DISCUSSION PAPERS

THE SCHMITTER PROBLEM

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At the ASTIN Colloquium in Montreux, HANS SCHMITTER posed the following problem.

PROBLEM

Consider the class $\mathscr{F}$ of distributions with range $[0, b]$, mean μ and variance σ^2. Let $\Psi_{\theta, F}(u)$ denote the probability of ultimate ruin under a compound Poisson claim process with given premium loading θ, initial capital u and individual claim size $d \cdot f \cdot F$. For fixed θ and μ, which $F \in \mathscr{F}$ maximizes $\psi_{\theta, F}(u)$ for a particular given u? In particular, is F diatomic?

PRACTICAL BACKGROUND OF THE PROBLEM

H. SCHMITTER describes the following practical background in which the problem arises.

The problem of determining bounds for ruin probabilities arises when an insurer decides his reinsurance retentions in order to increase the stability of an account. He may not only choose between various forms of reinsurance (quota share, surplus, excess loss etc.) but he usually combines them in what is called a reinsurance program. When evaluating reinsurance programs he needs to compare their prices and the effectiveness of the protection they offer. The reinsurance price is the difference between the gross (i.e. before reinsurance) and the net (i.e. retained, after reinsurance) expected profit. The effectiveness of the protection, on the other hand, can be measured by the probability of ruin: the lower the probability of ruin of the retained account the more effective the reinsurance program. Computing ruin probabilities is often criticized as being pointless because their absolute values are said to be irrelevant. However, if two reinsurance programs both reduce the expected profit of the ceding company by the same amount the one leading to the smaller probability of ruin is likely to be preferable.

The ruin probability depends on the initial reserve (known to the ceding company), the security loading (defined as the expected retained profit, hence a function of the reinsurance program) and on the retained claim amount distribution In practice, the latter is hardly ever known, apart from the maximum retained claim which is given by an excess loss deductible or a policy limit. At best we have to our disposal estimates of the expected value and the variance. An exact computation of the ruin probability is, therefore, not possible and one has to accept the determination of upper and lower bounds.

So far we do not even know the least upper bound in the case where the expected value, the variance and the maximum claim are known Perhaps the answer to the above question is not an isolated problem but leads to further investigations and applications: Suppose that for several independent risks the expected profits, frequencies, expected values, variances and maximum claims are known. What is the least upper bound of the overall ruin probability for a given initial reserve? Is there a natural way of allocating parts of the initial reserve to the independent risks? A question often asked in practice.

DISCUSSION

At Montreux, GREG TAYLOR pointed out that F more dangerous than G in stop-loss order implies that $\Psi_{0,F}(u) \geq \Psi_{0,G}(u)$ for all u (GOOVAERTS and DE VYLDER, 1984; TAYLOR, 1985)

Hence the problem is reduced to seeking an extremal distribution in $\mathcal{F}$ in terms of stop-loss order. However an extremal distribution in terms of stop-loss order does not exist in class $\mathcal{F}$.

The problem was further discussed at the " 1990 Risk Theory seminar at the Mathematisches Forschungsinstitut of the Federal Republic of Germany, in Oberwolfach"

MARC GOOVAERTS pointed out that an upper bound can be obtained by the criterion of danger which satisfies the range $[0, b]$, μ but not σ^2 where now danger is defined as in BÜHLMANN et al (1977). One can deduce a distribution which is more dangerous than all of those belonging to the class of distributions with prescribed range, mean μ but with a minimal variance, larger then σ^2in analogy to KAAS and GOOVAERTS (1986).

But only danger as well as first order stop-loss ordening will give rise to inequalities between ruin probabilities If we have $E(X) = E(Y)$ and $E((X-t)_+) < E((Y-t)_+)$ $\forall t$ then $\Psi_{0,F_X}(u) \leq \Psi_{0,F_Y}(u)$ uniformly for all 0 and u. The problem of finding $\sup_{F_X \in \mathcal{F}} E((X-t)_+)$ does not give rise to a uniform (in t) extremal distribution.

