An Application of Game Theory:

Property Catastrophe Risk Load

Abstract

Two well-known methods for calculating risk load -- Marginal Surplus and Marginal Variance -- are applied to output from catastrophe modeling software. Risk loads for these *marginal methods* are calculated for sample new and renewal accounts. Differences between new and renewal pricing are examined. For new situations, both current methods allocate the full marginal impact of addition of a new account to that new account. For renewal situations, a new concept is introduced -- "renewal additivity". Neither marginal method is renewal additive. A new method is introduced, inspired by game theory, which splits the mutual covariance between any two accounts evenly between those accounts. The new method is extended and generalized to a proportional sharing of mutual covariance between any two accounts. Both new approaches are tested in new and renewal situations.

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(1) Introduction

The calculation of risk load continues to be a topic of interest in the actuarial community -- see Bault [1] for a recent survey of well-known alternatives. One area of great need, where the CAS literature is somewhat scarce, is calculation of risk loads for property catastrophe insurance.

Many of the new catastrophe modeling products produce occurrence size-of-loss distributions for a series of simulated events. These output files might contain an event identifier, event probability, and modeled loss amount for that event for the selected portfolio of exposures. Given such output files for a portfolio *before* and *after* the addition of a new account, one could calculate the portfolio variance and standard deviation before and after. The difference will be called the *marginal impact* of that new account on the portfolio variance or standard deviation[1].

Two of the more well-known risk load methods from the CAS literature -- what shall be called "Marginal Surplus" (MS) from Kreps [3] and "Marginal Variance" (MV) from Meyers [6] -- use the marginal change in portfolio standard deviation (respectively variance) due to the addition of a new account to calculate the risk load for that new account. However, problems arise when these marginal methods are used to calculate the risk loads for the renewal of the accounts in a portfolio. These problems can be traced to the **order dependency** of the marginal risk load methods.

Order dependency is a perplexing issue. Many marginal risk load methods -- whether based on variance, standard deviation, or even a selected percentile of the loss distribution – suffer from it. It is also not just an actuarial issue; even the financial community struggles with it. "Value at Risk" (VAR) is an attempt by investment firms to capture their risk in a single number. It is a selected percentile of the return distribution (e.g., 95th) for a portfolio of financial instruments over a selected time frame (e.g., 30 days). VAR can be calculated for the entire portfolio or for a desired subset (e.g., asset class). But so-called "marginal" or "component" VAR has to this point eluded satisfactory solution in the finance community precisely because of what will be termed *renewal additivity*. Those finance professionals charged with assessing how much VAR a certain financial instrument or asset class contributes to the total VAR are staring straight at the same unresolved order dependency issues. As the finance and insurance worlds blend more and more, perhaps actuaries will combine forces with the "rocket scientists" and CFA's to determine a solution.

The remainder of this paper is organized as follows. Section 2 describes the basic characteristics of a catastrophe occurrence size-of-loss distribution. Sections 3 and 4 describe the application of the MV and MS methods to a simplified occurrence size-of-loss distribution. Sections 5 and 6 calculate risk loads both in assembling or building up a portfolio of risks and in subsequently

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[1] The variance and standard deviation are "between account" and "between event," and ignore any parameter uncertainty associated with the modeled loss amount for a given event and

renewing that portfolio. Section 7 discusses the differences between build-up and renewal results.

Section 8 introduces a new concept to the theory of property catastrophe risk loads -- renewal additivity. However, the concept is not new to the field of game theory. Section 9 introduces game theory concepts underlying a new approach. Section 10 extends and generalizes the effect of the new approach to sharing of covariance between accounts. Section 11 concludes by applying the new approaches to the examples.

(2) The Catastrophe Occurrence Size-of-loss Distribution

For demonstration purposes throughout the paper, a simplified version of an occurrence size-of-loss distribution will be used. It captures the essence of typical catastrophe modeling software output, while keeping the examples understandable[2].

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account.

[2] In particular, only single event or occurrence size-of-loss distributions will be considered. Many models also produce multi-event or aggregate loss distributions. Occurrence size-of-loss distributions only reflect the *largest* event which occurs in a given year. Aggregate loss distributions reflect the sum of losses for all events in a given year. Clearly, the aggregate table provides a more complete picture, but for purposes of the exposition here, the occurrence table works well and the formulas are substantially less complex.

A modeled event denoted by identifier **i** is considered an independent Poisson process with occurrence rate[3] λ_i. To simplify the mathematics, following Meyers [6], the binomial approximation with probability of occurrence $\mathbf{p_i}$ [where $\lambda_i = -\ln(1 - \mathbf{p_i})$] will be employed. This is a satisfactory approximation for small λ_i [4].

