the policyholder is alive, the insurance company should hold a portfolio, where the number of stocks is determined as the probability ${}_{T-t}p_{\lambda+t}$ of survival to T conditioned on survival to t times the factor $\Phi(z_t)$; the latter is recognized as the hedge from the Black-Scholes formula of a European Call Option. If the policyholder dies before the maturity date T , the insurer immediately cashes the reserve, as is apparent in the definition of η . These interpretations are easily carried over to the situation where the insurance portfolio consists of more than one individual. In this case, the numbers of stocks and bonds held are adjusted in accordance with the conditional expected number of survivors to T , that is $(l_\lambda - N_t) {}_{T-t}p_{\lambda+t}$. Thus, the risk-minimizing strategies reflect the actual development in the insurance portfolio, and bring to the surface the uncertainty associated with the insured lives. For example, we obtain expressions for the intrinsic risk processes, which serve as characterizations of the non-hedgeable risk inherent in a portfolio of unit-linked contracts. In Section 6 we present some numerical results in the set-up of Examples 4.5 and 4.6 obtained by Monte Carlo simulation.

4.2. Term insurance

Now consider the term insurance with single premium π^1 paid at time 0. The payments generated by this contract are described by the discounted claim

$$H_T = \int_0^T g(u, S_u) B_u^{-1} dN_u \tag{4.19}$$

An important step is the construction of the decomposition for the intrinsic value process for H_T . First of all, observe that

$$\begin{aligned} V_t^* &= E^*[H_T | \mathcal{F}_t] = \int_0^t g(u, S_u) B_u^{-1} dN_u + E^* \left[\int_t^T g(u, S_u) B_u^{-1} dN_u | \mathcal{F}_t \right] \\ &= \int_0^t g(u, S_u) B_u^{-1} dN_u + \int_t^T B_t^{-1} F^{g_u}(t, S_t) (l_x - N_t) {}_{u-t}p_{\lambda+t} \mu_{\lambda+u} du, \end{aligned}$$

where

$$F^{gu}(t, S_t) = E^* \left[e^{-\int_t^u r_\tau d\tau} g(u, S_u) \mid \mathcal{G}_t \right]$$

is the unique arbitrage-free price at time t of the simple u -claim $g(u, S_u)$ in the complete model with filtration \mathbf{G} . Secondly, by calculations similar to the ones in the previous section, we see that

$$d(B_t^{-1} F^{gu}(t, S_t)) = F_s^{gu}(t, S_t) dS_t^*$$

Using the general Itô formula and the Fubini Theorem for Itô processes, see Ikeda and Watanabe (1981), V^* can now be rewritten as

$$\begin{aligned} V_t^* &= V_0^* + \int_0^t (-B_\tau^{-1} F^{g_\tau}(\tau, S_\tau) \mu_{\lambda+\tau} (l_\lambda - N_\tau)) d\tau \\ &\quad + \int_0^t \left(g(\tau, S_\tau) B_\tau^{-1} - \int_\tau^T B_\tau^{-1} F^{gu}(\tau, S_\tau)_{u-\tau} p_{\lambda+\tau} \mu_{\lambda+u} du \right) dN_\tau \\ &\quad + \int_0^t \left(\int_\tau^T B_\tau^{-1} F^{gu}(\tau, S_\tau)_{u-\tau} p_{\lambda+\tau} \mu_{\lambda+u} du \right) (l_\lambda - N_{\tau-}) \mu_{\lambda+\tau} d\tau \\ &\quad + \int_0^t \left((l_\lambda - N_{\tau-}) \int_\tau^T F_s^{gu}(\tau, S_\tau)_{u-\tau} p_{\lambda+\tau} \mu_{\lambda+u} du \right) dS_\tau^*. \end{aligned}$$

Upon gathering terms, and using $F^{g_t}(t, S_t) = g(t, S_t)$, we obtain a decomposition corresponding to Lemma 4.1:

Lemma 4.8 *For the claim H_T in (4.19) the process V^* defined by $V_t^* = E^*[H_T | \mathcal{F}_t]$ has the decomposition*

$$V_t^* = V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t \nu_u^H dM_u,$$

where (ξ^H, ν^H) are given by

$$\xi_t^H = (l_\lambda - N_{t-}) \int_t^T_{u-t} p_{\lambda+t} \mu_{\lambda+u} F_s^{gu}(t, S_t) du, \tag{4.20}$$

$$\nu_t^H = g(t, S_t) B_t^{-1} - \int_t^T F^{gu}(t, S_t) B_t^{-1} p_{\lambda+t} \mu_{\lambda+u} du \tag{4.21}$$

