

BOUNDS ON MODIFIED STOP-LOSS PREMIUMS  
IN CASE OF KNOWN MEAN AND  
VARIANCE OF THE RISK VARIABLE

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ABSTRACT

In case of a stop-loss treaty the reinsurer takes over that part of the risk that exceeds a given amount  $y_1$ . We will deduce bounds on a modified stop-loss treaty where the liability of the reinsurer is limited to  $y_2 - y_1$  in case the claim amount exceeds  $y_2$ . Upper and lower bounds of this modified stop-loss premium are obtained as a simple application of results obtained earlier by the first author.

INTRODUCTION

In case of a stop-loss treaty the insurer takes over that part of the risk that exceeds a given amount  $y_1$ . We now suppose that the stop-loss treaty is modified in such a way that the liability of the reinsurer is limited to  $y_2 - y_1$  in case the claim amount exceeds the amount  $y_2$ . Hence, the risk of the reinsurer can be cast into the form

$$Y = \begin{cases} 0 & X \leq y_1 \\ X - y_1 & y_1 < X \leq y_2 \\ y_2 - y_1 & y_2 < X. \end{cases}$$

The net premium then equals:

$$E(Y) = \int_{y_1}^{y_2} (x - y_1) dF_X(x) + (y_2 - y_1) \int_{y_2}^{\infty} dF_X(x)$$

which can still be cast into the following form:

$$E(Y) = \int_a^b \max \{ \min (x - y_1, y_2 - y_1), 0 \} dF_X(x)$$

where  $y_2 \geq y_1$ ,  $F_X(a) = 0$ ,  $F_X(b) = 1$ .

Let  $\psi(x) = \max \{ \min (x - y_1, y_2 - y_1), 0 \}$ , then with  $y_1, y_2, m, m_2$  real numbers, we have to consider the following primal problems:

$$\begin{aligned} p_1(m, m_2; y_1, y_2) \\ = \sup \left( \int_a^b \psi(x) dF(x) \mid \int_a^b x dF(x) = m, \int_a^b x^2 dF(x) = m_2, \int_a^b dF(x) = 1 \right) \end{aligned}$$

$$q_1(m, m_2; y_1, y_2) = \inf \left( \int_a^b \psi(x) dF(x) \middle| \int_a^b x dF(x) = m, \int_a^b x^2 dF(x) = m_2, \int_a^b dF(x) = 1 \right)$$

where the supremum (infimum) is taken over the distributions  $F$  on  $[a, b]$  satisfying the constraints indicated after the slash.

We remark that in case  $y_1 < a$  or  $y_2 > b$  the solution of the problem at hand coincides with the solution obtained in DE VYLDER and GOOVAERTS (1982a). This paper contains the basis for our present analysis and the same notation will be used.

Let us first consider the case  $y_1 < a$ . We have:

$$\int_a^b \psi(x) dF(x) = m - y_1 - \int_{y_2}^b (x - y_2) dF(x).$$

Consequently:

$$\sup \int_a^b \psi(x) dF(x) = m - y_1 - \inf \int_{y_2}^b (x - y_2) dF(x).$$

Hence:

$$p_1(m, m_2; y_1, y_2) = m - y_1 - q_1(m, m_2)$$

and

$$q_1(m, m_2; y_1, y_2) = m - y_1 - p_1(m, m_2)$$

where  $q_1(m, m_2)$  and  $p_1(m, m_2)$  are the values of the corresponding problems in DE VYLDER and GOOVAERTS (1982a), with  $e$  changed in  $y_2$ .

In case  $y_2 > b$ , on the other hand, we get:

$$\int_a^b \psi(x) dF(x) = \int_{y_1}^b (x - y_1) dF(x)$$

such that:

$$p_1(m, m_2; y_1, y_2) = p_1(m, m_2)$$

and

$$q_1(m, m_2; y_1, y_2) = q_1(m, m_2)$$

where  $p_1(m, m_2)$  and  $q_1(m, m_2)$  are the values of the corresponding problems in the cited reference, with  $e$  changed in  $y_1$ . Consequently, without loss of generality we can restrict ourselves to values  $y_1, y_2$  such that:

$$a < y_1 < y_2 < b.$$

The distribution  $F$  for which the supremum and infimum are obtained are 3-atomic at most, see e.g., DE VYLDER (1982). If  $\alpha$  and  $\beta$  are two different atoms of the 2-atomic probability distribution  $F$  satisfying the first-order moment

constraint  $\int x dF = m$ , then the corresponding probability masses  $p_\alpha, p_\beta$  must necessarily be:

$$p_\alpha = \frac{m - \beta}{\alpha - \beta}, \quad p_\beta = \frac{m - \alpha}{\beta - \alpha}.$$

