PSEUDO COMPOUND POISSON DISTRIBUTIONS IN RISK THEORY

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ABSTRACT

Using Laplace transforms and the notion of a pseudo compound Poisson distribution, some risk theoretical results are revisited A well-known theorem by FELLER (1968) and VAN HARN (1978) on infinitely divisible distributions is generalized. The result may be used for the efficient evaluation of convolutions for some distributions. In the particular arithmetic case, alternate formulae to those recently proposed by DE PRIL (1985) are derived and shown more adequate in some cases. The individual model of risk theory is shown to be pseudo compound Poisson. It is thus computable using numerical tools from the theory of integral equations in the continuous case, a formula of Panjer type or the Fast Fourier transform in the arithmetic case. In particular our results contain some of DE PRIL'S (1986/89) recursive formulae for the individual life model with one and multiple causes of decrement. As practical illustration of the continuous case we construct a new two-parametric family of claim size density functions whose corresponding compound Poisson distributions are analytical finite sum expressions. Analytical expressions for the finite and infinite time ruin probabilities are also derived.

KEYWORDS

Pseudo compound Poisson, integral equation, infinite divisibility; multiple decrement model, ruin probability.

1. PSEUDO COMPOUND POISSON DISTRIBUTIONS

In order to investigate probability density "functions" such as

$$f(x) = \exp(-\lambda)\,\delta(x) + (1-\exp(-\lambda))\,\mu\exp(-\mu x),$$

$$\lambda,\ \mu > 0,\ \delta(x) \text{ the Dirac function,}$$

we need the theory of "generalized functions" or "distributions" in the sense of L. SCHWARTZ (1950/51/65/66). In this paper we refer to the presentation by DOETSCH (1976) (English translation is available) To avoid a conflict of terminology between Function Theory and Statistics we use the term generalized function. This is a linear and continuous functional on the space of infinitely differentiable functions on $\mathbb{R}$ with compact support In this paper generalized functions are usually written without argument as $f, g, \ldots$ Sometimes and especially in applications we will abuse notation and write $f(x)$

instead of f, e.g. we write $\delta(x)$ for the Dirac function instead of δ. Integrals are always understood in the Lebesgue sense.

Let $\mathcal{L}$ be the space of all locally integrable functions on $[0, \infty)$ (i.e. integrable in every finite subinterval of $[0, \infty)$), and let $\mathcal{L}'$ be the space of all generalized functions on $\mathbb{R}$. For $f \in \mathcal{L}$, $s \in \mathbb{C}$, the *Laplace transform* of $f(x)$ is defined to be

$$Lf(s) = \int_0^\infty \exp(-st) f(t)\, dt.$$

This mathematical object is extended as follows to an appropriate subspace of $\mathcal{D}'$ (see DOETSCH, § 12). Let D^k, $k = 1, 2, \ldots$, be the k-th derivative operator acting on the space $\mathcal{L}'$. A generalized function f is said to be of *finite order k* if $f = D^k h(x)$ for a continuous function $h(x)$ defined on $\mathbb{R}$, and k is the smallest integer with this property. For example, the Dirac function

$$\delta = D^2 h(x),\ h(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

is of order 2. Restrict now $\mathcal{L}'$ to the subspace $\mathcal{L}'_0$ of generalized functions of finite order whose associated continuous functions $h(x)$ satisfy the conditions

$$h(x) = 0 \quad \text{for} \quad x < 0,$$

$$Lh(s) \text{ converges absolutely for } Re(s) > \sigma,$$

$$\sigma \text{ dependent on } h.$$

For $f = D^k h(x) \in \mathcal{L}'_0$, $s \in \mathbb{C}$, the *Laplace transform* is defined to be

$$Lf(s) = s^k Lh(s) \tag{1.1}$$

and is an analytical function for $Re(s) > \sigma$. The space $\mathcal{L}$ is embedded in $\mathcal{L}'$ as follows. The generalized function defined by $f \in \mathcal{L}$ is the functional

$$\int_{-\infty}^{\infty} f(x)\varphi(x)\, dx,\ \varphi(x) \text{ infinitely differentiable on } \mathbb{R} \text{ with compact support.}$$

A function $f \in \mathcal{L}$ with a Laplace transform in the classical sense has the same Laplace transform in the generalized sense (DOETSCH, Satz 12.2). Moreover the inverse of the Laplace transform is unique up to a zero (generalized) function in both the classical and generalized sense (DOETSCH, Satz 5.1, and p 72) Here the zero function $z(x)$ in $\mathcal{L}$ is a function such that

$$\int_0^t z(x)\, dx = 0, \quad \text{for all} \quad t \geq 0.$$

The convolution operator on $\mathcal{L}'_0$ is defined as follows. If $f = D^m h(x)$, $g = D^n k(x)$, then

$$f * g = D^{m+n}(h * k)(x).$$

The operations on the classical Laplace transform extend to the generalized case Some operations used in this paper are summarized in the next Table

(Generalized) function	*Laplace transform*
f, g	$Lf(s), Lg(s)$
$af+bg,\ a, b \in \mathbf{R}$	$aLf(s)+bLg(s)$
$f * g$	$Lf(s)\ Lg(s)$
xf	$-(d/ds)\,Lf(s)$
$\exp(-ax)f,\ a \in \mathbf{R}$	$Lf(s+a)$
$f'(x)$	$sLf(s)-f(0^+)$
$\delta(x)$ (Dirac function)	1

To illustrate the consistency of the Table with definition (1.1) we derive the formula for the Laplace transform of the n-th derivative $f^{(n)}$ of a function $f \in \mathcal{C}$. From the theory of generalized functions (e.g. DOETSCH, § 14) one knows that

$$D^n f = f^{(n)} + f^{(n-1)}(0^+)\,\delta + \ldots + f(0^+)\,\delta^{(n-1)}$$

Since $L\delta^{(k)}(s) = s^k$ it follows with (1.1) that

$$\begin{aligned} s^n Lf(s) &= LD^n f(s) \\ &= Lf^{(n)}(s) + f^{(n-1)}(0^+) + \ldots + f(0^+)\,s^{n-1}, \end{aligned}$$

which provides after rearrangement the desired formula. The differential rule for a generalized function $f \in \mathcal{C}_0$ looks somewhat different, namely

$$LD^n f(s) = s^n Lf(s).$$

From now on our main concern is probabilistic. The set of locally integrable probability density functions $f \in \mathcal{C}$ is denoted by $\mathcal{C}P$ The distribution corresponding to $f(x)$ is