Using Theorem 3.3 we have now proved:

Theorem 4.9 For the term insurance given by the contingent claim (4.19) the unique admissible risk-minimizing strategy is given by

$$\begin{aligned}\xi_t^* &= (I_\lambda - N_{t-}) \int_t^T F_t^{g_u}(t, S_t) {}_{u-t}p_{\lambda+t} \mu_{\lambda+u} du, \\ \eta_t^* &= \int_0^t g(u, S_u) B_u^{-1} dN_u + (I_\lambda - N_t) \int_t^T B_t^{-1} F_t^{g_u}(t, S_t) {}_{u-t}p_{\lambda+t} \mu_{\lambda+u} du \\ &\quad - \xi_t^* S_t^*, \quad 0 \leq t \leq T.\end{aligned}$$

The intrinsic risk process R^{φ^*} is given by

$$R_t^{\varphi^*} = (I_\lambda - N_t) \int_t^T E^* \left[(\nu_u^H)^2 | \mathcal{F}_t \right] {}_{u-t}p_{\lambda+t} \mu_{\lambda+u} du,$$

where ν^H is taken from (4.21).

To give the resulting portfolio an interpretation, note that $\varphi = (\xi, \eta)$ is determined such that

$$V_t^\varphi = \int_0^t g(u, S_u) B_u^{-1} dN_u + E^* \left[\int_t^T g(u, S_u) B_u^{-1} dN_u | \mathcal{F}_t \right].$$

Thus, V_t^φ is determined as the sum of the benefits set aside to deaths already occurred and the expected discounted value of payments associated with future deaths

As in the case of the pure endowment, the term ν_t^H denotes the immediate loss due to the death of one of the insured persons. Here, the insurer has to set aside the sum insured $g(t, S_t)$ immediately upon a death within the portfolio at time t . In connection with the incurred death, the insurance company adjusts its expectations regarding the further development of the insurance portfolio. Since the number of survivors has been reduced by one, the insurer now reduces his reserves by the amount

$$\int_t^T F_t^{g_u}(t, S_t) B_t^{-1} {}_{u-t}p_{\lambda+t} \mu_{\lambda+u} du,$$

which is the expected discounted value of future payments conditional on survival to time t .

Example 4.10 Consider a unit-linked term insurance contract with guarantee in the case of a standard Black-Scholes market. Let the contract function be on the form $g(u, s) = \max(s, Ke^{hu})$, that is the guarantee is adjusted in

accordance with some constant force of inflation δ . The functions $F^{gu}(t, s)$ are determined by

$$F^{gu}(t, S_t) = Ke^{\delta u} e^{-r(u-t)} \Phi\left(-z_t^{(u)} + \sigma\sqrt{u-t}\right) + S_t \Phi(z_t^{(u)}), \quad (4.22)$$

with

$$z_t^{(u)} = \frac{\log(S_t/Ke^{\delta u}) + (r + \sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}.$$

Using Theorem 4.9 we find the risk-minimizing strategy

$$\begin{aligned} \xi_t &= (l_\lambda - N_{t-}) \int_t^T u-t p_{\lambda+t} \mu_{\lambda+u} \Phi(z_t^{(u)}) du, \\ \eta_t &= (l_\lambda - N_t) \int_t^T u-t p_{\lambda+t} \mu_{\lambda+u} Ke^{-(r-\delta)u} \Phi(-z_t^{(u)} + \sigma\sqrt{u-t}) du \\ &\quad + \int_0^t g(u, S_u) B_u^{-1} dN_u - \Delta N_t \int_t^T u-t p_{\lambda+t} \mu_{\lambda+u} \Phi(z_t^{(u)}) S_t^* du. \end{aligned}$$

The intrinsic risk process is also determined by that theorem upon inserting the functions F^{gu} from (4.22) in (4.21).

5. EXTENDING THE FINANCIAL MARKET

In the previous sections we have analyzed a model where the financial market consists of two assets only, namely a risk-free asset B (the bond) and a risky asset S (the stock). That model, which also describes the development of a given portfolio of insured lives, is incomplete. We considered two different basic types of insurance products, and in both cases risk-minimizing strategies were constructed and the corresponding intrinsic risk processes were determined. Due to incompleteness, the risk could not be eliminated completely and thus some uncertainty regarding the course of the insured lives in the portfolio (the intrinsic risk) remains with the insurance company.