If  $\alpha, \beta, \gamma$  are different atoms of the 3-atomic probability distribution  $F$ , satisfying the moment constraints  $\int x dF = m, \int x^2 dF = m_2$ , then the corresponding probability masses can only be:

$$p_\alpha = \frac{s^2 + (m - \beta)(m - \gamma)}{(\alpha - \beta)(\alpha - \gamma)}, \quad p_\beta = \frac{s^2 + (m - \alpha)(m - \gamma)}{(\beta - \alpha)(\beta - \gamma)}, \quad p_\gamma = \frac{s^2 + (m - \alpha)(m - \beta)}{(\gamma - \alpha)(\gamma - \beta)}.$$

The domain of the parameters  $m$  and  $m_2 = s^2 + m^2$  is defined as:

$$C' = \{(m, m_2) | a \leq m \leq b, 0 \leq s^2 \leq (m - a)(b - m)\}.$$

## 2. DEMONSTRATION OF THE MAIN RESULT

### Theorem

For  $(m, m_2)$  belonging to the domain  $C'$ , defined above, the problems  $p_1(m, m_2; y_1, y_2)$  and  $q_1(m, m_2; y_1, y_2)$  with  $a < y_1 < y_2 < b$  have the value and solution indicated in Table 1 and Table 2 at the end of this note.

### Demonstration

Let  $E$  be the curve with parametric equations:

$$X = x, \quad Y = x^2, \quad Z = \max\{\min(x - y_1, y_2 - y_1), 0\}, \quad a \leq x \leq b.$$

The curve  $E$  is shown in fig. 1. She consists of three parts  $E_1, E_2, E_3$ . The parametric representation in each of the three indicated regions is the following:

$$\begin{array}{lllll} E_1 & X = x & Y = x^2 & Z = 0 & a \leq x \leq y_1 \\ E_2 & X = x & Y = x^2 & Z = x - y_1 & y_1 \leq x \leq y_2 \\ E_3 & X = x & Y = x^2 & Z = y_2 - y_1 & y_2 \leq x \leq b. \end{array}$$

As far as the problem  $p_1(m, m_2; y_1, y_2)$  is concerned we get immediately three domains, namely  $D_1, D_2, D_3$ . We successively obtain:

$$(1) \quad D_1 = \{(m, s^2) | 1) a \leq m \leq y_2, (m - a)(y_2 - m) \leq s^2 \leq (m - a)(b - m) \\ 2) y_2 \leq m \leq b, (m - y_2)(b - m) \leq s^2 \leq (m - a)(b - m)\}.$$

The equation of the plane through the three points  $A, P_2$  and  $B$  enables us to construct an upper bound or a solution of the problem  $p_1(m, m_2; y_1, y_2)$  in  $D_1$ :

$$Z = (y_2 - y_1) \frac{Y - (y_2 + b)X + a(y_2 + b - a)}{(a - b)(y_2 - a)}.$$

Consequently in  $D_1$ :

$$p_1(m, m_2; y_1, y_2) = (y_2 - y_1) \frac{(m - a)(b + y_2 - m - a) - s^2}{(b - a)(y_2 - a)}$$

(2)  $D_2 = \{(m, s^2) | y_2 \leq m \leq b, 0 \leq s^2 \leq (m - y_2)(b - m)\}.$

In this case it is readily seen that:

$$p_1(m, m_2; y_1, y_2) = y_2 - y_1$$

(3)  $D_3 = \{(m, s^2) | a \leq m \leq y_2, 0 \leq s^2 \leq (m - a)(y_2 - m)\}.$

We consider a point  $Q_1$  on  $E_1$  with coordinates  $(x, x^2, 0)$  and a point  $Q_2$  on  $E_2$  with coordinates  $(y, y^2, y - y_1)$  and determine the equation of the plane through  $Q_1$  and  $Q_2$ , tangent on  $E_1$  in  $Q_1$  and tangent on  $E_2$  in  $Q_2$ . The equation reads:

