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SOME TRANSIENT RESULTS ON THE M/SM/1 SPECIAL SEMI-MARKOV MODEL IN RISK AND QUEUEING THEORIES

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We consider a usual situation in risk theory for which the arrival process is a Poisson process and the claim process a positive $(J - X)$ process inducing a semi-Markov process. The equivalent in queueing theory is the M/SM/1 model introduced for the first time by Neuts (1966)

For both models, we give an explicit expression of the probability of non-ruin on $[0, t]$ starting with u as initial reserve and of the waiting time distribution of the last customer arrived before t. "Explicit expression" means in terms of the matrix of the aggregate claims distributions.

1. THE SPECIAL SEMI-MARKOV MODEL IN RISK THEORY

In a usual situation of the theory of risk, let $(A_n, n \geqslant 1)$ be the claim inter-arrival times process, $(B_n, n \geqslant 1)$ the claim amounts process. Moreover, we suppose that m "types" of claims are possible represented by the set:

$$I = \{1, 2, \ldots, m\} \text{ (with } 1 \leqslant m < \infty). \tag{1.1}$$

The process starts just after payment of an initial claim of type $J_0 = i$ and after this payment, the fortune of the company is supposed to be u $(u \geqslant 0)$. The process $(J_n, n \geqslant 0)$ represents the sequence of the successive types of claims. For the simplicity of notations, we also introduce the random variables A_0 and B_0 such that:

$$A_0 = B_0 = 0 \qquad \text{a.s.} \tag{1.2}$$

If the claim arrivals process is not explosive, let N_t denote the total number of claims in $(0, t)$ (thus excluded the initial claim) and define:

$$X(t) = \sum_{n=0}^{N(t)} B_n \text{ (total amount of claims paid on } (0, t)) \tag{1.3}$$

$$Z_t = J_{N(t)} \text{ (type of the last claim occurred before or at } t). \tag{1.4}$$

If we also suppose that the incomes of the company occur at a constant rate c $(c > 0)$, then the "fortune" $Z(t)$ of the company at time t is given by

$$Z(t) = u + ct - X(t). \tag{1.5}$$

The matrix mxm $\mathfrak{F}$ of the "distribution" functions of the aggregate claims at time t will be, by definition:

$$\mathfrak{F}(x, t) = (F_{ij}(x, t)) \tag{1.6}$$

where

$$F_{ij}(x, t) = P[X(t) \leqslant x, J_{N(t)} = j \mid J_0 = i] \quad (i, j = 1, \ldots, m). \tag{1.7}$$

Probabilistic assumptions

We assume that the processes introduced satisfy the following assumptions:
1. The claim arrival process is a Poisson process of parameter λ.
2. The process $((J_n, B_n), n \geqslant 0)$ is a positive $(J-X)$ process (see JANSSEN (1970)); this means that

$$P[B_n \leqslant x, J_n = j \mid (J_k, B_k), k \leqslant n-1] = Q_{J_{n-1}j}(x) \text{ a.s.} \tag{1.8}$$

where the matrix $\mathbb{Q}$, defined by $\mathbb{Q}(x) = (Q_{ij}(x))$ is a matrix of mass functions such that:

$$\text{i. } Q_{ij}(x) = 0 \text{ for all } x \leqslant 0 \text{ for all } i, j \in I \tag{1.9}$$

$$\text{ii. } \sum_{j=1}^{m} Q_{ij}(+\infty) = 1 \text{ for all } i \in I. \tag{1.10}$$

From the semi-Markov theory (PYKE (1961)), it is well-known that

1° if $p_{ij} = \lim_{x\to\infty} Q_{ij}(x)$ and $\mathbf{P} = (p_{ij})$, then the process $(J_n, n \geqslant 0)$ — i.e. the process of claim types—is a homogeneous Markov chain with $\mathbf{P}$ as transition matrix.

