

MUTUAL REINSURANCE AND HOMOGENEOUS LINEAR ESTIMATION

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ABSTRACT

The technique of risk invariant linear estimation from NEUHAUS (1988) has been applied in the construction of a mutual quota share reinsurance pool between the subsidiary companies of the Storebrand Insurance Company, Oslo. The paper describes the construction of the reinsurance scheme.

1. INTRODUCTION

The Storebrand Insurance Company is the largest non-life insurer in Norway. Non-life business is written by four wholly owned stock companies, each covering a certain geographic area. The regional companies enjoy a large degree of autonomy, while certain areas, like tariffication and reinsurance, are managed centrally.

All but one of the regional companies are small, measured even by Norwegian standards. This makes their profitability subject to large fluctuations, even after deduction of external reinsurance. In 1987, the company top management issued a request to devise a way of stabilising the regional companies' profitability. The idea of additional reinsurance was launched at an early stage, and all the traditional forms of reinsurance were discussed. During the discussions a number of guidelines were formulated.

1. The reinsurance should give protection against large claims, as well as large claim numbers (typically caused by spells of bad weather).
2. No additional external reinsurance was to be bought.
3. The reinsurance should be fair, it should not take the accountability off the regional companies (other than correcting for "random" fluctuations).
4. The reinsurance should be very easy to administer.
5. Compulsory participation for the 4 regional companies.

Guidelines 1 and 4 quickly disqualified excess of loss reinsurance and surplus reinsurance. Guideline 3 disqualified stop-loss reinsurance. Left over was quota share reinsurance. The solution arrived at was a mutual quota-share pool, described briefly as follows.

- a. Each regional company cedes a certain share of its business (premium and losses) to the pool. Business to be ceded is own account business, i.e. after deduction of external reinsurance.
- b. The total losses ceded to the pool are redistributed amongst the participating companies in the same proportion as premium was ceded to the pool.
- c. The premium ceded to the pool is returned in its entirety, thus leaving the regional companies' premium unaltered.

The arrangement described is essentially a loss pool, since only losses (not premium) are affected. A desirable side effect of this property is that the regional companies' expense ratio is left unchanged; thus eliminating the need for reinsurance commission.

The mutual quota share pool is a very traditional way of reinsurance, which does not necessarily make it a poor way of reinsurance. In the following chapter a mathematical model is given, within which the mutual quota share is optimal.

2. OPTIMAL REINSURANCE IN THE BÜHLMANN-STRAUB MODEL

Let us number the regional companies by $i = 1, \dots, I$. For company i , define $P_i =$ premium for own account, $S_i =$ losses for own account, $X_i = S_i/P_i =$ loss ratio for own account. Note that "for own account" in this context means business net of external reinsurance, but before application of the mutual quota-share treaty.

We make the assumptions of the BÜHLMANN-STRAUB model (BÜHLMANN & STRAUB, 1970). These assumptions are that there exists a latent parameter θ_i so that

$$(2.1) \quad E(X_i | \theta_i) = b(\theta_i),$$

$$(2.2) \quad \text{Var}(X_i | \theta_i) = v(\theta_i)/P_i,$$

where b and v are real-valued functions of θ_i . It is then assumed that the parameters θ_i are i.i.d. random variables, and that

$$(2.3) \quad E(b(\theta_i)) = \beta,$$

$$(2.4) \quad E(v(\theta_i)) = \phi,$$

$$(2.5) \quad \text{Var}(b(\theta_i)) = \lambda.$$

These assumptions obviously fit the problem to be solved very well. The function $b(\theta_i)$ is interpreted as the underlying (long-run) loss ratio of company i , and the aim of the exercise is to estimate this quantity.