$$Z = z_1 X + z_2 Y + z_3$$

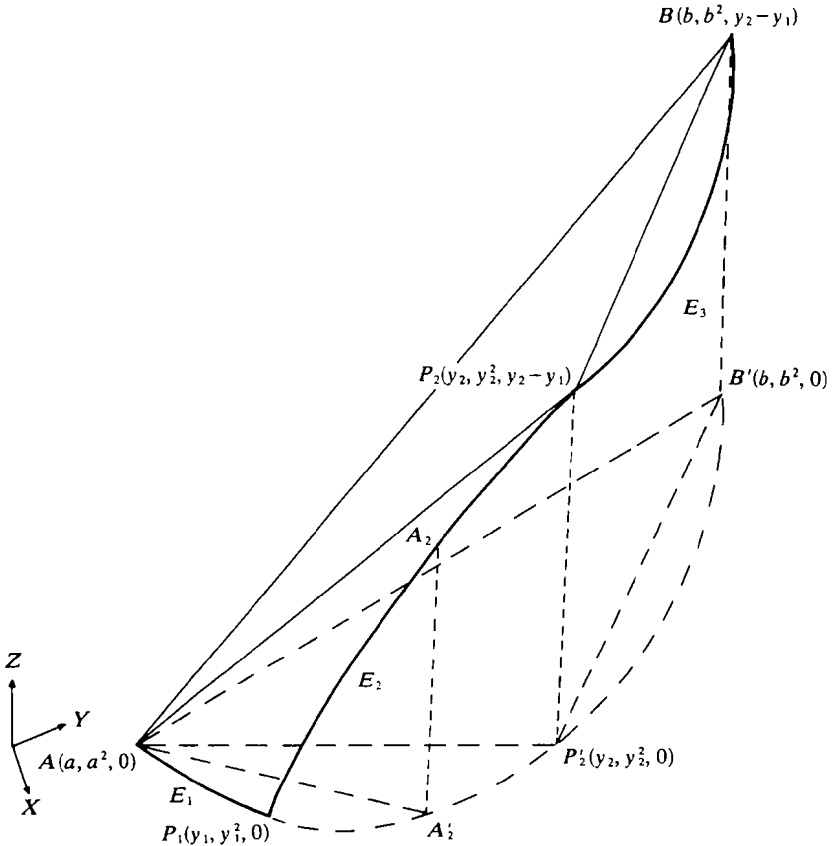


FIGURE 1 Curve E

with:

$$\begin{aligned} 0 &= z_1x + z_2x^2 + z_3 & Q_1 \in \text{plane} \\ y - y_1 &= z_1y + z_2y^2 + z_3 & Q_2 \in \text{plane} \\ 1 &= z_1 + 2z_2y & \text{tangent in } Q_2 \\ 0 &= z_1 + 2z_2x & \text{tangent in } Q_1. \end{aligned}$$

Solving the first three equations of this system of equations with respect to  $z_1$ ,  $z_2$ ,  $z_3$  gives:

$$z_2 = \frac{y_1 - x}{(x - y)^2}, \quad z_1 = \frac{x^2 + y^2 - 2yy_1}{(x - y)^2}.$$

Of course  $z_1$ ,  $z_2$  still need to satisfy the last equation. This gives:

$$(y - x)(y + x - 2y_1) = 0.$$

Hence with  $Q_1(x, x^2, 0)$  on  $E_1$  corresponds the point  $Q_2(2y_1 - x, (2y_1 - x)^2, y_1 - x)$  on  $E_2$ .

Now we have to consider two cases according to the position of the point  $A_2(2y_1 - a, (2y_1 - a)^2, y_1 - a)$  corresponding to  $A(a, a^2, 0)$ .

In case  $y_1 - a \leq y_2 - y_1$ ,  $A_2$  is lying under  $P_2$ , and we can consider a partition of  $D_3$  in  $D_{31}$  and  $D_{32}$ .

In case  $y_1 - a \geq y_2 - y_1$ , the point  $A_1$  on  $E_1$  corresponding with  $P_2$  is lying to the right of  $A$  and we have to consider a partition  $D'_{31}$  and  $D'_{32}$  as in fig. 3.

Let us examine now both cases separately.