It is solved by constructing a polynomial of degree two above $(X-t)_+$ which is tangent to this function in 2 points. The abscissas of these points will be the mass points (a recent reference is e g GOOVAERTS et al., 1990) These results are known but they cannot be used to obtain an upper bound for the infinite time ruin probability because the extremal distribution depends on the value of t.

One finds the following solution: A risk X with spectrum (r, s) exists with mean μ and variance σ^2 if and only if $s = r'$, where $r' = \mu+[\sigma^2/(\mu-r)]$

The following mass points of the extremal distributions are obtained: $(0, 0')$ in case $0 \leq t \leq 1/2\,0'$, $(t+\sqrt{(\mu-t)^2+\sigma^2},\ t-\sqrt{(\mu-t)^2+\sigma^2})$ in case $1/2\,0' \leq t \leq 1/2(b+b')$ and (b, b') in case $1/2(b+b') \leq t \leq b$ This indicates that even for the simple extremal stop-loss problem no uniform extremal distribution exists. Also BROCKETT and COX (1985, 1986) present explicit solutions to the above problem when $n = 1, 2$ or 3 moments are given using

Tchebycheff systems of functions. KEMPERMANN (1970) also solves this problem in general.

A problem closely related to the one stated by SCHMITTER and as intriguing is the following: consider $S = X_1 + \ldots X_N$ under the classical assumptions and find $\operatorname{Sup}_{F_X \in \mathcal{F}} E((S-t)_+)$.

This problem can be solved for the case $F_X \in \mathcal{F}_1$ (= a set of distributions with given μ and b), (see BUHLMANN, GAGLIARDI, GERBER, and STRAUB, 1977) An attempt to solve the above problem ($F_X \in \mathcal{F}$) has been presented by KAAS and GOOVAERTS (1984), cited above.

Also at Oberwolfach, P BROCKETT demonstrated that the $F \in \mathcal{F}$ which minimizes the adjustment coefficient R of the claim process lies in the class D_2 of diatomic distributions. Since $\Psi_{0,F}(u) \sim$ const. e^{-Ru} for large u, this implies that the required F lies in D_2 for sufficiently large u. It does not, however, identify F for smaller values of u. In fact, the extremal F for large u can be identified as follows:

$$\text{Mass } p = (b-\mu)^2/[\sigma^2+(b-\mu)^2] \text{ at } \mu-\sigma^2/(b-\mu); \text{ and Mass } 1-p \text{ at } b.$$

Similar results can be obtained for maximizing the adjustment coefficient. These results can also be found in DE VYLDER, GOOVAERTS and HAEZENDONCK (1984), BROCKETT and COX (1983, 1986) and KEMPERMANN (1970, 1971).

GREG TAYLOR suggested that, to the extent that Schmitter's problem related to premium rating (as SCHMITTER had said it did), that problem was probably not the most relevant for solution. In practice, the assumption of unimodality of F would almost always be reasonable, and this additional restriction on F could be expected to decrease the upper bound on $\Psi_{0,F}(u)$ substantially

Moreover, this additional condition does not add to the difficulty of the problem The history of this goes back to VERBEEK (1977), who dealt with the extremal unimodal stop-loss premium with fixed mean and upper bound, and TAYLOR (1977) who extended the results to the context of an arbitrary finite number of linear constraints on the unimodal distribution. Much extension has subsequently been made by GOOVAERTS (and co-authors) and BROCKETT and COX

The relevant result for Schmitter's problem if unimodality is required is that the extremal distribution must lie in the class $\mathcal{S}_3$ of step functions with 3 levels (with possible equality of 2 or 3 levels).

BROCKETT and COX (1985, 1986) demonstrate that the unimodal process lies in the class $\mathcal{S}_2$. As in the case where unimodality is not required, they give an explicit optimal solution to bounding the adjustment coefficient. They give the corresponding solution for an arbitrary finite number of linear constraints on F, and it is again true that his extremal distribution solves Schmitter's problem for sufficiently large u.

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