2° The random variables B_n, $n \geqslant 0$ are not independent, but only conditionally dependent given the Markov chain $(J_n, n \geqslant 0)$ — often called the "imbedded Markov chain".

3. The processes $(A_n, n \geqslant 0)$ and $((J_n, B_n), n \geqslant 0)$ are independent.

The main problem

The event "ruin before t" occurs if the trajectory of $Z(t')$ on $(0, t)$ goes under the time axis before t. More precisely, if $\phi_{ij}(u, t)$ represents the probability of non-ruin on $[0, t]$, starting with $J_0 = i$ and an initial fortune u, and such that $J_{N(t)} = j$, we have, by definition:

$$\phi_{ij}(u, t) = P[Z(t') \geqslant 0, 0 \leqslant t' \leqslant t, J_{N(t)} = j \mid J_0 = i] \tag{1.11}$$

or equivalently by (1.5):

$$\phi_{ij}(u, t) = P[\sup_{0\leqslant\tau\leqslant t} (X(\tau) - c\tau) \leqslant u, J_{N(t)} = j \mid J_0 = i]. \tag{1.12}$$

If we are not interested by the last type observed before t, we have enough with

$$\phi_i(u, t) = \sum_{j=1}^{m} \phi_{ij}(u, t) \tag{1.13}$$

and if $(p_1, \ldots, p_m)$ is an initial distribution on J_0, we have to compute

$$\phi(u, t) = \sum_{j=1}^{m} p_i \, \phi_i(u, t). \tag{1.14}$$

The problem solved in this paper is to find an explicit expression of the matrix ϕ, defined by

$$\boldsymbol{\phi}(x, t) = (\phi_{ij}(x, t)) \tag{1.15}$$

in terms of the matrix $\mathfrak{F}$.

2. THE ANALOGOUS MODEL IN QUEUEING THEORY: THE M/SM/1 MODEL

As quoted by several authors (PRABHU (1961), SEAL (1972), JANSSEN (1977)), a risk model can easily be interpreted as a queueing model and vice versa. It suffices to see the process $(A_n, n \geqslant 1)$ as the one of the interarrival times between two successive customers (i.e. customers $(n-1)$ and n) in a queueing system with one server and as discipline rule FIFO; then, the process $(B_n, n \geqslant 1)$ represents the successive service times (i.e. B_n is the service time of the customer number $(n-1)$, $n \geqslant 1$).

We also suppose that at $t=0$, the customer number 0 just begins his service. Moreover, we have m types of customers and J_n represents the type of customer n. Here N_t gives the "number" of the last customer arrived before or at t. With the same probabilistic assumptions as those of the preceding paragraph, the main problem considered in the queueing optic is to get an explicit expression of the distribution of $W_{N(t)}$ where W_n $(n \geqslant 0)$ represents the waiting of the nth customer. More precisely, we must express the matrix $\mathbf{W}$ in terms of $\mathfrak{F}$ where it is defined by

$$\mathbf{W}(x, t) = (\mathrm{W}_{ij}(x, t)) \tag{2.1}$$

with

$$\mathrm{W}_{ij}(x, t) = P[\mathrm{W}_{N(t)} \leqslant x, J_{N(t)} = j \mid J_0 = i]. \tag{2.2}$$

This model is noted M/SM/1 in the queueing literature (Poisson arrivals and semi-Markov service times) introduced by NEUTS (1966).

3. THE DISTRIBUTION OF AGGREGATE CLAIMS

Introduce the usual notation in semi-Markov theory: for any matrix mxm of mass functions $\mathbf{L}$, we note by $\mathbf{L}^{(\bar{n})}$ the n-fold convolution of the matrix $\mathbf{L}$, that is

$$\mathbf{L}^{(\bar{0})}(x) = (U_0(x)),\ \mathbf{L}^{(\bar{1})}(x) = (L_{ij}(x)) \tag{3.1}$$