(i)  $2y_1 - a \leq y_2$

$$D_{31} = \{(m, s^2) | a < m \leq 2y_1 - a, 0 \leq s^2 \leq (2y_1 - a - m)(m - a)\}.$$

The equation of the plane through  $(x, x^2, 0)$  tangent to  $E_1$  and also tangent to  $E_2$  is:

$$Z = \frac{1}{(x - y)^2} [(x^2 + y^2 - 2yy_1)(X - x) + (y_1 - x)(Y - x^2)]$$

of course with  $y = 2y_1 - x$ , or:

$$Z = \frac{1}{4(y_1 - x)} (-2xX + Y + x^2).$$

Hence the equation of the envelope of this set of planes reads:

$$4(y_1 - X + 2Z)Z = -2(X - 2Z)X + Y + (X - 2Z)^2.$$

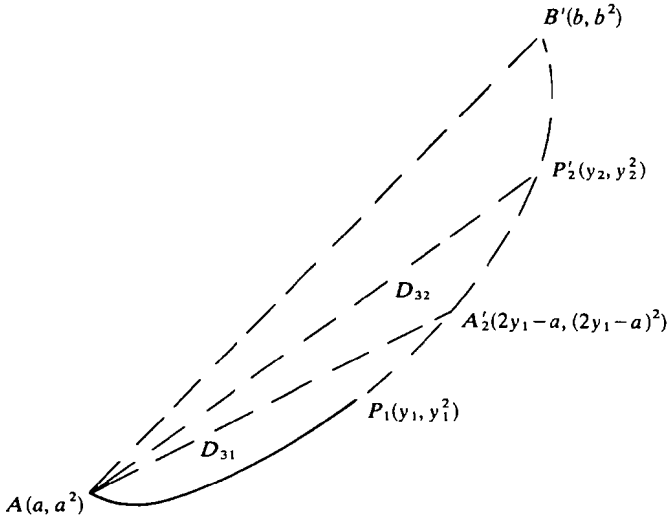


FIGURE 2 Partition of  $D_3$  in case  $2y_1 - a \leq y_2$

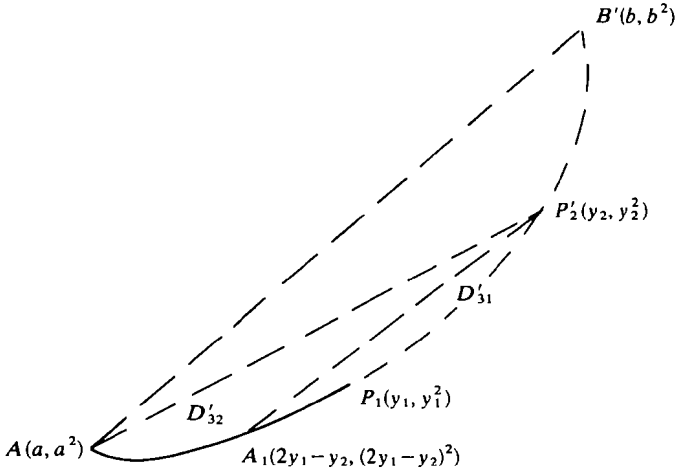


FIGURE 3 Partition of  $D_3$  in case  $2y_1 - a \geq y_2$ .

Consequently:

$$p_1(m, m_2; y_1, y_2) = \frac{1}{2}(m - y_1) + \frac{1}{2}s_{my_1}$$

with

$$s_{my_1}^2 = (y_1 - m)^2 + s^2.$$

Let us consider now:

$$D_{32} = \{(m, s^2) | 1) a \leq m \leq 2y_1 - a, (m - a)(2y_1 - a - m) \leq s^2 \leq (m - a)(y_2 - m) \\ 2) 2y_1 - a \leq m \leq y_2, 0 \leq s^2 \leq (m - a)(y_2 - m)\}.$$

The equation of the plane through  $A(a, a^2, 0)$  and through  $Q_2$  and tangent on  $E_2$  in  $Q_2$  is obtained by eliminating  $z_1, z_2, z_3$  from the following system of equations:

$$\begin{aligned} Z &= z_1 X + z_2 Y + z_3 \\ 0 &= z_1 a + z_2 a^2 + z_3 \\ y - y_1 &= z_1 y + z_2 y^2 + z_3 \\ 1 &= z_1 + 2z_2 y. \end{aligned}$$

This gives:

$$(Z - X + a)(Y - 2Xa + a^2) + (y_1 - a)(X - a)^2 = 0.$$

And consequently:

$$p_1(m, m_2; y_1, y_2) = m - a - \frac{(y_1 - a)(m - a)^2}{s^2 + (m - a)^2}.$$

(ii)  $2y_1 - a \geq y_2$

The point  $A_1$  corresponding to  $P_2$  has the following set of coordinates:

$$(2y_1 - y_2, (2y_1 - y_2)^2, 0).$$

Consequently in:

$$D'_{31} = \{(m, s^2) | 2y_1 - y_2 \leq m \leq y_2, 0 \leq s^2 \leq (m - 2y_1 + y_2)(y_2 - m)\}$$

we obtain the same upper bound as in  $D_{31}$ .