$$F(x) = \int_0^x f(t)\,dt.$$

It s well-known that Panjer's recursive formula plays an important role in computational risk theory. For $f \in \mathcal{C}P$ we are interested in the analogous integral equation

$$(1\,2) \qquad xf(x) = \lambda \int_0^x yh(y)\,f(x-y)\,dy, \qquad \lambda \in \mathbb{R},$$

where $h \in \mathcal{C}$ is not necessarily positive In applications of risk theory the assumption $0 < F(0) < 1$ is almost always fulfilled. We consider therefore the subset $\mathcal{C}P_0$ of all functions $f \in \mathcal{C}P$ with $0 < F(0) < 1$ and for which there is a unique solution $h \in \mathcal{C}$ with

$$\int_0^\infty h(x)\,dx = 1,$$

such that (1.2) is almost everywhere fulfilled From results by STEUTEL (1970) and VAN HARN (1978) the set $\mathcal{L}P_0$ contains all infinitely divisible densities on $(0, \infty)$ (see Corollary 2). It has been shown in the arithmetic case that there are interesting non-infinitely divisible distributions on $\mathbb{N}$ for which the arithmetic version of (1.2) is fulfilled, e.g. the individual model of risk theory with multiple causes of decrement (HÜRLIMANN (1989b)) Are there analogous continuous candidates in $\mathcal{L}P_0$ and what is exactly this set? A practical answer is postponed to the end of this Section. From a mathematical point of view, the set $\mathcal{L}P_0$, given that it contains non-infinitely divisible distributions, is appealing, since it leads to a natural generalization of the characterization by FELLER (1968) and VAN HARN (1978) of infinitely divisible distributions with non-vanishing zero-probability.

THEOREM 1 Let $f(x)$ be in the class $\mathcal{L}P_0$. Then in the space $\mathcal{L}_0$ the following representation holds almost everywhere

$$f(x) = \sum_{k=0}^{\infty} \exp(-\lambda)\,\lambda^k/k!\; h^{*k}(x)$$

where $h^{*0}(x) = \delta(x)$, $\lambda = -\ln\{F(0)\}$ and $h(x)$ is almost everywhere the unique solution of the integral equation

$$xf(x) = \lambda \int_0^x yh(y)\, f(x-y)\,dy. \tag{1.3}$$

PROOF. The integral equation (1 3) can be rewritten as

$$xf(x) = \lambda(f * u)(x) \qquad \text{with} \qquad u(x) = xh(x).$$

Applying the Laplace transform we get

$$(d/ds)\,Lf(s) = \lambda\, Lf(s)\cdot(d/ds)\,(Lh(s))$$

It follows that

$$Lf(s) = c\cdot\exp(\lambda\, Lh(s)).$$

By Laplace inversion in the space $\mathcal{L}_0$ we get almost everywhere

$$f(x) = c\sum_{k=0}^{\infty} \lambda^k/k!\; h^{*k}(x).$$

In this formula we see that $p = f - c\delta \in \mathcal{L}_0$ comes from a function $p \in \mathcal{L}$
By integration

$$F(x) = c + \int_0^x p(t)\,dt,$$

which shows that $c = F(0)$. Put $\lambda = -\ln\{F(0)\}$ to get the result.

The above result suggests the following definition.

DEFINITION. A probability density function $f(x)$ defined on $(0, \infty)$ is said to be of *pseudo compound Poisson* type if $f \in \curlyvee P_0$. We call the associated $h(x)$ a *pseudo density*.

INTERPRETATION In risk theory and when it is actually non-negative the function $h(x)$ plays the role of claim size density

The following equivalent formulation of Theorem 1 can be more adequate for practical evaluations In particular it generalizes the result by STRÖTER (1985)

COROLLARY 1. Let $f(x)$ be pseudo compound Poisson with parameter λ and pseudo density $h(x)$ Define $p(x) = f(x) - \exp(-\lambda)\,\delta(x)$.

Then $p(x)$ satisfies the integral equation

$$xp(x) = \lambda \exp(-\lambda)\, xh(x) + \lambda \int_0^x yh(y)\, p(x-y)\, dy \tag{1 4}$$

PROOF. Introduce $f(x) = \exp(-\lambda)\,\delta(x) + p(x)$ in the integral equation (1.3) to obtain immediately (1 4).

In view of its importance both in theory and practice (see e g. STEUTEL (1979)) we recall the definition of infinite divisibility.

DEFINITION. A random variable X, taking values in $\mathbb{R}$, is called *infinitely divisible* if for every $n \in \mathbb{N}$ there exist independent, identically distributed random variables $Y_{1,n}, \ldots, Y_{n,n}$ such that the following equality in distribution is valid·

$$X \overset{d}{=} Y_{1,n} + \ldots + Y_{n,n}.$$

Equivalently $P(z)^{1/n} = E[z^X]^{1/n}$, $Lf(s)^{1/n} = E[\exp(-sX)]^{1/n}$ or $\varphi(t)^{1/n} = E[\exp(itX)]^{1/n}$ is respectively a probability generating function, a Laplace transform or a characteristic function for every $n \in \mathbb{N}$ The associated probability density and distribution are also called infinitely divisible

The special case of Theorem 1 for infinitely divisible distributions on $[0, \infty)$ has been identified in other forms by STEUTEL (1970) and VAN HARN (1978) in the general and KATTI (1967) and FELLER (1968) in the arithmetic case

COROLLARY 2. Let X be a random variable defined on $[0, \infty)$ with locally integrable density $f(x)$ such that $0 < F(0) < 1$. Then the following conditions are equivalent·

(a) X is infinitely divisible;
(b) X is compound Poisson with parameter λ and jump density $h(x)$ and $f(x)$ is solution of the integral equation (1.3);
(c) The solution $h(x)$ of the integral equation (1.3) is positive

PROOF. In the arithmetic case the equivalence of (a) and (c) has been shown by KATTI (1967) (other proof by STEUTEL (1970, p. 83)) The equivalence of (a) and (b) was shown by FELLER (1968, vol. 1, 3rd edition, p. 290) (other proof by GERBER and VALDERRAMA OSPINA (1987)). In the continuous case the equivalence of (a) and (b) is due to VAN HARN (1978, theorem 1 6.6) for the compound Poisson representation and STEUTEL (1970) (see also VAN HARN, Corollary 1 6.3) for the integral equation representation. The equivalence of (b) and (c) follows from Theorem 1.

Next we display a subclass of $\nearrow P_0$ which is big enough for our applications. In particular we will show by construction in Section 4 that the class $\nearrow P_0$ contains more functions than the infinitely divisible ones.