For fixed values of the parameters β, ϕ, λ , the best linear estimator of $b(\theta_i)$ (with respect to mean squared error) is the credibility estimator

$$(2.6) \quad \bar{b}_i = z_i X_i + (1 - z_i) \beta,$$

where

$$(2.7) \quad z_i = P_i / (P_i + \kappa),$$

$$(2.8) \quad \kappa = \phi/\lambda.$$

To simplify notation, define

$$(2.9) \quad c_i = 1 - z_i,$$

and note the relation

$$(2.10) \quad P_i c_i = \kappa z_i.$$

For fixed values of ϕ , λ , and unknown β , the best linear unbiased estimator of $b(\Theta_i)$, based on X_1, \dots, X_I , is

$$(2.11) \quad \bar{b}_i = z_i X_i + (1 - z_i) \hat{\beta},$$

where

$$(2.12) \quad \hat{\beta} = [\sum_j z_j]^{-1} \sum_j z_j X_j.$$

Proofs of the optimality of (2.6), (2.11) may be found in BÜHLMANN (1970).

A risk exchange between the I companies is given by the transformation

$$(2.13) \quad (S_1, \dots, S_I) \rightarrow (\tilde{S}_1, \dots, \tilde{S}_I) = (P_1 \bar{b}_1, \dots, P_I \bar{b}_I).$$

This risk exchange is defined by replacing each company's loss ratio X_i with the estimate \bar{b}_i . It is optimal in the sense of minimum mean squared error estimation of the "underlying loss ratio" $b(\Theta_i)$. That the risk exchange coincides with a mutual quota share treaty may be seen by

$$(2.14) \quad \begin{aligned} \tilde{S}_i &= P_i \bar{b}_i = P_i \{z_i X_i + c_i \hat{\beta}\} = z_i S_i + P_i c_i z^{-1} \sum_j z_j (S_j / P_j) \\ &= z_i S_i + \kappa z_i z^{-1} \sum_j c_j \kappa^{-1} S_j = z_i S_i + (z_i / z) \sum_j c_j S_j = z_i S_i + (z_i / z) S, \end{aligned}$$

where we have defined $z = \sum_j z_j$, $S = \sum_j c_j S_j$. The variable S is just the total losses ceded to the pool. The risk exchange (2.13) replaces the losses of company i with the sum of the retained share and a share of the pool, the share of the pool being z_i/z . To see that this share is equal to the proportion of premium ceded to the pool, note that $z_i/z = P_i c_i / \sum_j P_j c_j$.

A direct consequence of (2.14) is the identity

$$(2.15) \quad \sum_i \tilde{S}_i = \sum_i S_i,$$

which makes (2.13) a proper risk exchange in the sense of BÜHLMANN & JEWELL (1979). GISLER (1987) mentions the property (2.15); it ensures that no claims are "lost" when homogeneous credibility estimation is applied.

3. CHOICE OF MODEL

Let us consider one line of business. The risk exchange (2.13) is characterised by the value of "action parameter" κ , entering into the credibility factors z_i , see (2.7). Ideally one should use $\kappa = \phi/\lambda$, where ϕ , λ are the true variances. Since it is preposterous to try to separate empirically the variance components ϕ and λ from just 4 replications (companies), and since the author does not subscribe to subjectivism, we applied the minimax approach of NEUHAUS (1988), which is sketched in the sequel.

For a fixed $k > 0$, define the risk exchange $S \rightarrow \tilde{S}(k)$ by

$$(3.1) \quad \tilde{S}_i(k) = z_i(k) S_i + (z_i(k)/z(k)) S(k),$$

where $z_i(k) = P_i/(P_i+k)$, $z(k) = \sum_j z_j(k)$, $c_j(k) = 1 - z_j(k)$, $S(k) = \sum_j c_j(k) S_j$.

This risk exchange has obviously the same structure as (2.13), only κ is replaced by k . Let $\tilde{X}_i(k) = \tilde{S}_i(k)/P_i$ be the loss ratio after reinsurance.