$$p_1(m, m_2; y_1, y_2) = \frac{1}{2}(m - y + s_{my_1}).$$

Let us consider next:

$$D'_{32} = \{(m, s^2) | 1) 2y_1 - y_2 \leq m \leq y_2, (y_2 - m)(m - 2y_1 + y_2) \leq s^2 \\ \leq (m - a)(y_2 - m) \\ 2) a \leq m \leq 2y_1 - y_2, 0 \leq s^2 \leq (m - a)(y_2 - m)\}.$$

We then have to determine the equation of the plane going through  $P_2(y_2, y_2^2, y_2 - y_1)$ , through  $Q_1(x, x^2, 0)$  and tangent on  $E_1$  in  $Q_1$ . This equation is obtained by eliminating  $z_1, z_2$  and  $z_3$  from the following system of equations:

$$\begin{aligned} Z &= z_1 X + z_2 Y + z_3 \\ y_2 - y_1 &= z_1 y_2 + z_2 y_2^2 + z_3 \\ 0 &= z_1 x + z_2 x^2 + z_3 \\ 0 &= z_1 + 2z_2 x. \end{aligned}$$

This gives:

$$Z(2y_2X - Y - y_2^2) = X^2(y_2 - y_1) - Y(y_2 - y_1)$$

and of course:

$$p_1(m, m_2; y_1, y_2) = (y_2 - y_1) \frac{s^2}{s^2 + (m - y_2)^2}.$$

As far as the atoms of the extremal distributions are concerned the solution can be obtained, completely similar to the solutions obtained in DE VYLDER and GOOVAERTS (1982a).

Now we come to the solution of the problem  $q_1(m, m_2; y_1, y_2)$ . In this case we have to consider the following three domains.

$$(1) \quad D_4 = \{(m, s^2) \mid 1) a \leq m \leq y_1, (m - a)(y_1 - m) \leq s^2 \leq (m - a)(b - m) \\ 2) y_1 \leq m \leq b, (m - y_1)(b - m) \leq s^2 \leq (m - a)(b - m)\}.$$

In order to obtain the solution of the problem  $q_1(m, m_2; y_1, y_2)$  we have to determine the equation of the plane through  $A$ ,  $P_1$  and  $B$ . The equation reads:

$$Z = (y_2 - y_1) \frac{Y - (y_1 + a)X + ay_1}{(b - a)(b - y_1)}.$$

Hence:

$$q_1(m, m_2; y_1, y_2) = (y_2 - y_1) \frac{s^2 + (m - a)(m - y_1)}{(b - a)(b - y_1)}$$

$$(2) \quad D_5 = \{(m, s^2) \mid a \leq m \leq y_1, 0 \leq s^2 \leq (m - a)(y_1 - m)\}.$$

In this case it is readily seen that:

$$q_1(m, m_2; y_1, y_2) = 0$$

$$(3) \quad D_6 = \{(m, s^2) \mid y_1 \leq m \leq b, 0 \leq s^2 \leq (m - y_1)(b - m)\}.$$

We have to determine the equation of the plane through  $Q_2(y, y^2, y - y_1)$  tangent on  $E_2$  in  $Q_2$  and through  $Q_3(z, z^2, y_2 - y_1)$  tangent on  $E_3$ . The equation of this plane reads:

$$Z = z_1X + z_2Y + z_3$$

where:

$$y - y_1 = z_1y + z_2y^2 + z_3$$

$$y_2 - y_1 = z_1z + z_2z^2 + z_3$$

$$0 = z_1 + 2zz_2$$

$$1 = z_1 + 2z_2y.$$



Solving this equation with respect to  $z_1$  and  $z_2$  gives:

$$z_2 = \frac{y_2 - z}{(y - z)^2}, \quad z_1 = \frac{y^2 + z^2 - 2yy_2}{(y - z)^2}.$$

These solutions have to satisfy  $0 = z_1 + 2z_2$ , hence:

$$z = 2y_2 - y.$$

Consequently with the point  $Q_2(y, y^2, y - y_1)$  on  $E_2$  corresponds the point  $Q_3(2y_2 - y, (2y_2 - y)^2, y_2 - y_1)$  on  $E_3$ . We have to consider two cases, namely  $2y_2 - y_1 \leq b$  and  $2y_2 - y_1 \geq b$ .

(i)  $2y_2 - y_1 \leq b$

In the present situation we consider a partition of  $D_6$  as shown in fig. 4.