THEOREM 2. Let $\nearrow P'$ be the subclass of $\nearrow P$ consisting of functions $f(x)$ which satisfy the following conditions:

(i) $0 < F(0) < 1$.
(ii) The associated generalized function $f - F(0)\,\delta \in \nearrow_0$ comes from a continuous function $f(x) - F(0)\,\delta(x)$ defined on $[0, \infty)$.

Then $\nearrow P'$ is contained in $\nearrow P_0$.

PROOF Let $f \in \nearrow P'$. The function $p(x) = f(x) - F(0)\,\delta(x)$ is by assumption continuous on $[0, \infty)$ Consider the Volterra integral equation of the second kind

$$a(x) = \lambda^{-1} \exp(\lambda)\, xp(x) - \exp(\lambda) \int_0^x a(y)\, p(x-y)\, dy,\ \lambda = -\ln\{F(0)\}.$$

Since $p(x-y)$ and $xp(x)$ are continous on $\{0 \le x \le a, 0 \le y \le x\}$ respectively $\{0 \le x \le a\}$, this equation can be solved uniquely for $a(x)$ (see e g JERRI (1985), p. 194 and p 201) Set $h(x) = a(x)/x$. After algebraic manipulation one sees that $h(x)$ is the unique solution of the integral equation (1.4). Since $f(x) = F(0)\,\delta(x) + p(x)$, one checks easily that $h(x)$ is also the unique solution of the integral equation (1.3). Provided that

$$\int_0^\infty h(x)\, dx = 1,$$

we have shown that $f \in \nearrow P_0$. This point is proved as follows. Since $h(x)$ is solution of (1 3) one shows that

$$\int_0^\infty h(x)\,dx = c < \infty$$

Then $\tilde{h}(x) = h(x)/c$ is the unique solution of the integral equation

$$xf(x) = \lambda c \int_0^\infty y\tilde{h}(y)\, f(x-y)\,dy.$$

Since $\int_0^\infty \tilde{h}(x)\,dx = 1$ one has $f \in \mathcal{P}_0$. But from Theorem 1 one has then

$$\lambda c = -\ln\{F(0)\}.$$

By definition of λ above one has indeed $c = 1$.

REMARKS

(1) In Theorem 1 and Corollary 2 the condition $F(0) > 0$ is necessary. The infinitely divisible exponential density $f(x) = \mu \exp(-\mu x)$ leads to the solution $h(x) = \exp(-\mu x)/x$, but $\int_0^\infty h(x)\,dx = \infty$. This density is not compound Poisson, but the weak limit of the compound Poisson densities $f_\lambda(x) = \exp(-\lambda)\,\delta(x) + (1-\exp(-\lambda))\,\mu \exp(-\mu x)$ as $\lambda \to \infty$, with claim size densities $h_\lambda(x) = \exp(-\mu x)\,(1-\exp(-ax))/\lambda x$, $a = (\exp(\lambda)-1)\,\mu$. This result will be derived in Section 4. In general $p(x)$ with $P(0) = 0$ is infinitely divisible if and only if $f_\lambda(x) = \exp(-\lambda)\,\delta(x) + (1-\exp(-\lambda))\,p(x)$ is infinitely divisible with $F(0) = \exp(-\lambda)$ and $p(x)$ is the weak limit of the f_λ's as $\lambda \to \infty$. (FELLER (1968), vol 2, 2nd edition, p. 303).

(2) In the arithmetic case the integral equation (1.3) is to be replaced by the well-known Panjer recursive formula

$$kp(k) = \lambda \sum_{s=1}^{k} sh(s)\,p(k-s) \tag{1.5}$$

An independent and more elementary proof of the results in this mathematically simpler case in presented in HÜRLIMANN (1989a, 1989b). Observe that Laplace transforms are to be replaced by the geometric transform (= probability generating function in case of arithmetic distributions, see GIFFIN (1975) for fundamentals)

(3) Methods to solve integral equations can be found in all parts of Applied Mathematics. Transform theory (see WIDDER (1971)), especially Laplace transforms, is a powerful tool to get closed analytical results An illustration is given in Section 4. Numerical methods were extensively studied by BAKER (1977) and more recently equation (1 4) has been solved in the insurance context by STRÖTER (1985). It is worthwile to mention that the Laplace

transform approach simplifies the derivation of Theorem 1 1. of the latter author, which uses the method of successive approximation.

(4) Theorem 1 can be interpreted as a duality assertion. There is a duality between integrable densities on $[0, \infty)$ and pseudo densities, where the pseudo compound Poisson representation realizes this duality The subclass of infinitely divisible densities is just dual to the ordinary densities.

(5) Theorem 1 suggests many (also difficult) applications. It can be useful for the computational evaluation of convolutions (see next Section), as well as for the study of other properties of exact sampling distributions. A statistical application is given in HÜRLIMANN (1989a).

(6) With more technical refinements it should be possible to extend the results to arbitrary one-sided unbounded intervals $[a, \infty)$, $a > -\infty$, (see VAN HARN (1978) for the case of infinitely divisible distributions). It would be of great interest to generalize Theorem 1, if possible, to the whole real line and especially obtain a single characterizing functional equation valid on $\mathbb{R}$. Unfortunately, even for infinitely divisible distributions, the latter requirement is still an open problem, as reported by VAN HARN (1978), p. 189.

2. CONVOLUTIONS OF DISTRIBUTIONS

Let $X_1, X_2, \ldots, X_n$ be n mutually independent random variables on $[0, \infty)$ with a common integrable density $f(x)$ such that $0 < F(0) < 1$. In probability and statistical theory one is interested in the exact sample distribution of the mean. It is a straightforward rescaling of the distribution of the sum

$$X = X_1 + \ldots + X_n$$

whose density is given by the n-fold convolution

$$\bar{f}(x) = f^{*n}(x).$$

The evaluation of this function uses the recursive formula

$$f^{*(k+1)}(x) = \int_0^x f(y) f^{*k}(x-y)\, dy,$$

which is very time-consuming for large values of n, especially when $f(x)$ is not a simple function

Using Theorem 1 and the various methods for solving integral equations, an alternative general approach to this problem follows immediately. In the following we will often use $g(x) = \lambda h(x)$ instead of $h(x)$

COROLLARY 3. Let the X_i be defined on $[0, \infty)$ with $0 < F(0) < 1$. Assume $f \in \mathcal{P}_0$. Let $g(x)$ be the solution of the integral equation

$$xf(x) = \int_0^x yg(y)\, f(x-y)\, dy \tag{2.1}$$

Then the n-fold convolution $\bar{f}(x)$ is solution of the integral equation

$$x\bar{f}(x) = n \int_0^x yg(y)\, \bar{f}(x-y)\, dy \tag{2.2}$$

PROOF. In the proof of theorem 1 we have seen that

$$Lf(s) = F(0) \exp(Lg(s)),$$

and thus

$$L\bar{f}(s) = F(0)^n \exp(nLg(s)).$$

Therefore $\bar{f}(x)$ is pseudo compound Poisson with parameter $n\lambda$ and pseudo density $g(x)/\lambda$. The affirmation follows from Theorem 1.