The loss incurred by using k as action parameter is measured by the loss function

$$(3.2) \quad L(k, \phi, \lambda) = I^{-1} \sum_i E(\tilde{X}_i(k) - b(\Theta_i))^2,$$

the objective being to minimize (3.2). It can be shown that

$$(3.3) \quad L(k, \phi, \lambda) = I^{-1} [\phi \sum_{i,j} g_{ij}^2(k)/P_j + \lambda \sum_{i,j} (\delta_{ij} - g_{ij}(k))^2],$$

where we have defined for $1 \leq i, j \leq I$,

$$(3.4) \quad g_{ij}(k) = \delta_{ij} z_i(k) + c_i(k) z_j(k)/z(k).$$

Assume that data available are P'_1, \dots, P'_I and X'_1, \dots, X'_I , representing premiums and loss ratios for (one or more) previous periods. Then one may estimate β by

$$(3.5) \quad \beta^* = \sum_i w'_i X'_i,$$

where $w'_i = P'_i / \sum_j P'_j$. The estimator β^* is the overall loss ratio for the period observed. The statistic

$$(3.6) \quad V^* = \sum_i w'_i (X'_i - \beta^*)^2$$

has expectation

$$(3.7) \quad E(V^*) = \lambda \sum_i w'_i (1 - w'_i) + \phi \sum_i w'_i (1 - w'_i)/P'_i.$$

The total variance in the loss ratio is estimated by V^* .

As in NEUHAUS (1988), the parameter $k > 0$ may be chosen so that the risk exchange $S \rightarrow \bar{S}(k)$ becomes an equaliser rule with respect to the parameter set

$$(3.8) \quad \mathcal{N} = \{(\phi, \lambda) \mid V^* = \lambda \sum_i w'_i(1-w'_i) + \phi \sum_i w'_i(1-w'_i)/P'_i\},$$

i.e. $L(k, \phi, \lambda) = \text{constant}$ for $(\phi, \lambda) \in \mathcal{N}$

The calculations needed to find k are similar to those given in NEUHAUS (1988). The reason for choosing an equaliser rule is that it will be a (restricted) minimax rule with respect to the parameter set \mathcal{N} . Note that $\{\kappa \mid (\phi, \lambda) \in \mathcal{N}\} = \langle 0, \infty \rangle$.

4. EXAMPLE

Consider the line "Small to medium commercial risk". Table 1 gives the relevant statistics for the year 1987. It is found that $\beta^* = 1.211$, $V^* = 0.102$,

$$\sum_i w'_i(1-w'_i) = 0.595, \quad \sum_i w'_i(1-w'_i)/P'_i = 0.021.$$

The value $k = 42$ makes the risk exchange an equaliser rule across the parameter set

$$(4.1) \quad \mathcal{N} = \{(\phi, \lambda) \mid 0.102 = \lambda \cdot 0.595 + \phi \cdot 0.021\}.$$

The factors c_i (42) are displayed in the rightmost column of table 1. One sees that the large company should cede about one-third of its business to the pool, while the 3 small companies should cede about two-thirds of their business to the pool.

TABLE 1
STATISTICS FOR "SMALL TO MEDIUM COMMERCIAL RISK"

Company	P'_i	X'_i	w'_i	$w'_i(X'_i - \beta^*)^2$	$c_i(42)$
East	81.366	1.425	0.590	0.026	0.34
South	19.816	1.163	0.144	0.000	0.68
West	18.149	0.475	0.132	0.071	0.70
North	18.596	1.047	0.135	0.003	0.69
Total	137.927	1.211 = β^*		0.102 = V^*	

Figure 1 shows the square root of the different loss functions (3.2) dependent on the true κ , where it is assumed that $(\phi, \lambda) \in \mathcal{N}$ given by (4.1). The square root is displayed because it is measured in the same scale as the estimand. The loss functions of three risk exchanges are displayed,

