SHORT CONTRIBUTIONS

AN APPLICATION OF EXPONENTIAL DISPERSION MODELS IN PREMIUM RATING

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ABSTRACT

A practical method of allowing for covariates in compound Poisson modelling distributions is discussed.

1. INTRODUCTION

TAYLOR (1989) has proposed a method for premium rating by geographical area. In practice, this makes use of a generalized linear model (GLM) comprising a bivariate spline predictor linked through the reciprocal (link) function to the gamma modelling distribution. As explained by TAYLOR (1989, Section 4.4) the choice of gamma variate is justified on the basis that it represents the limiting form of the generalized Poisson variate for large expected number of claims; the latter distribution being characterised by a spike at zero, together with a continuous distribution on strictly positive support. The purpose of this note is to describe how the exact generalized Poisson variate can be implemented in this context without having to resort immediately to its limiting case and to counter the suggestion that standard regression packages do not appear to provide for this by indicating how this can be done in the GLIM package. The result also has wider application.

2. THE METHOD

The essential purpose of the distribution assumption in a GLM is to establish a log-likelihood so that the estimation of the parameters in the linear predictor can proceed by optimization. Specifically, the distribution of the independent response variables Y_i, the operating ratios adjusted for region (i) in this instance, is taken from the family of exponential dispersion models, JORGENSEN (1987). Thus the general form of the log-likelihood is

$$\sum_i l_i = \sum_i \left\{ \frac{y_i \theta_i - b(\theta_i)}{\phi} + c(y_i, \phi) \right\}$$

for specified functions $b(.)$ and $c( )$, where θ_i is the cannonical parameter and ϕ the constant dispersion parameter. The identity

$$E \left(\frac{\partial l_i}{\partial \theta_i} \right) = 0 \Rightarrow E(Y_i) = \mu_i = b'(\theta_i)$$

where dash denotes differentiation, so that θ_i is a function of μ_i provided b' has an inverse, which is defined to be the case. The identity

$$E \left(\frac{\partial^2 l_i}{\partial \theta_i^2} \right) + E \left\{ \left(\frac{\partial l_i}{\partial \theta_i} \right)^2 \right\} = 0 \Rightarrow \text{Var} (Y_i) = b''(\theta_i) \phi$$

the product of two terms Noting that $b''( )$ is a function of θ_i, the cannonical parameter, and hence a function of μ_i, $b''(\theta_i) = V(\mu_i)$ defines the so-called variance function $V( )$.

The unknown parameters β_j in the linear predictor $\eta_i = \sum_j x_{ij} \beta_j$ with known covariate structure (x_{ij}) enter the log-likelihood via the inverse link

$$\mu_i = g^{-1} \left(\sum_j x_{ij} \beta_j \right)$$

leading to the estimating equations

$$\frac{\partial L}{\partial \beta_j} = \sum_i \frac{\partial l_i}{\partial \beta_j} = \sum_i \frac{\partial l_i}{\partial \theta_i} \frac{\partial \theta_i}{\partial \mu_i} \frac{\partial \mu_i}{\partial \beta_j}$$

$$= \sum_i \frac{y_i - \mu_i}{\phi V(\mu_i)} \frac{\partial \mu_i}{\partial \beta_j} = 0$$

which are solved numerically to obtain the desired fit. The link function g is defined to be both monotonic and differentiable so that

$$g(\mu_i) = \eta_i \quad \text{and} \quad \frac{\partial \mu_i}{\partial \beta_j} = \frac{\partial \mu_i}{\partial \eta_i} x_{ij}$$

The overriding feature of these equations is that a knowledge of only the first and second moments of the modelling distribution is required in their construction Thus the estimating equations are characterised by the variance function of the exponential dispersion model and the nature of the predictor link assumption. By this means it is possible to extend the range of models used in the regression context beyond the standard senarios associated with the normal, Poisson, binomial and gamma distributions.

Two such standard types are of immediate interest in this problem, the Poisson distribution parameterized by the mean μ_i and the gamma distribution

parameterized by the mean μ_i and inverse scale parameter ν. For these two cases the following detail applies

	Poisson	gamma
scale parameter ϕ	1	ν^{-1}
cumulant function $b(\theta_i)$	$\exp(\theta_i)$	$-\log(-\theta_i)$
$c(y_i, \phi)$	$-\log(y_i!)$	$\nu \log(\nu y_i) - \log(y_i) - \log \Gamma(\nu)$
mean $\mu(\theta_i)$	$\exp(\theta_i)$	$-1/\theta_i$
variance function $V(\mu_i)$	μ_i	μ_i^2

In particular, the variance functions are of special interest in this context. They are members of the more general class of power variance functions $V(\mu_i) = \mu_i^\zeta$ with $\zeta = 1$ or 2. In addition, JORGENSEN (1987) has established that the power variance function of the type $V(\mu_i) = \mu_i^\zeta$ with $1 < \zeta < 2$ corresponds to compound Poisson distributions of the type discussed by TAYLOR (1989) thereby establishing the vital connection needed to model these distributions in a regression context.

Implementation is possible using the current version (release 3.77) of the GLIM computer package, BAKER and NELDER (1985), through the OWN model user facility, in which it is necessary, to supply explicit formulae for (a) the inverse link function, (b) the derivative of the link function, (c) the variance function, and (d) the contribution to the deviance attributed to each unit. It is suggested that the log-link may well be appropriate in this context while the explicit formula for the contribution to the deviance of the general unit (i) is

$$
d_i(y_i, \mu_i) = 2 \int_{\mu_i}^{y_i} \frac{y_i - t}{V(t)} \, dt
$$

$$
= 2 \left\{ -\frac{y_i}{1-\zeta} (\mu_i^{1-\zeta} - y_i^{1-\zeta}) + \frac{1}{2-\zeta} (\mu_i^{2-\zeta} - y_i^{2-\zeta}) \right\}
$$

The optimum value for the power variance parameter ζ in the range $(1, 2]$ is determined from the deviance profile, obtained by repeated fitting of the model under incremental changes in ζ

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