Let us have a look to the special arithmetic case. The n-fold convolution $\bar{p}(x) = p^{*n}(x)$ can be evaluated using the recursive Panjer formula

$$\begin{cases} \bar{p}(0) = p(0)^n \\ k\bar{p}(k) = n \sum_{s=1}^{k} sg(s)\, \bar{p}(k-s) \end{cases} \tag{2.3}$$

where $g(s)$ is itself computed recursively by

$$sg(s)\, p(0) = sp(s) - \sum_{i=1}^{s-1} ig(i)\, p(s-i) \tag{2.4}$$

At first sight it might appear that this two-stage nested recursive algorithm is computationally less efficient than the recursive formula proposed by DE PRIL (1985), Theorem 1

$$\begin{cases} \bar{p}(0) = p(0)^n \\ k\bar{p}(k)\, p(0) = \sum_{s=1}^{k} [(n+1)s-k]\, p(s)\, \bar{p}(k-s) \end{cases} \tag{2.5}$$

In some cases it might be that only $g(k)$ is known and $p(k)$ must be computed recursively using Panjer's formula (1.5). Then the formula (2.3) is simpler and more direct than formula (2.5)

EXAMPLES. The choice

$$g(k) = \frac{p \cdot \Gamma(a+k-1)\, c^{k-1}}{\Gamma(a)\, k!\, (1+c)^{a+k-1}}, \quad k = 1, 2, \ldots, p > 0,\ c > 0,\ a \geq 0 \tag{2.6}$$

leads to Hoffmann/Thyrion's family proposed as claim number distribution by KESTEMONT and PARIS (1985/87). A similar choice would be the ETNB distribution

$$(2.7)\quad g(k) = \frac{\Gamma(k+a)\ \beta^k}{\Gamma(a)k!\ [(1-\beta)^{-a-1}]}, \quad k = 1, 2, \ldots, -1 < a < 0, 0 < \beta \leq 1,$$

studied as probability density (however) by WILLMOT (1988). In these examples it is more direct to apply formula (2.3) to compute exact n-fold convolutions than to use De Pril's formula (2.5).

3. THE INDIVIDUAL MODEL OF RISK THEORY

Consider n mutually independent random variables $X_1, X_2, \ldots, X_n$, not necessarily identically distributed as in Section 2. Suppose each X_i has a range contained in the interval $[0, \infty)$, which may be arithmetic or not. In risk theory the sum

$$X = X_1 + X_2 + \ldots + X_n,$$

called individual model, can be interpreted as the aggregate claims in a finite period on a portfolio of n independent contracts. Let $F(x) = \Pr(X \leq x)$, $F_i(x) = \Pr(X_i \leq x)$, $i = 1, 2, \ldots, n$, and assume that $0 < F_i(0) < 1$ for all i.

THEOREM 2. Assume the probability densities $f_i \in \mathcal{P}_0$, $i = 1, \ldots, n$. Then the individual model of risk theory is pseudo compound Poisson with parameter

$$(3.1)\qquad \lambda = -\ln\{F(0)\} = -\sum_{i=1}^{n} \ln\{F_i(0)\},$$

and pseudo density

$$(3.2)\qquad h(x) = \left(\sum_{i=1}^{n} g_i(x)\right)/\lambda,$$

where each $g_i(x)$ is unique solution of the integral equation

$$(3.3)\qquad xf_i(x) = \int_0^x y g_i(y)\, f_i(x-y)\, dy$$

PROOF. Clearly $f = f_1 * f_2 * \ldots * f_n$. In the proof of Theorem 1 we have seen that

$$Lf_i(s) = F_i(0)\ \exp(Lg_i(s)), \qquad i = 1, 2, \ldots, n$$

It follows that

$$Lf(s) = \sum_{i=1}^{n} Lf_i(s) = \prod_{i=1}^{n} F_i(0)\ \exp\left(\sum_{i=1}^{n} Lg_i(s)\right)$$

Taking inverse Laplace transforms in the space $\mathcal{H}_0$ the result follows immediately

For simplicity restrict the following discussion to the arithmetic case First of all formulae for $g_i(x)$ must be obtained, or the $g_i(x)$ must be computed by other means, using for example Panjer's recursive formula (3.3). Then the probability density function of the individual model can be computed using Panjer's recursion, valid in the generalized case

$$f(x) = \begin{cases} \prod_{i=1}^{n} f_i(0), & x = 0, \\ (-\ln\{f(0)\}/x) \sum_{y=1}^{x} yh(y) f(x-y), & x > 0. \end{cases} \tag{3.4}$$

Compared to the collective model of risk theory the extra cost for preparing $h(x)$ may be substantial since many values of $g_i(x)$, $i = 1, 2, \ldots$, are involved in the computation. A sound procedure would be to approximate the pseudo density, as suggested by DE PRIL (1987/89) (see Example 1 below), by a more tractable function $h^*(x)$ and compute the approximate density

$$f^*(x) = \begin{cases} \prod_{i=1}^{n} f_i(0), & x = 0, \\ (-\ln\{f(0)\}/x) \sum_{y=1}^{x} yh^*(y) f^*(x-y), & x > 0 \end{cases} \tag{3.5}$$

Another possibility to reduce the computational effort is to apply the Fast Fourier Transform, inverting the Fourier transform of the pseudo compound Poisson representation according to the formula

$$\bar{f} = \{f(0)/n\}\, \mathrm{FFT}^{-}(\exp(\mathrm{FFT}^{+}(\bar{g})))$$

Here FFT^{+}, $\{1/n\}\,\mathrm{FFT}^{-}$ denote Fast Fourier Transform, respectively the inverse transform, and n is the size of the vectors $\bar{f}$, $\bar{g}$ associated to the functions $f(x)$, $g(x)$. Since one has to take into accout a relatively long support of $h(x)$, the FFT-method has been shown superior to Panjer's recursion in many cases (cf BÜHLMANN (1984)), and the error bound in the distribution as well as in associated stop-loss premiums are controllable (BUHLMANN (1984), HURLIMANN (1986))