The present section is devoted to a brief investigation of the situation where the financial market is extended by a third tradeable asset that is related to the specific insured lives. As in Section 4, focus will be on the pure endowment, but all results can be repeated for the term insurance and the endowment insurance as well. Furthermore we restrict the analysis to the case where the risk-free interest rate r is assumed to be constant.

In addition to the assets (B, S) with prices processes defined by (2.1) and (2.2), respectively, we introduce an asset with price process $Z = (Z_t)_{0 \leq t \leq T}$, where

$$Z_t = (l_\lambda - N_t) {}_{T-t}p_{\lambda+t} e^{-r(T-t)}. \quad (5.1)$$

The initial value $Z_0 = l_{\lambda} T p_{\lambda} e^{-iT}$ is equal to the price at time 0 of l_{λ} standard pure endowment contracts with sum insured 1 calculated on a valuation basis consisting of the mortality hazard function μ_{λ} and the risk-free interest rate r . Assuming that premiums are paid as a single premium at time 0, Z_t represents, at any time $0 \leq t \leq T$, the traditional prospective reserve for the portfolio. This reserve is calculated as the conditional expected value of future benefits, given the current number of survivors $(l_{\lambda} - N_t)$. The introduction of this extra investment possibility is motivated by the existence of reinsurance markets, where the direct insurer is able to reduce his total risk by selling some part of the insurance portfolio. Trading on the reinsurance markets will typically be controlled by certain restrictions such as short-selling constraints and upper limits for the amount reinsured. However, in the present formulation we do not impose any restrictions on the trading of any of the three assets

As an example, let us now consider an insurer facing the contingent claim arising from the portfolio of pure endowment unit-linked contracts with sum insured $g(S_T)$ for the portfolio, that is

$$H = (l_{\lambda} - N_T) B_T^{-1} g(S_T), \tag{5.2}$$

and assume that the insurer is allowed to trade continuously on the extended market (B, S, Z) . Note that the asset Z depends on the uncertainty from the insured lives only and evolves independently of the other assets (B, S) . The insurance claim H , however, depends on both sources of uncertainty.

Define the deflated price processes S^* and Z^* by $S^* = S/B$ and $Z^* = Z/B$, respectively. In this new setup a trading strategy is a sufficiently integrable process $\varphi = (\xi, \vartheta, \eta)$, where ξ and ϑ are \mathbf{F} -predictable and η is \mathbf{F} -adapted. At any time t , ϑ_t , ξ_t and η_t are the number of units held of standard pure endowment contracts, stocks, and bonds respectively, and the (discounted) value process V^{φ} is now given by

$$V_t^{\varphi} = \xi_t S_t^* + \vartheta_t Z_t^* + \eta_t$$

We set out by verifying that the measure P^* defined by (2.3) is a martingale measure for S^* and Z^* . It already follows from the calculations in Section 4 that S^* is an (\mathbf{F}, P^*) -martingale, and the process Z^* is obviously also an (\mathbf{F}, P^*) -martingale, since

$$(l_{\lambda} - N_t) T_{-t} p_{\lambda+t} = E^*[(l_{\lambda} - N_T) | \mathcal{F}_t].$$

From the decomposition for the intrinsic value process V^* for (5.2) and a similar representation result for Z^* with respect to M , we obtain

$$V_t^* = V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t \vartheta_u^H dZ_u^*,$$

with

$$(\xi_t^H, \vartheta_t^H) = \left((l_\lambda - N_{t-})_{T-t} p_{\lambda+t} F_\lambda^g(t, S_t), e^{r(T-t)} F^g(t, S_t) \right). \quad (5.3)$$

The intrinsic value process V^* has now been rewritten as a sum of two integrals with respect to the price processes S^* and Z^* . This implies that the contingent claim H associated with the pure endowment can be replicated by means of self-financing strategies in terms of the three assets (B, S, Z) . We can summarize this result by

Theorem 5.1 *Consider the pure endowment with present value (5.2) and assume that standard pure endowment contracts with sum insured 1 are traded freely on a financial market with constant short rate of interest. A self-financing admissible (risk-minimizing) strategy φ^* is given by*

$$\xi_t^* = (l_\lambda - N_{t-})_{T-t} p_{\lambda+t} F_\lambda^g(t, S_t), \quad (5.4)$$

$$\vartheta_t^* = e^{r(T-t)} F^g(t, S_t), \quad (5.5)$$

$$\eta_t^* = V_t^* - \xi_t^* S_t^* - \vartheta_t^* Z_t^*, \quad 0 \leq t \leq T \quad (5.6)$$

Furthermore, the intrinsic risk process R^{φ^*} is identically 0.

