

BOUNDS ON STOP-LOSS PREMIUMS FOR COMPOUND DISTRIBUTIONS

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ABSTRACT

Upper and lower bounds are derived for the stop-loss premium of compound distributions with fixed claim number distribution and known mean, variance and range for the claim severity distribution.

1. INTRODUCTION

In this paper we investigate bounds on stop-loss premiums for compound distributions

$$(1) \quad S = X_1 + X_2 + \cdots + X_N$$

where the claim number distribution F_N is fixed (e.g., Poisson (λ)) and where the claim severity distribution F_X is restricted to have

$$(2) \quad \begin{aligned} F_X(-0) &= 0 \\ F_X(b) &= 1 \\ E[X] &= \mu \end{aligned}$$

for some finite b and $\mu \in [0, b]$.

The stop-loss premium of S with stop-loss point t will be denoted by

$$(3) \quad \pi(t; F_X) = \int_t^\infty (x-t) d \left[\sum_{n=0}^\infty \Pr(N=n) F_X^{n*}(x) \right]$$

BÜHLMANN *et al.* (1977), introducing the concept of stop-loss ordering, derived bounds for $\pi(t; F_X)$.

In fact, when the random variable X^- equals μ with probability one, and X^+ has range $\{0, b\}$ and mean μ ,

$$(4) \quad \pi(t; F_{X^-}) \leq \pi(t; F_X) \leq \pi(t; F_{X^+})$$

uniformly in t and for all X satisfying (2).

To prove that this is true first observe that α and β exist such that

$$(5) \quad \begin{aligned} F_{X^-}(x) &\leq F_X(x) & \text{for } x < \alpha \\ F_X(x) &\geq F_X(x) & \text{for } x \geq \alpha \\ F_{X^+}(x) &\leq F_X(x) & \text{for } x \geq \beta \\ F_{X^+}(x) &\geq F_X(x) & \text{for } x < \beta. \end{aligned}$$

As $E[X^-] = E[X] = E[X^+]$, this means that X^- is less dangerous than X and X^+ is more dangerous. As a more dangerous distribution has higher stop-loss premiums, we have

$$(6) \quad X^- \propto X \propto X^+$$

where \propto denotes stop-loss order. Stop-loss order is preserved under compounding, so

$$(7) \quad \sum_{i=1}^N X_i^- \propto \sum_{i=1}^N X_i \propto \sum_{i=1}^N X_i^+$$

which is equivalent to (4) holding for all real t . For a more detailed proof, see GOOVAERTS *et al.* (1984)

Since X^- and X^+ satisfy all requirements for X , the bounds in (4) are best possible, and X^- and X^+ are extremal distributions.

It is not possible to give such extreme distributions when the variance of X is also fixed, say

$$(8) \quad \text{Var}(X) = \sigma^2$$

With the techniques of GOOVAERTS *et al.* (1984) one may compute extreme values of stop-loss premiums, but unfortunately the corresponding distributions depend on the value of the stop-loss point chosen. There is no severity distribution in this class that is smallest or largest in the sense of stop-loss order.

In Section 2 we exhibit random variables Z^- and Z^+ that give bounds like (4), uniformly in t . These bounds are not the best possible, since Z^- and Z^+ have variances different from σ^2 . They are, however, the greatest lower and least upper bound with respect to dangerousness.

In Section 3 we give a numerical illustration using the examples of GERBER (1982).

2. ANALYTICAL BOUNDS ON DISTRIBUTION FUNCTIONS

In GOOVAERTS and KAAS (1986) extreme values are given for distributions F_X with range $[0, b]$ and the first few moments fixed. When X has mean μ and variance σ^2 , we have

$$(10) \quad F^l(x) \leq F_X(x) \leq F^u(x)$$

with the values of F^l and F^u given in the following table, where $z = (x - \mu)/\sigma$ and $d = b\mu - \mu^2 - \sigma^2 \geq 0$ are used for notational convenience.