EXAMPLE 1. The simplest individual life model has been considered by DE PRIL (1986/87). Let n_{ij} be the number of policies with amount at risk i and mortality rate q_j, $i = 1, \ldots, a$, $j = 1, \ldots, b$ Let $p_j = 1 - q_j$ the corresponding survival probabilities, $n_j = \sum_{i=1}^{a} n_{ij}$ the number of policies with mortality

rate q_j, $n = \sum_{j=1}^{b} n_j$ the total number of policies, and $m = \sum_{i=1}^{a} \sum_{j=1}^{b} i \cdot n_{ij}$ the maximum possible amount of aggregate claims. Furthermore let X_{ij} be the random variable representing the claim produced by a policy with amount at risk i and mortality rate q_j. Its probability density function is given by

$$f_{ij}(x) = \begin{cases} p_j, & x = 0 \\ q_j, & x = i \\ 0, & \text{else} \end{cases} \tag{3.6}$$

Following the device given by the arithmetic version of Theorem 2 we search for unique functions $g_{ij}(x)$ such that

$$x f_{ij}(x) = \sum_{y=1}^{x} y g_{ij}(y) f_{ij}(x-y)$$

In the lemma below they are shown to be

$$g_{ij}(x) = \begin{cases} (-1)^{k-1}/k \cdot (q_i/p_j)^k, & x = ik, \quad k = 1, 2, \ldots \\ 0, & \text{else} \end{cases} \tag{3.7}$$

It follows that this individual model is pseudo compound Poisson with parameter

$$\lambda = -\sum_{j=1}^{b} n_j \ln(p_j) = -\ln\{f(0)\}$$

and pseudo density

$$h(x) = 1/\lambda \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij} g_{ij}(x).$$

Insert these formulae in (3.4). Then one has

$$f(0) = \prod_{j=1}^{b} (p_j)^n$$

For $x > 0$ one obtains with $y = ik$:

$$xf(x) = \sum_{i=1}^{\min(a,x)} \sum_{k=1}^{[x/i]} A(i,k) f(x-ik), \qquad x = 1, 2, \ldots, m \tag{3.8}$$

with

$$A(i,k) = (-1)^{k+1} i \sum_{j=1}^{b} n_{ij} (q_j/p_j)^k.$$

This has been derived differently by DE PRIL (1986). For computational reasons REIMERS (1988) has proposed to reverse the order of summation:

$$xf(x) = \sum_{k=1}^{x} \sum_{\iota=1}^{\min(a,\lfloor x/k \rfloor)} A(\iota, k)\, f(x-\iota k) \tag{3.9}$$

To save computer time it is advisable to truncate the first summation taking only 4-5 terms as proposed by De Pril and Vandenbroek (1987). An analysis of the magnitude of error involved in this approximation step is given by De Pril (1988).

Lemma The Panjer recurrence relation equations

$$xf(x) = \sum_{y=1}^{x} yg(y)\, f(x-y)$$

where

$$f(x) = \begin{cases} p, & x = 0 \\ q, & x = \iota, \\ 0, & \text{else} \end{cases} \qquad 0 < q < 1, \qquad p+q = 1,$$

have the unique solution

$$g(x) = \begin{cases} (-1)^{k-1}/k \cdot (q/p)^k, & x = \iota k, \quad k = 1, 2, \ldots \\ 0, & \text{else} \end{cases}$$

Proof One uses induction. For this rewrite the recurrence equations in form (2.4):

$$xg(x)\, f(0) = xf(x) - \sum_{y=1}^{\iota-1} yg(y)\, f(x-y).$$

For $x = 1, \ldots, \iota-1$ one obtains $g(x) = 0$ For $x = i$ the equation reads

$$\iota g(\iota) p = \iota q.$$

Hence one has $g(i) = q/p$. Let now $x > \iota$ and assume the formula for $g(y)$ correct for all $y < x$. If $x = ik$ is a multiple of i, then the right-hand side of the equation gives a contribution only for $x-y = i$, that is $y = (k-1)i$. The equation reads

$$ikg(x)p = -(k-1)\, \iota g((k-1)i)q$$

and the correct value of $g(x)$ is checked by induction assumption. When x is not a multiple of ι the right-hand side vanishes and hence $g(x) = 0$.

Example 2. Consider the individual life multiple decrement model which has applications in pension theory for example (see Bowers et al. (1986)). Let m be the number of causes of decrement and let the vector $\underline{s} = (s_1, \ldots, s_m)$ represent amounts at risk, s_j being a sum at risk due to cause j. The vector $\underline{s}$ is assumed to take values in a finite set $A \subset \mathbb{Z}^m$. Let $n_{\underline{s}k}$ be the number of policies with risk

sum structure $\underline{s}$ and probabilities of decrement $q_k^{(j)}$ due to cause, j, $j = 1, \ldots, m$, $k = 1, \quad , b$ Let $p_k^{(\tau)} = 1 - \sum_{j=1}^{m} q_k^{(j)}$ be the survival probability due to all causes of decrement Denote by $n_k = \sum_{\underline{s} \in A} n_{\underline{s}k}$ the number of policies with survival probability $p_k^{(\tau)}$ and by $n = \sum_{k=1}^{b} n_k$ the total number of policies The maximum possible amount of aggregate claims is denoted by M and is equal to

$$M = \sum_{k=1}^{b} \sum_{\underline{s} \in A} \max_{1 \le j \le m} (s_j)\, n_{\underline{s}k}.$$

Moreover let the random variable $X_{\underline{s}k}$ represent the claim produced by a policy with risk sum structure $\underline{s}$ and probabilities of decrement $q_k^{(j)}$, $j = 1, \ldots, m$, $k = 1, \ . \ , b$ Its probability density function, denoted by $f_{\underline{s}k}(x)$, is given by

$$f_{\underline{s}k}(x) = \begin{cases} p_k^{(\tau)}, & x = 0 \\ q_k^{(j)}, & x = s_j, \quad j = 1, \ . \ , m, \\ 0, & \text{else} \end{cases} \tag{3.10}$$