For an individual account or portfolio of accounts, the model produces a modeled loss for each event $\mathbf{L_i}$. A table containing the event identifiers **i**, the event probabilities $\mathbf{p_i}$ and modeled losses $\mathbf{L_i}$ will be referred to as an "occurrence size-of-loss distribution."

From Meyers [6], the formulas for expected loss and variance are:

$$E[L] = \Sigma_i \{ L_i * p_i \} \qquad [2.1]$$

$$Var[L] = \Sigma_i \{ L_i^2 * p_i * (1 - p_i) \} . \qquad [2.2]$$

[Σ_i *= sum over all events*]

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[3]This entails a loss in generality, as it implies the loss for a given event and account to be fixed and known.

[4]An event with a probability of 0.001 (typical of the more severe modeled events) would have = 0.0010005.

The formula for covariance of an existing portfolio **L** (with losses $\mathbf{L_i}$) and a new account n (with losses $\mathbf{n_i}$) is :

$$\text{Cov}\,[L, n] = \Sigma_i \{ L_i * n_i * p_i * (1 - p_i) \} \qquad [2.3]$$

(Note that Cov [L, n] is always greater than zero when each of L_i, n_i, p_i and (1 - p_i) are greater than zero.)

The total variance of the combined portfolio [L + n] is then

$$\text{Var}\,[L] + \text{Var}\,[n] + 2 * \text{Cov}\,[L, n] \qquad [2.4]$$

(3) <u>The Marginal Surplus (MS) Method</u>

This is a translation to property catastrophe of the method described in Rodney Kreps' "Reinsurer Risk Loads from Marginal Surplus Requirements" [3].

Consider:

$\mathbf{L_0}$ = losses from a portfolio before a new account is added

$\mathbf{L_1}$ = losses from a portfolio after a new account is added

$\mathbf{S_0}$ = Standard deviation of $\mathbf{L_0}$

$\mathbf{S_1}$ = Standard deviation of $\mathbf{L_1}$

$\mathbf{R_0}$ = Return on the portfolio before new account is added

$\mathbf{R_1}$ = Return on the portfolio after new account is added

Borrowing from Mr. Kreps, assume needed surplus V is given by

$$z * \text{Standard Deviation of loss}^{5} - \text{expected Return} \qquad [3.1]$$

where ***z*** is, to cite Mr. Kreps (p. 197), "a distribution percentage point corresponding to the acceptable probability that the actual result will require even more surplus than allocated." Then

$$\mathbf{V_0 = z * S_0 - R_0}$$

$$\mathbf{V_1 = z * S_1 - R_1} \qquad [3.2]$$

The difference in returns $\mathbf{R_1 - R_0 = r}$, the risk load charged to the new account. The marginal surplus requirement is then

$$\mathbf{V_1 - V_0 = z * [\, S_1 - S_0 \,] - r} \qquad [3.3]$$

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[5]Mr. Kreps sets needed surplus equal to z * standard deviation of return - expected return. If one assume premiums and expenses are invariant, then Var[Return] = Var[P - E - L] = Var[L].

Based on the required return ***y*** on that marginal surplus (which is based on management goals, market forces and risk appetite), the MS risk load would be:

$$\mathbf{r} = \mathbf{y} * \mathbf{z} / (1 + \mathbf{y}) * [\, \mathbf{S_1} - \mathbf{S_0} \,] \qquad [3.4]$$

(4) <u>The Marginal Variance (MV) Method</u>

This is based on Glenn Meyers' "The Competitive Market Equilibrium Risk Load Formula for Catastrophe Ratemaking" [6].

For an existing portfolio **L** and a new account **n**, the MV risk load **r** would be:

$$\mathbf{r} = \lambda * \text{Marginal Variance of adding } \mathbf{n} \text{ to } \mathbf{L}$$

$$= \lambda * \{ \text{Var}[\, \mathbf{n} \,] + 2 * \text{Cov}[\, \mathbf{L}, \mathbf{n} \,] \} \qquad [4.1]$$

where λ is a multiplier similar to y * z / (1 + y) from the MS method, although dimensioned to apply to variance rather than standard deviation[6].

(5) <u>Building Up a Portfolio of 2 Accounts</u>

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[6]Mr. Meyers develops a variance based risk load multiplier by converting a standard deviation based multiplier using the following formula ([6], p. 573):

Table 12 shows the occurrence size-of-loss distribution and risk load calculations for building up (assembling) a portfolio of 2 accounts, (X) and (Y). It is assumed (X) is written first, and is the only risk in the portfolio until (Y) is written.