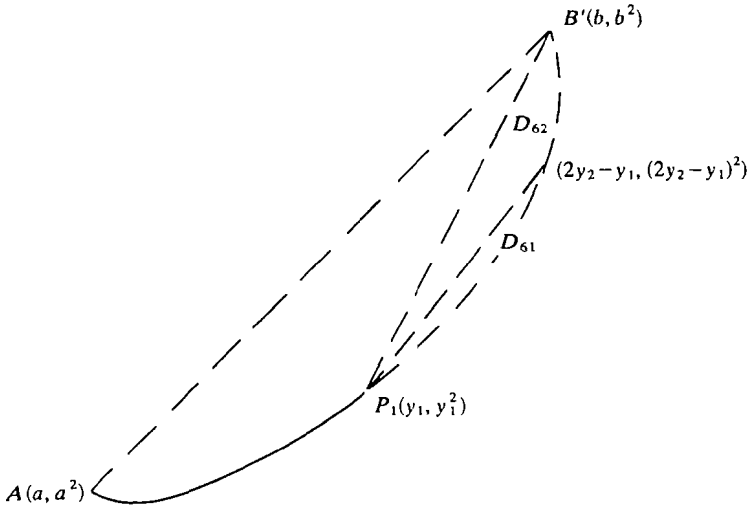


FIGURE 4 Partition of  $D_6$  in case  $2y_2 - y_1 \leq b$

We have:

$$D_{61} = \{(m, s^2) | y_1 \leq m \leq 2y_2 - y_1, 0 \leq s^2 \leq (m - y_1)(2y_2 - y_1 - m)\}.$$

Next we have to deduce the equation of the envelope of the set of planes:

$$Z = y_2 - y_1 + \frac{y^2 + z^2 - 2yy_2}{(y - z)^2} (X - z) + \frac{y_2 - z}{(y - z)^2} (Y - z^2)$$

with  $z = 2y_2 - y$ .

Substitution gives:

$$4(y - y_2)(Z - y_2 + y_1) = 2(y - 2y_2)(X - (2y_2 - y)) + Y - (2y_2 - y)^2.$$

The equation of the envelope is obtained by eliminating  $y$  between this equation and the next one, obtained by taking the derivative with respect to  $y$  in the preceding equation

$$y = 2Z + 2y_1 - X.$$

Hence the equation of the envelope becomes:

$$(2Z + 2y_1 - X - y_2)^2 = (y_2 - X)^2 + Y - X^2.$$

Finally

$$q_1(m, m_2; y_1, y_2) = \frac{1}{2}(y_2 + m - 2y_1 - s_{my_2}).$$

Next we consider:

$$D_{62} = \{(m, s^2) \mid 1) y_1 \leq m \leq 2y_2 - y_1, (m - y_1)(2y_2 - y_1 - m) \leq s^2 \leq (m - y_1)(b - m) \\ 2) 2y_2 - y_1 \leq m \leq b, 0 \leq s^2 \leq (m - y_1)(b - m)\}.$$

In the present situation the envelope is obtained by considering a set of planes through  $(y_1, y_1^2, 0)$  and tangent on  $E_3$ . We get:

$$\begin{aligned} Z &= z_1 X + z_2 Y + z_3 \\ 0 &= z_1 y_1 + z_2 y_1^2 + z_3 \\ y_2 - y_1 &= z_1 z + z_2 z^2 + z_3 \\ 0 &= z_1 + 2z_2 z. \end{aligned}$$

Eliminating  $z_1, z_2$  and  $z_3$  gives:

$$Z = 2z \frac{y_2 - y_1}{(z - y_1)^2} (X - y_1) - \frac{y_2 - y_1}{(z - y_1)^2} (Y - y_1^2).$$

Consequently the envelope of this set of planes depending on  $z$  is obtained by eliminating  $z$  between this equation and the derivative with respect to  $z$ .

$$(z - y_1) Z = (y_2 - y_1)(X - y_1).$$

This results in

$$-Z(2y_1 X - y_1^2 - Y) = (y_2 - y_1)(X - y_1).$$

Hence

$$q_1(m, m_2; y_1, y_2) = (y_2 - y_1) \frac{(m - y_1)^2}{s^2 + (m - y_1)^2}$$

(ii)  $2y_2 - y_1 \geq b$

In this case we have to consider a partition of  $D_6$  in  $D'_{61}$  and  $D'_{62}$ , as indicated in fig. 5.