Evaluate now the probability density function of aggregate claims using Panjer's recursive formula (3.4). We have clearly

$$f(0) = \prod_{k=1}^{b} (p_k^{(\tau)})^{n_k}.$$

For $x > 0$ it is necessary to evaluate first in a recursive manner the functions $g_{\underline{s}k}(x)$ such that

$$x f_{\underline{s}k}(x) = \sum_{y=1}^{x} y g_{\underline{s}k}(y)\, f_{\underline{s}k}(x-y), \qquad \underline{s} \in A, \qquad k = 1, \ . \ , b. \tag{3.11}$$

Then

$$h(x) = 1/\lambda \sum_{\underline{s} \in A} \prod_{k=1}^{m} n_{\underline{s}k}\, g_{\underline{s}k}(x), \tag{3.12}$$

$$\lambda = -\ln \{f(0)\},$$

is introduced in the recursive formula (3.4). It is important to note that the proposed algorithm requires a two-stage nested recursive computation. Up to the maximum possible amount of aggregate claims M prepare for each $y = 1, 2, \ . \ , M$ the finite number of elements $g_{\underline{s}k}(y)$ recursively solving (3.11) such that

$$(3\,13)\qquad g_{\underline{sk}}(y) = \left[y f_{\underline{sk}}(y) - \sum_{z=1}^{y-1} z g_{\underline{sk}}(z)\, f_{\underline{sk}}(y-z) \right] / (y p_k^{(\tau)})$$

Then apply Panjer's recursive formula (3 4) computing $h(y)$ using formula (3.12) As many of the values $f_{\underline{sk}}(y)$ indeed vanish the summation in (3.13) extends over at most m terms. To illustrate consider the double-decrement model with $m = 2$, for example death and withdrawal or death and disability as causes of decrement. Use for brevity the notation $\underline{s} = (\iota, \jmath)$ with $A = \{1 \leq \iota, \jmath \leq a\}$ Assuming $i < j$ (the other cases $\iota = \jmath$ and $\iota > \jmath$ are similar) the elements $g_{\underline{sk}}(x)$ are computed more efficiently by the recursive formulae

$$(3\,14)\qquad q_{\underline{sk}}(x) = \begin{cases} 0, & \text{if} \quad x \in \{1, \ldots, \iota - 1\} \quad \text{or} \\ & x \in \{\iota + 1, \ldots, \jmath - 1 \mid x \text{ not multiple of } \iota\} \\ (-1)^{r-1}/r \cdot (q_k^{(1)}/p_k^{(\tau)})^r, & \text{if} \quad x = r\iota \neq \jmath, \\ & r \in \{1, \ldots, [\jmath/\iota]\} \\ g_k^{(2)}/p_k^{(\tau)}, & \text{if} \quad x = \jmath \text{ is not multiple of } \iota \\ q_k^{(2)}/p_k^{(\tau)} + (-1)^{r-1}/r \cdot (q_k^{(1)}/p_k^{(\tau)})^r, & \text{if} \\ x = \jmath = r\iota \quad \text{for} \quad r \in \mathbb{N}, & \\ - [(x-\jmath)\, g_{\underline{sk}}(x-\jmath)\, q_k^{(2)} + & \\ + (x-\iota)\, g_{\underline{sk}}(x-\iota)\, q_k^{(1)}]/(x p_k^{(\tau)}), & \text{if} \quad x > \jmath \end{cases}$$

An alternative derivation and additional formulae concerning the individual model of risk theory can be found in De Pril (1989)

4 PARAMETRIC AGGREGATE CLAIMS MODELS

It is well-known that the compound Poisson gamma and the compound negative binomial exponential distributions can be expressed as analytical series, the latter one as a finite sum. Other cases are less well-known. For many practical purposes it is most desirable to have tractable parametric functions modeling aggregate claims. The classical approach to this problem uses asymptotic approximate formulae as Normal, Normal-Power, Wilson-Hilferty, three-parameter gamma, Haldane, Esscher transforms and others These approximations are attached with approximation errors which are usually difficult to control Furthermore the structure of the claim size density has been lost in these models Since it is often necessary to study claims frequency and claim size separately, parametric aggregate claims models with explicit structure of claim number and claim size distribution are of interest This can be achieved solving analytically integral equations of the form (1 4). The method is illustrated at a simple new case, namely a modified two parameter gamma aggregate claims model.

Let $f(x)$ be an aggregate claims density such that $0 < F(0) = \exp(-\lambda) < 1$. This assumption is in particular fulfilled for a Poisson claim number model with parameter λ and when there are no claims of amount ≤ 0 More generally

this can be assumed for infinitely divisible aggregate claims distributions defined on $[0, \infty)$ (see Corollary 1). Rewrite the density as

$$f(x) = \exp(-\lambda)\,\delta(x) + g(x) \tag{4.1}$$

The derivative $(d/ds)\,Lf(s)$ of a Laplace transform is denoted for short by $L'f(s)$ Solving the integral equation (1 4) is equivalent to solving a differential equation in the Laplace space and taking inverse Laplace transforms The differential equation reads

$$L'g(s) = \lambda Lg(s)\,L'h(s) + \lambda \exp(-\lambda)\,L'h(s) \tag{4 2}$$

Given the function $h(x)$ its general solution is

$$Lg(s) = c \cdot \exp(\lambda Lh(s)) - \exp(-\lambda). \tag{4.3}$$

where c is a constant. We have gained nothing since this is equivalent to the pseudo compound Poisson representation and is difficult to handle analytically. However specifying the function $g(x)$ it might be easier to find $h(x)$ according to the formula

$$L'h(s) = \exp(\lambda)\,L'g(s)/[\lambda(1 + \exp(\lambda)\,Lg(s))] \tag{4.4}$$

For the modified two-parameter gamma aggregate claims model, the task is to find the pseudo density $h(x)$ which corresponds to

$$g(x) = (1 - \exp(-\lambda))\,\mu^a x^{a-1} \exp(-\mu x)/\Gamma(a), \qquad a \geq 1, \qquad \mu > 0 \tag{4.5}$$

Setting $\omega = 1 - \exp(-\lambda)$ one gets

$$Lg(s) = \omega(1 + s/\mu)^{-a}, \qquad L'g(s) = -(a\omega/\mu)\,(1 + s/\mu)^{-a-1} \tag{4.6}$$

After straightforward calculation it follows that

$$L'h(s) = -a a^a/[\lambda(s+\mu)\,((s+\mu)^a + a^a)], \tag{4.7}$$

where a is the positive a-th root defined by

$$a^a = (\exp(\lambda) - 1)\,\mu^a \tag{4.8}$$

Inverse Laplace transformation yields

$$h(x) = \exp(-\mu x)/\lambda x \int_0^x L^{-1}[a a^a/(a^a + s^a)](y)\,dy \tag{4.9}$$