(5.1) MS Method

Pertinent values from Table 12 for the Marginal Surplus method are summarized here:

Table 5.1

Building Up (X) & (Y): Marginal Surplus	Account (X)	Account (Y)	Account (X) + Account (Y)	Portfolio (X + Y)
(1) Change in Standard Deviation	4,429	356	4,785	4,785
(2) Risk Load Multiplier	0.33	0.33	-	0.33
(3) Risk Load = (1) * (2)	$1,461.71	$117.43	$1,579.14	$1,579.14

* Item (1) is the change in portfolio standard deviation from adding each account, or *marginal* standard deviation.

λ = (Rate of Return * Std Dev Mult^2) / (2 * Avg Capital of Competitors)

* Item (2) is the Risk Load multiplier of 0.33. Using Mr. Kreps' formula, a return on marginal surplus **y** of 20% and a standard normal multiplier **z** of 2.0 (2 standard deviations, corresponding to a cumulative non-exceedance probability of 97.725%) would produce a risk load multiplier of

$$\mathbf{y} * \mathbf{z} / (1 + \mathbf{y}) = \quad 0.20 * 2 / 1.20 \quad = \quad 0.33 \text{ (rounded)} \qquad [5.1]$$

* Item (3) is the Risk Load, the product of Items (1) and (2).

Since (X) is the first account, the marginal standard deviation from adding (X) equals the standard deviation of (X) (Std Dev [X]) of 4,429. This gives a risk load of $1,461.71.

The marginal standard deviation from writing (Y) equals Std Dev [X + Y] - Std Dev [X], or $356, implying a risk load of $117.43.

The sum of these two risk loads (X) + (Y) is $1,461.71 + $117.43 = $1,579.14. This equals the risk load which this method would calculate for the combined account (X + Y).

(5.2) <u>MV Method</u>

Pertinent values from Table 12 for the Marginal Variance method are summarized here:

Table 5.2

Building Up (X) & (Y): ***Marginal Variance***	Account (X)	Account (Y)	Account (X) + Account (Y)	Portfolio (X + Y)
(1) Change in Variance	19,619,900	3,279,059	22,898,959	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,353.02	$226.13	$1,579.14	$1,579.14

* Item (1) is the change in portfolio variance from adding each account, or *marginal* variance.

* Item (2) is the Variance Risk Load multiplier λ of 0.000069. To simplify comparisons between the two methods (recognizing the difficulty of selecting a MV-based multiplier[7]), The MS multiplier was converted to a MV basis by dividing by Std Dev [X + Y]:

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[7]Mr. Meyers [6] (p. 572) admits that in practice "it might be difficult for an insurer to obtain the (lambdas) of each of its competitors." He goes on to suggest an approximate method to arrive at a usable lambda based on required capital being "Z standard deviations of the total loss distribution" (p. 574).

$$\lambda = 0.33 / 4{,}785 = 0.000069 \qquad [5.2]$$

This means the total risk load calculated for the portfolio by the two methods will be the same, although the individual risk loads for (X) and (Y) will differ between the methods.

* Item (3) is the Risk Load, the product of Items (1) and (2).

Since (X) is the first account, the marginal variance from adding (X) equals the variance of (X) (Var [X]) of 19,619,900. This gives a risk load of $1,353.02.

The marginal variance from writing (Y) equals Var [X + Y] - Var [X], or $3,279,059, implying a risk load of $226.13.

The sum of these two risk loads (X) + (Y) is $1,353.02 + $226.13 = $1,579.14. This equals the risk load which this method would calculate for the combined account (X + Y).

(6) Renewing the Portfolio of 2 Accounts

Table 13 shows the natural extension of the Build-up scenario -- renewal of these 2 accounts, in what could be termed a "static" or "steady state" portfolio (one with no new entrants).

As for applying these methods in the renewal scenario, renewing policy (X) is assumed equivalent to adding (X) to a portfolio of (Y); renewing (Y) is assumed equivalent to adding (Y) to a portfolio of (X).

(6.1) MS Method

Pertinent values from Table 13 for the Marginal Surplus method are summarized here:

Table 6.1

Renewing (X) & (Y): Marginal Surplus	Account (X)	Account (Y)	Account (X) + Account (Y)	Portfolio (X + Y)
(1) Change in Standard Deviation	4,171	356	4,526	4,785
(2) Risk Load Multiplier	0.33	0.33	-	0.33
(3) Risk Load = (1) * (2)	$1,376.27	$117.43	$1,493.70	$1,579.14
(4) Build-up Risk Load	$1,461.71	$117.43	$1,579.14	$1,579.14
(5) Difference	($85.45)	$0	($85.45)	$0

The marginal standard deviation for adding (Y) to (X) is 356, same as it was during Build-up -- see Section (5.1). The risk load of $117.43 is also the same.