(where $U_0(x)$ is the distribution function with a unit mass at 0) and for $\mathbf{L}^{(\bar{n})}$ we have:

$$L_{ij}^{(\bar{n})}(x) = \sum_k \int_{\mathsf{R}} L_{ik}^{(\overline{n-1})}(x-y)\, d\, L_{kj}(y),\ n \geqslant 1. \tag{3.2}$$

If

$$S_n = \sum_{i=0}^{n} B_i \tag{3.3}$$

it is clear, from (1.8), that

$$Q_{ij}^{(\bar{n})}(x) = \mathsf{R}[S_n \leqslant x, J_n = j \mid J_0 = i]. \tag{3.4}$$

From assumption (3), it follows then that:

$$\mathfrak{F}(x, t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \mathfrak{Q}^{(\bar{n})}(x) \tag{3.5}$$

expression given the matrix of distribution of aggregate claims by means of the semi-Markov kernel $\mathfrak{Q}$.

Let us remark that the assumption (1) gives:

$$\begin{aligned} &\mathsf{P}[X(t+s) \leqslant x, J_{N(t+s)} = j \mid X(s'), J_{N(s')}, s' \leqslant s, X(s) = y, \\ &J_{N(s)} = i] = F_{ij}(x-y, t) \end{aligned} \tag{3.6}$$

showing that the process $((X(t), J_{N(t)}), t \geqslant 0)$ is markovian.

4. LOADINGS OF PREMIUMS

To show how the concept of loading of premiums can be introduced in the special semi-Markov risk model considered here, let us suppose that the quantities—mean cost of a claim of type i—

$$\eta_i = \sum_j \int_0^{\infty} x\, d\, Q_{ij}(x), \qquad i \in I \tag{4.1}$$

are finite. Moreover, we suppose that the Markov chain $(J_n, n \geqslant 0)$ is ergodic and that $(\Pi_1, \ldots, \Pi_m)$ represents the unique stationary probability distribution. Starting with this distribution for J_0, we get, using (3.5):

$$\mathsf{P}[X(t) \leqslant x] = \sum_i \sum_j \Pi_i F_{ij}(x, t) \tag{4.2}$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^{m} \sum_{j=1}^{m} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \Pi_i Q_{ij}^{(\bar{n})}(x) \tag{4.3}$$

so that the mean of the aggregate claims at time t is given by

$$\text{(4.4)} \qquad \mathsf{E}[X(t)] = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \Pi_i \int_0^{\infty} x \; d\, Q_{ij}^{(\bar{n})}(x) \right).$$

The term under brackets is the expectation of S_n or, by (3.3)

$$\text{(4.5)} \qquad \sum_{k=1}^{n} \mathsf{E}(B_k).$$

As the process $(J_n, n \geqslant 0)$ is stationary, we have, for all k

$$\text{(4.6)} \qquad \mathsf{E}(B_k) = \sum_{l=1}^{m} \Pi_l \, \eta_l.$$

This gives:

$$\text{(4.7)} \qquad \mathsf{E}[X(t)] = \sum_{n=0}^{\infty} e^{-t\lambda} \frac{(\lambda t)^n}{n!} \, n \left(\sum_{=1}^{m} \Pi_l \, \eta_l \right)$$

or

$$\text{(4.8)} \qquad \mathsf{E}[X(t)] = \lambda \varphi t$$

with

$$\text{(4.9)} \qquad \varphi = \sum_{l=1}^{m} \Pi_l \, \eta_l.$$

It follows that the mean fortune at time t is given by:

$$\text{(}\ldots\text{)} \qquad (c - \lambda\varphi)t$$

is positive if and only if $c = \lambda\varphi(1 + \eta)$, with $\eta > 0$. The justification …ading η comes also from the fact that, except some degenerate cases, …sts a reserve u such that for all i, j, $\phi_{ij}(u)$ is positive—where $\phi_{ij}(u)$ … (u, t)—if and only if $\lambda\varphi < c$ (see JANSSEN (1970)).