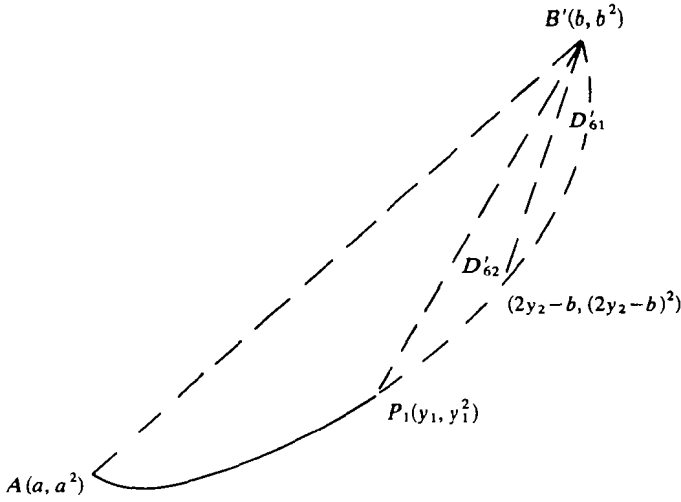


FIGURE 5 Partition of  $D_6$  in case  $2y_2 - y_1 \geq b$

In the domain

$$D'_{61} = \{(m, s^2) | 2y_2 - b \leq m \leq b, 0 \leq s^2 \leq (m - 2y_2 + b)(b - m)\}$$

the same result as in the case  $D_{61}$  applies.

Hence

$$q(m, m_2; y_1, y_2) = \frac{1}{2}(y_2 + m - 2y_1 - s_{my_2}).$$

On the other hand we have:

$$D'_{62} = \{(m, s^2) | 1) 2y_2 - b \leq m \leq b, (m - 2y_2 + b)(b - m) \leq s^2 \leq (m - y_1)(b - m) \\ 2) y_1 \leq m \leq 2y_2 - b, 0 \leq s^2 \leq (m - y_1)(b - m)\}.$$

We have to examine the set of planes through  $B(b, b^2, y_2 - y_1)$ , through a point of  $E_2$  and tangent on  $E_2$  in that point.

These planes are determined by the following equations:

$$\begin{aligned} Z &= z_1 X + z_2 Y + z_3 \\ y_2 - y_1 &= z_1 b + z_2 b^2 + z_3 \\ y - y_1 &= z_1 y + z_2 y^2 + z_3 \\ 1 &= z_1 + 2z_2 y. \end{aligned}$$

Hence the parametric representation of these planes reads:

$$Z = y_2 - y_1 + \frac{y^2 + b^2 - 2yy_2}{(y - b)^2} (X - b) - \frac{b - y_2}{(y - b)^2} (Y - b^2).$$

Taking the derivative with respect to  $y$  gives:

$$(y-b)(Z-y_2+y_1) = (y-y_2)(X-b).$$

Hence, the following equation is obtained for the envelope:

$$\left(\frac{Y-b^2}{X-b} - 2b\right)(Z-y_2+y_1) = \left(\frac{Y-b^2}{X-b} - b - y_2\right)(X-b)$$

such that:

$$q_1(m, m_2; y_1, y_2) = y_2 - y_1 + \frac{s^2 + (m-y_2)(m-b)}{s^2 + (m-b)^2} (m-b).$$