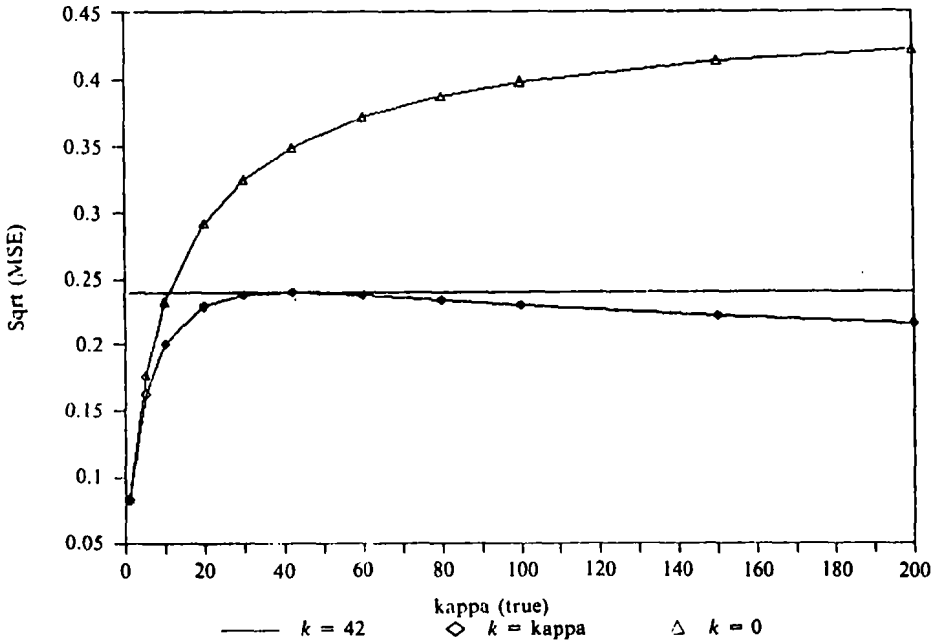


FIGURE 1

1. $k = 42$, giving the equaliser rule with respect to \mathcal{N}
2. $k = \kappa$, giving the optimal risk exchange (2.13),
3. $k = 0$, meaning no reinsurance at all.

It is seen that the choice $k = 42$ gives a constant loss function across \mathcal{N} and a considerable improvement over $k = 0$ (using $k = 0$ means judging each regional company only by its own loss ratio). The choice $k = \kappa$ is optimal, but the improvement it gives over $k = 42$ is very moderate over most of the parameter space displayed.

5. CONCLUDING COMMENTS

The aim of the paper has been to show that even a very traditional quota share pool reinsurance exhibits optimality properties when the shares are appropriately chosen.

Conceding that it is preposterous to separate empirically the variance components ϕ and λ , one may ask whether estimating $E(V^*)$ by V^* is any better; it is probably not, but the equaliser value of k does not depend on V^* , see NEUHAUS (1988).

The aim being to estimate the companies' loss ratio, should one include β^* in the estimator? Two arguments may be used against using β^* . The first argument

is that a linear estimator using β^* , being the empirical counterpart of (2.6), would not have the desirable property (2.15), thus it does not give a proper risk exchange. The second argument goes as follows: The parameter β should not be fixed but random, $\beta = \beta(\psi)$, and (2.1)-(2.5) should be conditional relations, given ψ . This is a hierarchical credibility model; let $\xi = \text{Var}(\beta(\psi))$. The optimal inhomogeneous estimator of $b(\theta_i)$ is then

$$(5.1) \quad \bar{b}_i = z_i(\kappa)X_i + (1 - z_i(\kappa)) \frac{\sum_j z_j(\kappa)X_j + \lambda\xi^{-1} E(\beta(\psi))}{\sum_j z_j(\kappa) + \lambda\xi^{-1}},$$

see SUNDT (1979). The estimator (2.11) is obtained by letting $\xi \rightarrow \infty$, which in the Bayesian context means using a vague prior distribution for $\beta(\psi)$.

One may contend that it is unnecessary to establish a reinsurance treaty in order to assess the 4 companies' underlying loss ratio, when simple calculation of the homogeneous unbiased linear estimator would do the job. But, as experience has shown, the bottom line after mutual reinsurance is accepted by everyone as true expression of a company's profitability. On the other hand, an actuary telling company management that "*well, the loss ratio is 120, but my model says it should have been 105*" is doomed to fail. The reinsurance treaty makes the same statement more credible.

The loss function (3.2) is an unweighted average of the 4 companies' loss functions. This loss function reflects the objective of estimating the companies' underlying loss ratio, regardless of their premium volume. In an economic environment, the loss function should be weighted to reflect the fact that a unit of error in assessing the loss ratio is most serious for the large companies. It is possible to find an equaliser rule for weighted loss function, and probably the optimal k would not be changed much, see NEUHAUS (1988).