5. EXPRESSION OF $\phi_{ij}(u, t)$

…ons made—(1), (2), (3)—are such that the method used by …and later by SEAL (1974) is valid. For the facility, let us suppose …nctions $Q_{ij}(x)$ have densities $q_{ij}(x)$ on $(0, \infty)$; then the PRABHU's becomes the integral system:

$$\ldots ct, t) = \phi_{ij}(u, t) + \sum_{k=1}^{m} \int_0^t \phi_{kj}(0, t - \tau \; d_x \, F_{ik}(u + c\tau, \tau)$$

$$i, j = 1, \ldots, m$$

where

$$d_x F_{ik}(u + c\tau, \tau) = c \frac{\partial F_{ik}}{\partial x} (u + c\tau, \tau)\, d\tau. \tag{5.2}$$

The system (5.1) gives the $\phi_{ij}(u, t)$ provided we know the values at $u = 0$. These can be computed using (5.1) with $u = 0$:

$$F_{ij}(ct, t) = \phi_{ij}(0, t) + \sum_{k=1}^{m} \int_0^t \phi_{kj}(0, t-\tau)\, d_x F_{ik}(c\tau, \tau) \tag{5.3}$$

$$i, j = 1, \ldots, m.$$

To write this system of Volterra integral equations in a more concise way, let us introduce the following matrices:

$$\Phi(t) = (\phi_{ij}(0, t)) = (\phi(0, t)) \tag{5.4}$$

$$\mathbf{F}(t) = (F_{ij}(ct, t)) = (\mathfrak{F}(ct, t)) \tag{5.5}$$

$$\mathbf{G}(t) = c\left(\frac{\partial F_{ij}}{\partial x} (ct, t)\right) \tag{5.6}$$

$$(\mathbf{A} * \mathbf{B})\,(t) = (\sum_{k=1}^{m} \int_0^t A_{ik}(t-v)\, B_{kj}(v)dv) \tag{5.7}$$

(with A and B mxm matrices)

$$\tilde{\mathbf{A}}(s) = (\int_0^\infty e^{-st} A_{ij}(t)dt) \tag{5.8}$$

(Laplace transform for matrices).

The system (5.3) takes the matrix form:

$$\mathbf{F}(t) = \Phi(t) + \mathbf{G} * \Phi(t) \tag{5.9}$$

and using Laplace transforms, we get

$$\tilde{\mathbf{F}}(s) = (\tilde{\mathbf{I}} + \mathbf{G}(s))\, \tilde{\Phi}(s) \tag{5.10}$$

and consequently:

$$\tilde{\Phi}(s) = (\mathbf{I} + \tilde{\mathbf{G}}(s))^{-1}\, \tilde{\mathbf{F}}(s) \tag{5.11}$$

provided the inverse matrix of $\mathbf{I} + \tilde{\mathbf{G}}(s)$ exists.

We can now show the main result and for simplicity, we suppo
derivatives $q_{ij}(x)$ of $Q_{ij}(x)$ exist for all i and j.

Proposition

If the quantity M defined by

(5.12) $$M = \sup \{q_{ij}(x),\ i, j \in I,\ x \geqslant 0\}$$

is finite, then

(5.13) $$\Phi(t) = \sum_{n=0}^{\infty} (-1)^n \mathbf{G}^{(n)} * \mathbf{F}(t)$$

(5.14) $$\Phi(u, t) = \mathbf{F}(u + ct, t) - \mathbf{G}_u * \sum_{n=0}^{\infty} (-1)^n \mathbf{G}^{(n)\ 1)} * \mathbf{F}(t)$$

where

(5.15) $$\mathbf{G}_u(t) = \left(c \frac{\partial F_{ij}}{\partial x} (u + ct, t)\right).$$