TABLE 1
BOUNDS FOR DISTRIBUTION FUNCTIONS WITH RANGE $[0, b]$,
MEAN μ AND VARIANCE σ^2

x	$F^l(x)$	$F^u(x)$
$0 < x \leq \frac{d}{b-\mu}$	0	$\frac{1}{1+z^2}$
$\frac{d}{b-\mu} \leq x \leq b - \frac{d}{\mu}$	$1 - \frac{\mu}{b} - \frac{d}{bx}$	$1 - \frac{\mu}{b} + \frac{d}{b(b-x)}$
$b - \frac{d}{\mu} \leq x \leq b$	$1 - \frac{1}{1+z^2}$	1

Now define the following two severity distributions:

$$(11) \quad F_{Z^+}(x) = \begin{cases} F^u(x) & 0 \leq x \leq \alpha \\ F^u(\alpha) & \alpha \leq x \leq \beta \\ F^l(x) & \beta \leq x \leq b \end{cases}$$

where α and β satisfy

$$(12) \quad \alpha = \mu + \frac{\sigma}{\sigma^2 + \mu(b-\mu)} \times [\sigma(b-2\mu) - \sqrt{\sigma^2(b-2\mu)^2 + (\sigma^2 + \mu(b-\mu))^2}]$$

$$\beta = \mu + \frac{\sigma}{\sigma^2 + \mu(b-\mu)} \times [\sigma(b-2\mu) + \sqrt{\sigma^2(b-2\mu)^2 + (\sigma^2 + \mu(b-\mu))^2}]$$

and

$$(13) \quad F_{Z^-}(x) = \begin{cases} F^l(x) & 0 \leq x < \mu \\ F^u(x) & \mu \leq x \leq b \end{cases}$$

With d as in table 1, we have $\alpha \leq d/(b-\mu)$ and $\beta \geq b - d/\mu$. To check that F_{Z^+} is well-defined and

$$E[Z^-] = E[Z^+] = \mu$$

is a laborious process but involves only elementary calculus. Since the distribution G with $dG(\alpha) = F^u(\alpha) = 1 - dG(\beta)$ has mean μ and variance σ^2 , we have $\text{Var}(Z^+) > \sigma^2$.

In fact, it may be shown that, writing $t^* = (t - \mu)/\sigma$ for all t ,

$$(14) \quad \text{Var}(Z^+) = \sigma^2 \left(1 + \ln \left(\frac{(1+b^{*2})(1+0^{*2})}{(1+\alpha^{*2})(1+\beta^{*2})} \right) \right)$$

In the same way, considering the distribution H with

$$dH(0) = F^l(\mu), \quad dH(\mu) = F^u(\mu) - F^l(\mu), \quad dH(b) = 1 - F^u(\mu),$$

which has mean μ and variance σ^2 , one shows that $\text{Var}(Z^-) < \sigma^2$. Because of (10) we have immediately that Z^- is less dangerous than any X , and Z^+ is more dangerous, so

$$(15) \quad \pi(t; Z^-) \leq \pi(t; X) \leq \pi(t; Z^+)$$

uniformly for all t and for all feasible X .

Now let W be a random variable with $dF_W(x) > 0$ for some x where also $F^l(x) < F_W(x) < F^u(x)$. It is easy to construct a feasible X with $F_X(x) = F_W(x)$ and x outside the spectrum of X : $dF_X(x) = 0$. But then either X is more dangerous than W , or X and W are not comparable because F_X and F_W have two more sign changes. So to be more dangerous than all X , F_W must be first above F^u , then constant between F^u and F^l , then below F^l . But then it is easy to see that F_{Z^+} and F_W have only one point of intersection, so Z^+ is less dangerous than W .

Reasoning along the same lines for Z^- we may conclude that among the distributions more dangerous than any feasible X , Z^+ is the least dangerous, whereas Z^- is the most dangerous less dangerous distribution. In this sense Z^+ and Z^- are optimal choices.