The insurer is now able to eliminate the risk associated with the insurance claims completely by following a strategy in accordance with Theorem 5.1. According to this result, the insurer should not only adjust the portfolio of stocks and bonds continuously – also the portfolio of reinsurance contracts should be continuously rebalanced. By some simple calculations involving (5.4) and (5.5), formula (5.6) can be rewritten as

$$\eta_t^* = -(l_\lambda - N_{t-})_{T-t} p_{\lambda+t} F_\lambda^g(t, S_t) S_t^* = -\xi_t^* S_t^*$$

Furthermore, φ^* satisfies $V_t^* = \vartheta_t^* Z_t^*$. Thus, the self-financing (and risk-minimizing) strategy consists of a number ϑ_t^* of shares of standard pure endowment contracts on the portfolio of insured lives, which is adjusted such that the value $\vartheta_t^* Z_t^*$ exactly equals the intrinsic value process V_t^* at any time $t \in [0, T]$. When allowing trading of reinsurance contracts, the criterion of risk-minimization simply states that all risk should be surrendered to the reinsurer. Furthermore, the number of stocks ξ_t^* to be held is the same as in the situation where standard insurance contracts are not traded. By the above calculations, we see that this position is financed by an equivalent short position η_t^* in the risk-free asset, that is, $\eta_t^* = -\xi_t^* S_t^*$.

We end this section by mentioning that P^* would not be a martingale measure for Z^* had we defined the price process $Z = (Z_t)_{0 \leq t \leq T}$ by

$$Z_t = (l_\lambda - N_t)_{T-t} p_{\lambda+t} e^{-\delta(T-t)}.$$

Here, the risk-free interest rate r has been replaced by some first order interest rate $\delta \neq r$. In this case, a martingale measure \hat{P} for (Z^*, S^*) could be defined by (2.15) with $h_t = (\delta - r)/\mu_{\lambda+t}$, provided that $h_t > -1$ for all t . This, in turn, would impose unique arbitrage-free prices for the unit-linked contracts that differ from those computed using the minimal martingale measure P^*

6. NUMERICAL RESULTS

We round off by presenting some Monte Carlo simulation results. We consider the pure endowment where the sum insured is due at the maturity date if the insured is then still alive. Premiums are assumed to be paid as a single premium at time 0. The contract functions from Example 4.5-4.6 will then be examined by evaluating the initial value of the intrinsic risk process V_0^* , the initial intrinsic risk R_0 and the risk-increase associated with some simple (piecewise constant) strategies. Since these quantities are proportional to the size of the portfolio l_λ , recall e.g. (4.3) and (4.10), we consider an insurance portfolio consisting of only one individual, that is, we take $l_\lambda = 1$. Furthermore we take the age of the policyholder to be $x = 45$ upon issue of the contract, and fix the term of the contract to be $T = 15$ years. We use the Gompertz-Makeham hazard function as mortality law of the policyholder

$$\mu_{\lambda+t} = 0.0005 + 0.000075858 \cdot 1.09144^{x+t}, \quad t \geq 0,$$

which is used in the Danish 1982 technical basis for men. With this mortality law, the conditional probability ${}_{15}p_{45}$ of surviving another 15 years given survival to age 45 is 0.8796. The basic financial market is standard Black-Scholes with parameters $\sigma = 0.25$ and $r = 0.06$, that is, the deterministic risk-free interest is 6% and the volatility of the stock is 25%. Furthermore, we take $S_0 = 1$ and $B_0 = 1$. The importance of the volatility parameter is illustrated by considering, in addition, the case of small market volatility ($\sigma = 0.15$) and large market volatility ($\sigma = 0.35$).