We show now that for integer values $a = n = 1, 2, 3, \ldots$ the function $h(x)$ has a *finite closed form.* Using properties of the Laplace transform it suffices to invert the functions

$$L'\bar{h}(s) = -1/[s(1 + s^n)] = s^{n-1}/(1 + s^n) - 1/s, \qquad n = 1, 2, \ldots \tag{4.10}$$

Set $\bar{h}(x) = \bar{h}_1(x) + \bar{h}_2(x)$ with $L'\bar{h}_1(s) = -1/s$, $L'\bar{h}_2(s) = s^{n-1}/(1 + s^n)$. It follows that $\bar{h}_1(x) = 1/x$, $x > 0$, and $\bar{h}_2(x) = -(1/x) \cdot L^{-1}[s^{n-1}/(1 + s^n)](x)$, $x > 0$. To find the latter inverse Laplace transform expand the rational function as a partial fraction (e.g. DOETSCH (1976), p. 89):

$$(4.11) \qquad s^{n-1}/(1+s^n) = 1/n \sum_{k=0}^{n-1} 1/(s-\exp(\iota(2k+1)\pi/n))$$

and re-group the complex conjugate terms. As n is odd or not one obtains two different formulae summarized as follows:

$$(4.12) \qquad s^{n-1}/(1+s^n) = (1/n)\left[(1-(-1)^n)/[2(1+s)] + \right.$$
$$\left. + \sum_{k=0}^{[n/2]-1} 2(s-a_{k,n})/(s^2-2a_{k,n}s+1)\right]$$

where $a_{k,n} = \cos[(2k+1)\pi/n]$. For later use set $\beta_{k,n} = |\sin[(2k+1)\pi/n]|$. From a table of Laplace transforms (e.g. DOETSCH (1976)) one has

$$L^{-1}[1/(s^2-2as+1)](x) = (1/\beta)\exp(ax)\sin(\beta x).$$

It follows that

$$(4.13) \qquad L^{-1}[(2s-2a)/(s^2-2as+1)](x) = 2\exp(ax)\cos(\beta x)$$

whenever $a^2+\beta^2 = 1$. Using these results one gets after some algebraic manipulation the pseudo density in form of a finite sum:

$$(4.14) \qquad h(x) = (\exp(-\mu x)/\lambda x)\left[n-(1-(-1)^n)\exp(-ax)/2 - \right.$$
$$\left. - \sum_{k=0}^{[n/2]-1} 2\exp(a_{k,n}ax)\cos(\beta_{k,n}ax)\right]$$

with $a = (\exp(\lambda)-1)^{1/n}\mu$. In particular for lower dimensions one has the pseudo densities

$$n = 1: \; h(x) = \exp(-\mu x)(1-\exp(-ax))/(\lambda x),$$
$$a = \mu(\exp(\lambda)-1),$$
$$(4.15) \qquad n = 2: \; h(x) = 2\exp(-\mu x)(1-\cos(ax))/(\lambda x),$$
$$a = \mu\sqrt{\exp(\lambda)-1},$$
$$n = 3: \; h(x) = [\exp(-\mu x)/(\lambda x)][3-\exp(-ax) -$$
$$- 2\exp(ax/2)\cos\{(\sqrt{3}/2)ax\}],$$
$$a = \mu\sqrt[3]{\exp(\lambda)-1}$$

We apply now Corollary 2. For $n = 1, 2$ we have $h(x) > 0$ and the corresponding model (4.1) is infinitely divisible and thus compound Poisson For $n = 3$ one may have $h(x) < 0$. Hence (4.1) is not infinitely divisible and thus only pseudo compound Poisson In particular we have shown that the classe $\mathcal{P}'$ is bigger than the class of infinitely divisible probability density functions defined on $(0, \infty)$ As known to the author the present model $n = 1$

is among the few examples of compound Poisson models allowing *finite analytical sum expressions* for the main risk theoretical quantities of interest. In particular it is comparable to the Poisson exponential aggregate claims model concerning mathematical simplicity.

Furthermore analytical expressions for the finite and infinite time ruin probabilities can be derived. We have computed the simple case $n = 1$ (details of calculation in appendix). Assume a stationary evolution of the portfolio. In this context $P = (1+\theta)\lambda m$ represents the premiums received continuously per unit of time, with θ the security loading, m the expected claim size, and λ measures the expected number of claims per unit of time Then the probability of ruin $\psi(x, t)$ before time t given the initial reserves x is

$$\psi(0, t) = 1/(1+\theta) - (1-\exp(-\lambda t))\ \exp(-\mu P t)/(\mu P t), \tag{4.16}$$

and for $x > 0$,

$$\begin{aligned}\psi(x, t) = {} & (1-\exp(-\lambda t))\ \exp(-\mu(x+Pt)) + \\ & + \theta/(1+\theta)\cdot\exp(-\mu x)\cdot[\lambda/(\lambda+P\mu) - \\ & - \exp(-\mu P t)\cdot\{1-\exp(-\lambda t)\cdot P\mu/(\lambda+P\mu)\}] + \\ & + \exp(-\mu(x+Pt))\ \sum_{k=2}^{\infty}\ (-\lambda t)^k/k!\ \sum_{j=1}^{k-1}\ 1/j.\end{aligned} \tag{4.17}$$

Taking limits as $t \to \infty$ it follows that the infinite time ruin probabilities are

$$\begin{aligned}\psi(0) &= 1/(1+\theta), \\ \psi(x) &= \theta/(1+\theta)\cdot\exp(-\mu x)\cdot\lambda/(\lambda+P\mu), \qquad x > 0.\end{aligned} \tag{4.18}$$

The obtained results will practically be more useful if one fits the claim size density by a linear combination of densities as follows:

$$h(x) = \sum_{i=1}^{r}\ c_i h_i(x), \qquad c_1 + \ldots + c_n = 1, \tag{4 19}$$

$$\begin{aligned}h_i(x) &= \exp(-\mu_i x)\cdot\{1-\exp(-a_i x)\}/\lambda x, \\ a_i &= (\exp(\lambda)-1)\mu_i.\end{aligned}$$

From the proof of Theorem 1 we know that the aggregate claims density $f(x, t)$ up to time t satisfies the Laplace representation

$$(Lf)(s) = \exp(-\lambda t)\ \exp(\lambda t L h(s)) = \prod_{i=1}^{r}\ \exp(-\lambda c_i t)\cdot\exp(\lambda c_i t L h_i(s)).$$