3. NUMERICAL ILLUSTRATION

In order to assess the quality of the bounds derived in the previous section, we give a numerical example. In GERBER (1982) methods are described to bound as well as to approximate stop-loss premiums of compound Poisson distributions. His method to obtain a lower bound using mass concentration does not always give an arithmetic discrete distribution, so we used the method of matching (two) moments, which is much more exact with the same computational effort. To obtain Gerber's uniform (1, 3) claim severity distribution as a special case, we took $b = 3$, $\mu = 2$ and $\sigma^2 = \frac{1}{3}$ in our examples, the claim numbers being Poisson (λ) with $\lambda = 1, 10$ and 100 .

TABLE 2
BOUNDS FOR STOP-LOSS PREMIUMS WITH CLAIM-RANGE $[0, 3]$, MEAN 2,
VARIANCE $\frac{1}{3}$ AND CLAIM NUMBER POISSON (1)

Stop-Loss Point t	Gerber's Exact Value	Upper Bound (4)	Upper Bound (15)	Lower Bound (15)	Lower Bound (4)
0	2 000	100 0%	100 0%	100 0%	100 0%
2	$8\,277 \times 10^{-1}$	124 1	113 5	89 3	88.9
4	$2\,689 \times 10^{-1}$	147 2	124 2	78 4	77.1
6	$7\,184 \times 10^{-2}$	149 4	134 8	67 3	65 0
8	1.627×10^{-2}	288 4	182.9	56 7	53 5
10	$3\,254 \times 10^{-3}$	364.0	227 3	46 2	42 3
12	5.815×10^{-4}	365 6	261 1	36 7	32 6
14	$9\,346 \times 10^{-5}$	921 3	356 9	28 7	24 6
16	$1\,366 \times 10^{-5}$	1166 9	495 0	22 1	18 3
18	$1\,840 \times 10^{-6}$	1154 0	597 4	16 8	13 3
20	$2\,302 \times 10^{-7}$	3578 6	807 5	12 5	9 5

TABLE 3

BOUNDS FOR STOP-LOSS PREMIUMS WITH CLAIM RANGE $[0, 3]$, MEAN 2, VARIANCE $\frac{1}{3}$ AND CLAIM NUMBER POISSON (10)

Stop-Loss Point t	Gerber's Exact Value	Upper Bound (4)	Upper Bound (15)	Lower Bound (15)	Lower Bound (4)
15	5 757	105 4%	103 0%	99 0%	99 0%
20	2 626	118 2	109 2	95 6	95 3
25	$9\,321 \times 10^{-1}$	141 6	121 2	91 6	91 6
30	$2\,563 \times 10^{-1}$	175 9	141 1	81 9	80 7
35	$5\,507 \times 10^{-2}$	257 4	172 0	74 7	74 7
40	$9\,383 \times 10^{-3}$	380 3	218 8	61 2	59 2
45	$1\,289 \times 10^{-3}$	551 5	289 4	53 1	53 1
50	$1\,449 \times 10^{-4}$	1028 0	396 5	40 2	37 9
55	$1\,355 \times 10^{-5}$	1762 1	560 8	33 3	33 3
60	$1\,067 \times 10^{-6}$	2829 9	816 8	23 5	21 5
65	$7\,175 \times 10^{-8}$	6465 7	1226 1	18 6	18 6

TABLE 4

BOUNDS FOR STOP-LOSS PREMIUMS WITH CLAIM RANGE $[0, 3]$, MEAN 2, VARIANCE $\frac{1}{3}$ AND CLAIM NUMBER POISSON (100)

Stop-Loss Point t	Gerber's Exact Value	Upper Bound (4)	Upper Bound (15)	Lower Bound (15)	Lower Bound (4)
180	$2\,177 \times 10^{-1}$	104 4%	102 2%	99.1%	99 1%
200	8 304	117 7	109 2	96 1	96 0
220	1 959	152 0	126.4	89 2	88 9
240	$2\,647 \times 10^{-1}$	230 0	161 6	78 1	77 6
260	$1\,992 \times 10^{-2}$	420 9	229 6	64 2	63 5
280	$8\,395 \times 10^{-4}$	922 3	364 7	49 3	48 4
300	2.319×10^{-5}	2846 4	725 9	33 8	33 4

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