A separate risk exchange was set up for each line of business. The obvious reason was to spare the accounting staff for troublesome allocation problems. Stabilising each line of business also had the positive side effect of reducing regional demands for immediate remedial action (premium increases or discounts) in the wake of fluctuating loss ratios.

A more complicated model is needed if one wants to design a risk exchange for the I companies, which spans all lines of business. Probably the simplest model would be of the form

$$(5.2) \quad b_{ij} = \mu + \alpha_i + \beta_j,$$

where

- b_{ij} is the underlying loss ratio for company i , line j ,
- μ is a fixed mean,
- α_i is a random parameter characterising company i ,
- β_j is a random parameter characterising line j .

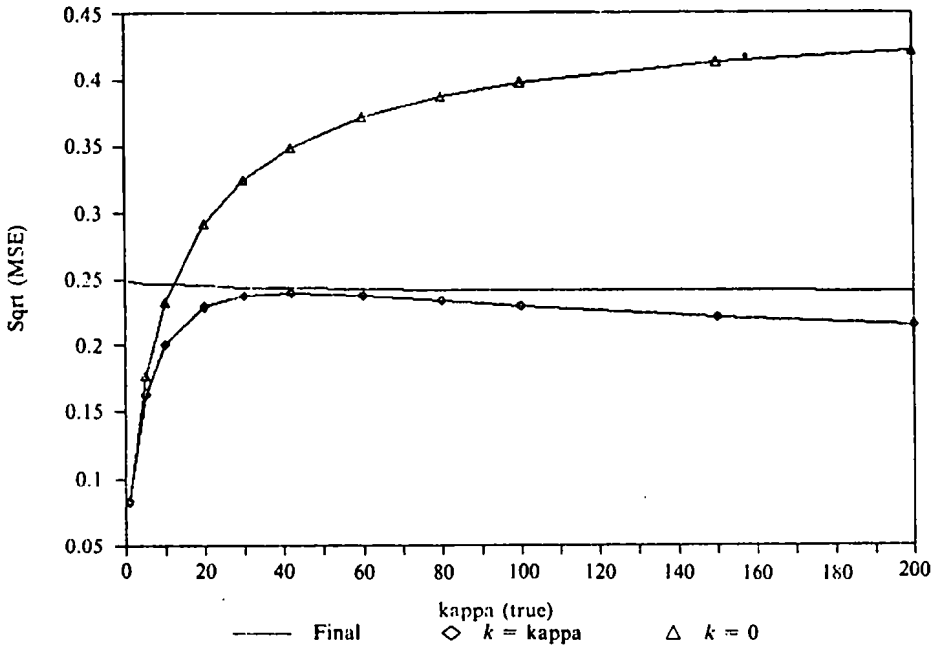


FIGURE 2

An estimator of the underlying loss ratio of company i is

$$(5.3) \quad \bar{b}_i = \bar{\mu} + \bar{\alpha}_i + \left[\sum_j p_{ij} \right]^{-1} \sum_j p_{ij} \bar{\beta}_j,$$

where $\bar{\mu}$, $\bar{\alpha}_i$, $\bar{\beta}_j$ are calculated by the credibility method described in BUCHANAN et al. (1989). Unfortunately, this method lacks the transparency which makes the estimators (2.6) and (2.11) so attractive.

A point of lengthy discussions was the choice of reinsured shares, although all but one company finally accepted the recommended shares. Figure 2 shows the loss function of the final scheme, compared with the optimal loss function ($k = \kappa$) and the loss function without reinsurance. We did not analyse whether the final scheme, being (very slightly) sub-optimal in the sense of minimaxing (3.2) over (4.1), has any other nice properties, such as Pareto-optimality. Here is a field for further analysis. Incidentally, if there is anything like empirical Pareto-optimality, the author has experienced it: Whatever modification of the scheme was suggested during the discussions, someone was certain to object.

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