Define $f_i(x, t)$ as solution of the Laplace equation

$$(Lf_i)(s) = \exp(-\lambda c_i t)\cdot\exp(\lambda c_i t L h_i(s)).$$

As we have shown, one obtains by inversion

$$f_i(x, t) = \exp(-\lambda c_i)\,\delta(x) + (1-\exp(-\lambda c_i t))\cdot\mu_i\exp(-\mu_i x). \tag{4.20}$$

The direct calculation of the convolutions

$$f(x, t) = f_1(x, t) * \ldots * f_r(x, t)$$

yields the formula (use induction):

$$(4.21) \quad f(x, t) = \exp(-\lambda t)\,\delta(x) + \sum_{i=1}^{r} (1 - \exp(-\lambda c_i t)) \times \left[\prod_{j \neq i} (\mu_j - \mu_i \exp(-\lambda c_j t))/(\mu_j - \mu_i)\right] \times \mu_i \exp(-\mu_i x)$$

In this model the net stop-loss premiums to the priority M can be expressed as finite analytical sums, namely

$$(4.22) \quad SL(F, M) = \int_M^\infty (x - M)\, f(x, t)\, dx = \sum_{i=1}^{r} (1 - \exp(-\lambda c_i t)) \times \left[\prod_{j \neq i} (\mu_j - \mu_i \exp(-\lambda c_j t))/(\mu_j - \mu_i)\right] \times \exp(-\mu_i M)/\mu_i$$

Analytical formulae for the finite and infinite time ruin probabilities can also be derived

APPENDIX
CALCULATION OF RUIN PROBABILITIES

Assume an aggregate claims distribution function up to time t of the form

$$F(x, t) = 1 - (1 - \exp(-\lambda t)) \cdot \exp(-\mu x).$$

Then the probability of survival to time t, denoted by $U(x, t) = 1 - \psi(x, t)$, can be calculated using Seal's formulae (e.g GERBER (1979)).

$$U(0, t) = \theta/(1 + \theta) + (1/Pt) \int_{Pt}^{\infty} (1 - F(z, t))\, dz$$

$$U(x, t) = F(x + Pt, t) - P \int_0^t U(0, t - w)\, f(x + Pw, w)\, dw$$

One obtains

$$U(0, t) = \theta/(1 + \theta) + (1 - \exp(-\lambda t)) \cdot \exp(-\mu P t)/(\mu P t).$$

Further calculate

$$U(x,t) = 1-(1-\exp(-\lambda t))\cdot\exp(-\mu(x+Pt)) -$$
$$- P\int_0^t [\theta/(1+\theta)+(1-\exp(-\lambda(t-w))\times$$
$$\times \exp(-\mu P(t-w))/(\mu P(t-w))]\times$$
$$\times [\exp(-\lambda w)\,\delta(x+Pw)+(1-\exp(-\lambda w))\times$$
$$\times \exp(-\mu(x+Pw))]\,dw.$$

Since $x+Pw > 0$ for $w \in (0, t)$ the term in $\delta(x+Pw)$ does not contribute to the integral. For clearness write

$$U(x,t) = 1-(1-\exp(-\lambda t))\cdot\exp(-\mu(x+Pt))+I_1+I_2,$$

with

$$I_1 = -P\int_0^t \theta/(1+\theta)\cdot(1-\exp(-\lambda w))\times$$
$$\times \mu\exp(-\mu(x+Pw))\,dw,$$
$$I_2 = -P\int_0^t (1-\exp(-\lambda(t-w)))\cdot(1-\exp(-\lambda w))\times$$
$$\times \exp(-\mu(x+Pt))/(P(t-w))\cdot dw$$

The evaluation of the first integral gives

$$I_1 = \theta/(1+\theta)\cdot\exp(-\mu x)\cdot\left[-P\mu\int_0^t \exp(-\mu P w)\,dw + \right.$$
$$\left. + P\mu\int_0^t \exp(-(\lambda+P\mu)w)\,dw\right]$$
$$= \theta/(1+\theta)\ \exp(-\mu x)\ [\exp(-\mu P t) -$$
$$- 1+P\mu/(\lambda+P\mu)\cdot(1-\exp(-(\lambda+P\mu)t))]$$
$$= \theta/(1+\theta)\ \exp(-\mu x)\cdot[\exp(-\mu P t)\times$$
$$\times \{1-\exp(-\lambda t)\cdot P\mu/(\lambda+P\mu)\}-\lambda/(\lambda+P\mu)]$$

To evaluate the second integral expand the first exponential function in a Taylor series to get

$$I_2 = -\exp(-\mu(x+Pt))\sum_{k=0}^{\infty}(-1)^k\lambda^{k+1}/(k+1)!$$
$$\int_0^t (1-\exp(-\lambda w))\ (t-w)^k\,dw$$

By induction one shows the recursive relation

$$\int_0^t \exp(-\lambda w)\ (t-w)^k\, dw$$

$$= t^k/\lambda - k/\lambda \int_0^t \exp(-\lambda w) \times (t-w)^{k-1}\, dw, \qquad k > 0,$$

with starting value

$$\int_0^t \exp(-\lambda w)\, dw = (1-\exp(-\lambda t))/\lambda.$$

It follows that

$$\int_0^t (1-\exp(-\lambda w))\cdot(t-w)^k\, dw = t^{k+1}/(k+1) - k!$$

$$\left[\exp(-\lambda t)/(-\lambda)^{k+1} - \sum_{j=0}^{k} t^j/j!\,(-\lambda)^{k+1-j}\right]$$

Introduced above one obtains

$$I_2 = \exp(-\mu(x+Pt))\cdot[S_1+S_2+S_3]$$

with

$$S_1 = \sum_{k=0}^{\infty} \{1/(k+1)\}\cdot(-\lambda t)^{k+1}/(k+1)!,$$

$$S_2 = -\sum_{k=0}^{\infty} 1/(k+1) \sum_{j=0}^{\infty} (-\lambda t)^j/j!,$$

$$S_3 = \sum_{k=0}^{\infty} 1/(k+1) \sum_{j=0}^{k} (-\lambda t)^j/j!.$$

But one has

$$S_1+S_2+S_3 = -\sum_{k=0}^{\infty} 1/(k+1)\cdot \sum_{j=k+2}^{\infty} (-\lambda t)^j/j!$$

$$= -\sum_{j=2}^{\infty} (-\lambda t)^j/j! \sum_{k=1}^{j-1} 1/k,$$

the last equality being obtained by interchanging the order of summation Therefore formula (4.17) is shown.

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