The value at time 0 of the intrinsic value process V^* , given by

$$V_0^* = l_\lambda \cdot {}_T P_\lambda \cdot F^g(0, S_0), \tag{6.1}$$

is evaluated by simply inserting the parameters (r, σ) and $S_0 = 1$ in the function F^g determined in Example 4.5 and 4.6. Results are listed in Table 1 for different choices of guarantees, the pure unit-linked insurance corresponds to guarantee $K = 0$. The initial intrinsic risk R_0 is given by

$$R_0 = E^* \left[l_\lambda \cdot {}_T P_\lambda \cdot \int_0^T (e^{-ru} F^g(u, S_u))^2 \cdot {}_{T-u} p_{\lambda+u} \mu_{\lambda+u} du \right], \tag{6.2}$$

and since we have no explicit expression for the expected value of $(F^g(u, S_u))^2$, we apply Monte Carlo simulation combined with numerical integration in order to evaluate (6.2)

The price process for the stock S under P^*

$$S_t = e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t} \quad (6.3)$$

can be simulated by simply simulating a standard Brownian motion and inserting this in (6.3). Let $n = 100$ be the number of time intervals per time unit (one year) and denote by $\Delta t = 1/n$ the mesh of this partition. Also let M denote the number of paths of S to be simulated and let $\varepsilon_j^{(m)}$, $m = 1, \dots, M$, $j = 1, \dots, T \cdot n$ be a sequence of simulated independent standard normal variables. The simulated versions $\hat{S}^{(m)}$ of (6.3) are determined as

$$\hat{S}_k^{(m)} = \exp\left(\left(r - \frac{1}{2}\sigma^2\right)k \cdot \Delta t + \sum_{j=1}^k \sigma \sqrt{\Delta t} \varepsilon_j^{(m)}\right), \quad k = 1, \dots, T \cdot n, m = 1, \dots, M,$$

where $\hat{S}_k^{(m)}$ has same distribution as $S_{k \cdot \Delta t}$. The initial risk R_0 is now approximated numerically by applying Monte Carlo simulation for the integral (6.2) which is discretized using the so-called summed Simpson rule, see e.g. Schwarz (1989). In all computations we apply the step size $\Delta t = 1/100$. In Table 1 we have also presented the estimate for R_0 and the standard error on this estimate based on $M = 300000$ simulated paths for $\sigma = 0.15$ and 0.25 and $M = 500000$ for $\sigma = 0.35$.

TABLE I

THE INITIAL INTRINSIC VALUES AND RISKS ASSOCIATED WITH UNIT-LINKED PURE ENDOWMENT CONTRACTS FOR VARIOUS CHOICES OF GUARANTEE AND VOLATILITY

	Guarantee (K)	V_0^*	R_0	(std.dev.)	$\sqrt{R_0}/V_0^*$
$\sigma = 0.15$	0	0.8796	0.131		0.411
	0.5 exp(rT)	0.8996	0.134	(0.0002)	0.407
	exp(rT)	1.0807	0.173	(0.0002)	0.385
($M = 300000$)	2 exp(rT)	1.7993	0.446	(0.0001)	0.371
$\sigma = 0.25$	0	0.8796	0.194	--	0.501
	0.5 exp(rT)	0.9580	0.205	(0.001)	0.474
	exp(rT)	1.2066	0.261	(0.001)	0.422
($M = 300000$)	2 exp(rT)	1.9161	0.538	(0.001)	0.383
$\sigma = 0.35$	0	0.8796	0.365		0.687
	0.5 exp(rT)	1.0255	0.380	(0.005)	0.608
	exp(rT)	1.3213	0.449	(0.005)	0.513
($M = 500000$)	2 exp(rT)	2.0511	0.743	(0.005)	0.423

The unrestricted risk-minimizing strategies are not applicable in practice, since they are based on the assumption of continuously adjustable portfolios. However, the expressions can be used as a guide in practical portfolio administration. For example, the insurer could apply a piecewise constant strategy on the form

$$\hat{\xi}_t = \sum_{j=1}^J I(t \in (t_{j-1}, t_j]) \xi_{t_{j-1}}, \tag{6.4}$$

where ξ denotes the unrestricted risk-minimizing strategy determined in Section 4. Thus, the portfolio of stocks is adjusted at fixed times $0 = t_0 < t_1 < \dots < t_{J-1} < t_J = T$, as an approximation to the continuously adjustable risk-minimizing strategy. Here, we have chosen $t_j = j$ and $t_j = j/12$, which implies trading once a year and once a month, respectively. In Table 2, we have listed the risk-increase associated with the piecewise constant strategies (6.4), obtained by evaluating the expression

$$\sum_{j=1}^J E^* \left[\int_{t_{j-1}}^{t_j} (\xi_{it} - \xi_{it-1})^2 \sigma^2 S_{it}^{*2} du \right].$$

In Møller (1996) optimal simple strategies are derived by means of some heuristic calculations