TABLES 1 AND 2

VALUE AND SOLUTIONS OF THE PRIMAL PROBLEM

Abbreviation  $s_{my}^2 = s^2 + (m-y)^2$

Domain of the parameters  $a \leq m \leq b, 0 \leq s^2 \leq (m-a)(b-m)$

Maximization Conditions	Value of the problem	Atoms
$a \leq m \leq y_2$ $(m-a)(y_2-m) \leq s^2 \leq (m-a)(b-m)$	$(y_2-y_1) \frac{(m-a)(b+y_2-m-a)-s^2}{(b-a)(y_2-a)}$	$a, y_2, b$
$y_2 \leq m \leq b$ $(m-y_2)(b-m) \leq s^2 \leq (m-a)(b-m)$	$(y_2-y_1) \frac{(m-a)(b+y_2-m-a)-s^2}{(b-a)(y_2-a)}$	$a, y_2, b$
$y_2 \leq m \leq b$ $0 \leq s^2 \leq (m-y_2)(b-m)$	$y_2 - y_1$	$y_2, m, b$
$a \leq m \leq y_2$ $0 \leq s^2 \leq (m-a)(y_2-m)$		
(i) $2y_1 - a \leq y_2$ $a \leq m \leq 2y_1 - a$ $0 \leq s^2 \leq (2y_1 - a - m)(m - a)$	$\frac{1}{2}(m - y_1 + s_{my_1})$	$y_1 - s_{my_1}, y_1 + s_{my_1}$
$a \leq m \leq 2y_1 - a$ $(m-a)(2y_1 - a - m) \leq s^2$ $\leq (m-a)(y_2 - m)$	$m - a - \frac{(y_1 - a)(m - a)^2}{s^2 + (m - a)^2}$	$a, m + \frac{s^2}{m - a}$
$2y_1 - a \leq m \leq y_2$ $0 \leq s^2 \leq (m-a)(y_2 - m)$	$m - a - \frac{(y_1 - a)(m - a)^2}{s^2 + (m - a)^2}$	$a, m + \frac{s^2}{m - a}$
$2y_1 - y_2 \leq m \leq y_2$ $0 \leq s^2 \leq (m - 2y_1 + y_2)(y_2 - m)$	$\frac{1}{2}(m - y_1 + s_{my_1})$	$y_1 - s_{my_1}, y_1 + s_{my_1}$
(ii) $2y_1 - a \geq y_2$		
$2y_1 - y_2 \leq m \leq y_2$ $(y_2 - m)(m - 2y_1 + y_2) \leq s^2$ $\leq (m-a)(y_2 - m)$	$(y_2 - y_1) \frac{s^2}{s^2 + (m - y_2)^2}$	$m - \frac{s^2}{y_2 - m}, y_2$
$a \leq m \leq 2y_1 - y_2$ $0 \leq s^2 \leq (m-a)(y_2 - m)$	$(y_2 - y_1) \frac{s^2}{s^2 + (m - y_2)^2}$	$m - \frac{s^2}{y_2 - m}, y_2$

Minimization Conditions	Value of the problem	Atoms
$a \leq m \leq y_1$ $(m-a)(y_1-m) \leq s^2 \leq (m-a)(b-m)$	$(y_2-y_1) \frac{s^2+(m-a)(m-y_1)}{(b-a)(b-y_1)}$	$a, y_1, b$
$y_1 \leq m \leq b$ $(m-y_1)(b-m) \leq s^2 \leq (m-a)(b-m)$	$(y_2-y_1) \frac{s^2+(m-a)(m-y_1)}{(b-a)(b-y_1)}$	$a, y_1, b$
$a \leq m \leq y_1$ $0 \leq s^2 \leq (m-a)(y_1-m)$	$0$	$a, m, y_1$
$y_1 \leq m \leq b$ $0 \leq s^2 \leq (m-y_1)(b-m)$		
(i) $2y_2 - y_1 \leq b$ $y_1 \leq m \leq 2y_2 - y_1$ $0 \leq s^2 \leq (m-y_1)(2y_2 - y_1 - m)$	$\frac{1}{2}(y_2 + m - 2y_1 - s_{my_2})$	$y_2 - s_{my_2}, y_2 + s_{my_2}$
$y_1 \leq m \leq 2y_2 - y_1$ $(m-y_1)(2y_2 - y_1 - m) \leq s^2$ $\leq (m-y_1)(b-m)$	$(y_2-y_1) \frac{(m-y_1)^2}{s^2+(m-y_1)^2}$	$y_1, m + \frac{s^2}{m-y_1}$
$2y_2 - y_1 \leq m \leq b$ $0 \leq s^2 \leq (m-y_1)(b-m)$ (ii) $2y_2 - y_1 \geq b$	$(y_2-y_1) \frac{(m-y_1)^2}{s^2+(m-y_1)^2}$	$y_1, m + \frac{s^2}{m-y_1}$
$2y_2 - b \leq m \leq b$ $0 \leq s^2 \leq (m-2y_2+b)(b-m)$	$\frac{1}{2}(y_2 + m - 2y_1 - s_{my_2})$	$y_2 - s_{my_2}, y_2 + s_{my_2}$
$2y_2 - b \leq m \leq b$ $(m-2y_2+b)(b-m) \leq s^2$ $\leq (m-y_1)(b-m)$	$y_2 - y_1 + \frac{s^2+(m-y_2)(m-b)}{s^2+(m-b)^2} (m-b)$	$m - \frac{s^2}{b-m}, b$
$y_1 \leq m \leq 2y_2 - b$ $0 \leq s^2 \leq (m-y_1)(b-m)$	$y_2 - y_1 + \frac{s^2+(m-y_2)(m-b)}{s^2+(m-b)^2} (m-b)$	$m - \frac{s^2}{b-m}, b$

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# THE STATICS AND DYNAMICS OF INCOME

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