Proof: From (3.5), we deduce that

(5.16) $$\frac{\partial F_{ij}}{\partial x} (x, t) = \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} q_{ij}^{(n)}(x)$$

where

(5.17) $$q_{ij}^{(1)}(x) = q_{ij}(x)$$

and

(5.18) $$q_{ij}^{(n)}(x) = \sum_k \int_0^x q_{ik}^{(n-1)}(x - y)\, q_{kj}(y) dy,\ n > 1.$$

From (5.12), (5.18), it is clear that, for all $n \geqslant 1$

(5.19) $$q_{ij}^{(n)}(x) \leqslant M$$

so that from (5.16):

(5.20) $$\frac{\partial F_{ij}}{\partial x} (x, t) \leqslant M(1 - e^{-\lambda t}) \leqslant M.$$

From the definition (5.6), we get

$$\tilde{G}_{ij}(s) \leqslant c \int_0^{\infty} M e^{-st} dt = \frac{cM}{s}$$

[1] From now, this symbol means the n-fold convolution product for the definition (5.7).

$$\tilde{G}^2_{ij}(s) = \Sigma\, \tilde{G}_{ik}(s)\, \tilde{G}_{kj}(s) \leqslant m\, \frac{c^2M^2}{s^2}$$

$$\vdots$$

$$\tilde{G}^n_{ij}(s) = \Sigma\, \tilde{G}^{n-1}_{ik}(s)\, \tilde{G}_{kj}(s) \leqslant m^{n-1}\, \frac{c^nM^n}{s^n}.$$

Consequently, the matrix series $\overset{\infty}{\Sigma}\ \tilde{\mathbf{G}}^n(s)$ converges for all $s > m\, c\, M$. A well-known consequence of this fact is that the matrix $(\mathbf{I} + \tilde{\mathbf{G}}(s))^{-1}$ is invertible and

$$(\mathbf{I} + \tilde{\mathbf{G}}(s))^{-1} = \sum_{n=0}^{\infty} (-1)^n\, \tilde{\mathbf{G}}^n(s)$$

of course on $(m\, c\, M, \infty)$.

Using the matrix version of a theorem of DOETSCH (1974) and (5.11), we get (5.13).

The result (5.14) follows then from the relations (5.1) written under the matrix form and where $\phi(t)$ is under the form (5.13).

6. RESULTS FOR THE ACTUAL WAITING TIME AT TIME t OF THE M/SM/1 QUEUEING MODEL

The probabilistic assumptions made in the paragraph 1 imply that the process $((J_n, A_n, B_n), n \geqslant 0)$ is a two-dimensional $(J - X)$ process (JANSSEN, 1979) with kernel $(Q_{ij}(t, x))$ given by:

(6.1) $$Q_{ij}(t, x) = E(t) \cdot Q_{ij}(x)$$

where

(6.2) $$E(t) = \begin{cases} 0 & , t < 0 \\ 1 - e^{-\lambda t}, & t \geqslant 0. \end{cases}$$

If we suppose that the matrix $\mathbf{P}$ $(= \mathbb{Q}(+\infty))$ is ergodic with a stationary probability distribution $(\Pi_1, \ldots, \Pi_m)$, the dual kernel $(\hat{Q}_{ij}(t, x))$ of $(Q_{ij}(t, x))$ is given by (see JANSSEN (1979)):

(6.3) $$\hat{Q}_{ij}(t, x) = \frac{\Pi_j}{\Pi_i}\, Q_{ij}(t, x)$$

(6.4) $$= \frac{\Pi_j}{\Pi_i}\, E(t)\, Q_{ji}(x).$$

Let us now consider the M/SM/1 queueing model whose kernel is given by (6.4). The asymptotical study has been done for the first time by NEUTS (1966). Now the transient behaviour of $\hat{W}_{ij}(x, \tau)$—defined by (2.2)—can be easily deduced from the last paragraph and our duality results (JANSSEN, 1979). From the proposition 4 of this last reference, we get, for all $x > 0$ and all $t > 0$:

$$\Pi_i \int_0^t e^{\lambda t} \hat{W}_{ij}(x, d\tau) = \Pi_j \int_0^t e^{\lambda t} \phi_{ji}(x, d\tau) \tag{6.5}$$

so that

$$\hat{W}_{ij}(x, \tau) = \frac{\Pi_j}{\Pi_i} \phi_{ji}(x, \tau). \tag{6.6}$$

If Π_d represents the mxm diagonal matrix whose ith element on the principal diagonal is Π_i, (6.6) takes the form

$$\hat{\mathbf{W}}(x, \tau) = \Pi_d^{-1} \phi^{\tau}(x, \tau) \Pi_d \tag{6.7}$$

with

$$\hat{\mathbf{W}}(x, \tau) = (\hat{W}_{ij}(x, \tau)).$$

(6.7) with the aid of (5.14) gives an explicit expression of the distribution of the actual waiting time in a M/SM/1 model.

7. COMMENTS

a) For $m = 1$, the model considered becomes the classical Cramér's model of risk theory and the M/G/1 queueing model for which it is known (see PRABHU (1961), SEAL (1972)) that:

$$\phi(0, t) = \frac{1}{t} \int_0^t F(x, t)\, dx. \tag{7.1}$$

Using successive integrations by parts, it is possible to show—in this case—the equivalence of (7.1) and (5.13). It does not seem possible to have an analogous result for $m > 1$, in particular an extension of the analytically proof of DE VYLDER (1977) cannot be used as the variables (B_n) are no more exchangeable.

b) The effect of a suppression of the k^e type of claim is theoretically possible by comparing $\phi(u, t)$ and $\phi_k(u, t)$, representing the non-ruin probability with $(m-1)$ types of claims, k being excluded.

c) The main result can be extended to the non-Poisson case if we suppose that the process (J_n, A_n) is a semi-Markov process of kernel

$$(p_{ij}\, E_i(t))$$

where

$$E_i(t) = \begin{cases} 0 & , t < 0 \\ 1 - e^{-\lambda_i t}, & t \geqslant 0 \end{cases}$$

that is a regular continuous Markov process with a finite number of states.

d) The following remarks may be useful for numerical computation.

It is easy to show that

$$\mathbf{G}^{(n)} * \mathbf{F}(t) \leqslant m^n \frac{M^n t^n}{n!} \tag{7.2}$$

so that approximating $\phi(t)$ by the first $(N-1)$ terms of (5.13), we have for the absolue value of the error $R_N(t)$, the following upper bound:

$$| R_N(t) | \leqslant \frac{(mMt)^N}{N!} e^{mMt}. \tag{7.3}$$

For $m = 1$, we can say more. Indeed, let us suppose, without loss of generality, that $c = 1$ and $M \leqslant 1$. For c, that is well-known in risk theory; if $M > 1$, it suffices to introduce the random variables (B'_n), (A'_n) defined by $B'_n = M^{-1} B_n$ and $A'_n = M^{-1} A_n$ so that the process (A'_n) induces a Poisson one of parameter $\lambda' = M^{-1}\lambda$. Then, if $\phi'(u', t')$ is the probability of non-ruin for this model: $\phi(u, t) = \phi'(Mu, Mt)$. (7.4)

In this case, we have

$$\mathbf{G}^{(n)} * \mathbf{F}(t) - \mathbf{G}^{(n+1)} * \mathbf{F}(t) = \mathbf{G}^{(n)} * (U_0 - \mathbf{G}) * \mathbf{F}(t) \tag{7.5}$$

which is a non-negative quantity as $G(t) \leqslant 1$ (U_0 is the Heaviside function with a unit mass at 0).

Consequently, the series (5.13) is alternating so that the sign of the error R_N is this of $(-1)^N$ and

$$| R_N(t) | \leqslant \mathbf{G}^{(N)} * \mathbf{F}(t). \tag{7.6}$$

From (7.2), it follows that:

$$| R_N(t) | \leqslant \frac{t^N}{N!}. \tag{7.7}$$

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