TABLE 2
THE RISK INCREASE ASSOCIATED WITH SIMPLE STRATEGIES WITH YEARLY AND MONTHLY ADJUSTMENTS FOR UNIT-LINKED PURE ENDOWMENT CONTRACTS

	<i>K</i>	<i>R</i> ₀	<i>Yearly</i>	(<i>std. dev.</i>)	<i>Monthly</i>	(<i>std. dev.</i>)
$\sigma = 0.15$	0	0.131	0.0015	–	0.00012	–
	0.5 exp(<i>rT</i>)	0.134	0.0014	(1.5 · 10 ⁻⁶)	0.00012	(1.3 · 10 ⁻⁷)
	exp(<i>rT</i>)	0.173	0.0011	(1.6 · 10 ⁻⁶)	0.00009	(1.3 · 10 ⁻⁷)
(<i>M</i> = 1000000)	2 exp(<i>rT</i>)	0.446	0.0004	(1.4 · 10 ⁻⁶)	0.00003	(1.1 · 10 ⁻⁷)
$\sigma = 0.25$	0	0.194	0.0060	–	0.00051	–
	0.5 exp(<i>tT</i>)	0.205	0.0058	(1.9 · 10 ⁻⁵)	0.00050	(1.6 · 10 ⁻⁶)
	exp(<i>rT</i>)	0.261	0.0051	(1.9 · 10 ⁻⁵)	0.00044	(1.6 · 10 ⁻⁶)
(<i>M</i> = 1000000)	2 exp(<i>rT</i>)	0.538	0.0040	(1.9 · 10 ⁻⁵)	0.00034	(1.6 · 10 ⁻⁶)
$\sigma = 0.35$	0	0.365	0.0225	–	0.00187	–
	0.5 exp(<i>tT</i>)	0.380	0.0218	(3.1 · 10 ⁻⁴)	0.00186	(2.6 · 10 ⁻⁵)
	exp(<i>rT</i>)	0.449	0.0209	(3.1 · 10 ⁻⁴)	0.00178	(2.6 · 10 ⁻⁵)
(<i>M</i> = 1000000)	2 exp(<i>rT</i>)	0.743	0.0193	(3.1 · 10 ⁻⁴)	0.00160	(2.6 · 10 ⁻⁵)

With volatility parameter $\sigma = 0.25$, the ratio between the square root of the initial intrinsic risk $\sqrt{R_0}$ and the intrinsic value process V_0^* is 0.5 for the pure unit-linked life insurance, see Table 1. By increasing the size l_λ of the portfolio to 100, say, the corresponding ratio is reduced by the factor $\sqrt{100}/100 = 0.1$ to 0.05. As mentioned in the previous sections, V_0^* can be interpreted as a natural candidate for the single premium. In non-life insurance premiums are often increased by adding a safety loading, typically twice the standard deviation of the total liability. This procedure would lead to a safety loading about $2 \cdot 5\%$, that is 10% when $l_\lambda = 100$. Furthermore, it is noted that the minimal risk associated with the simple strategy (6.4) with trading once per year is only 0.006 higher than the minimum obtainable risk $R_0 = 0.194$. This corresponds to an increase of 3.1%. Thus, the uncertainty associated with the death of the policyholders seems to be by far the most important.

The results obtained for the unit-linked contract with guarantee different from 0 indicate lower values of the ratio between the square root of the minimal obtainable risk R_0 and the intrinsic value process V_0^* than in the pure unit-linked case. Furthermore, the ratio seems to be decreasing as a function of the guaranteed amount. Also the relative risk increase associated with simple strategies is smaller than the corresponding results for the pure unit-linked life insurance. These properties could be partly explained by considering the exact form of the sum insured, described by the underlying derivative

$$\max(S_T, K) = K + (S_T - K)^+$$

Obviously, the probability of the European Call Option $(S_T - K)^+$ being in the money will converge to zero as K converges to infinity. In this way the relative uncertainty associated with the sum insured should decrease when the guaranteed amount increases.

Table 1 also gives indications of the consequences of possible mis-specification of the volatility parameter σ . It is seen that all quantities listed here seem to be non-decreasing functions of the volatility. In particular, calculation of premiums based on the initial intrinsic value V_0^* only would neglect the increase in the ratio $\sqrt{R_0}/V_0^*$ as σ increases. Thus, this principle could result in premiums which are not adequate to cover the insurer's liabilities to the insured.

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