However, adding (X) to (Y) gives a marginal standard deviation of Std Dev [X + Y] - Std Dev [Y] , or 4,171. This gives a risk load for (X) of $1,376.27, which is (85.45) **less** than $1,461.71, the risk load for (X) calculated in Section (5.1).

The sum of these two risk loads is $1,376.27 + $117.43 = $1,493.70. This is also (85.45) **less** than the total risk load from Section (5.1).

(6.2) MV Method

Pertinent values from Table 13 for the Marginal Variance method are summarized here:

Table 6.2

Renewing (X) & (Y): ***Marginal Variance***	Account (X)	Account (Y)	Account (X) + Account (Y)	Portfolio (X + Y)
(1) Change in Variance	22,521,000	3,279,059	25,800,059	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load	$1,553.08	$226.13	$1,779.21	$1,579.14

= (1) * (2)				
(4) Build-up Risk Load	$1,353.02	$226.13	$1,579.14	$1,579.14
(5) Difference	$200.06	$0	$200.06	$0

The marginal variance for adding (Y) to (X) is 3,279,059, same as it was during Build-up -- see Section (5.2). The risk load of $226.13 is also the same.

However, adding (X) to (Y) gives a marginal variance of Var [X + Y] - Var [Y], or 22,521,000. The risk load is now $1,553.08, which is $200.06 **more** than the $1,353.02 calculated in Section (5.2).

The sum of these two risk loads is $1,553.08 + $226.13 = $1,779.21. This is also $200.06 **more** than the total risk load from Section (5.2).

(7) Exploring the Differences Between New and Renewal

Why are the total Renewal risk loads different from the total Build-up risk loads?

In Section (5.1) Build-up, the marginal standard deviation for (X), ∆Std Dev [X], was :

ΔStd Dev [X] = Std Dev [X]

$$= \text{SQRT} [\Sigma_i \{ X_i^2 * p_i * (1 - p_i) \}] , \qquad [7.1]$$

(X_i = modeled losses for X for event i)

while in Section (6.1) Renewal, the marginal standard deviation was

ΔStd Dev [X] = Std Dev [X + Y] - Std Dev [Y]

$$= \text{SQRT} [\Sigma_i \{ (X_i+Y_i)^2 * p_i * (1 - p_i) \}] -$$

$$\text{SQRT} [\Sigma_i \{ Y_i^2 * p_i * (1 - p_i) \}] \qquad [7.2]$$

For positive Y_i, this value is less than Std Dev [X][8]. Therefore, one *would* expect the Renewal risk load to be less than the Build-up.

Unfortunately, when the MS method is applied in the renewal of all the accounts in a portfolio, the sum of the individual risk loads will be less than the total portfolio standard deviation times the multiplier. This is because the sum of the marginal standard deviations (found by taking the difference in portfolio standard deviation with and without each account in the portfolio) is less than the total

[8]For example, assume Var [X] = 9, Var [Y] = 4, Cov [X, Y] = 1.5; then
ΔStd Dev [X] = Sqrt(Var [X]) = Sqrt (9) = 3 *for X alone*
ΔStd Dev [X] = Sqrt(9 + 4 + 2*1.5) - Sqrt(4) = 4 - 2 = 2 < 3. *for X added to Y*

portfolio standard deviation[9]. This is because **the square root operator is "sub-additive"**: the square root of a sum is less than the sum of the square roots[10].

What about marginal variance? In Section (5.2) Build-up, the marginal variance ΔVar [X] was

$$\begin{aligned} \Delta \text{Var}\,[X] &= \text{Var}\,[X] \\ &= \Sigma_i \ \{ X_i^2 * p_i * (1 - p_i) \} \,, \qquad [7.3] \end{aligned}$$

while in Section (6.2) Renewal the marginal variance was

$$\begin{aligned} \Delta \text{Var}\,[X] &= \text{Var}\,[X + Y] - \text{Var}\,[Y] \\ &= \{ \text{Var}\,[X] + 2 * \text{Cov}\,[X, Y] + \cancel{\text{Var}\,[Y]} \} - \cancel{\text{Var}\,[Y]} \qquad [7.4] \\ &= \text{Var}\,[X] + 2 * \text{Cov}\,[X, Y] \\ &> \text{Var}\,[X]. \end{aligned}$$

Since 2 * Cov [X, Y] is greater than zero, one *would* expect the Renewal risk load to be greater than the Build-up.

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[9]The same issue is raised in Mr. Gogol's discussion [2]. He suggests correcting for this sub-additivity by using a weighted average of the contract's own standard deviation and its last-in marginal standard deviation. The weight is chosen so the sum of these redefined marginal impacts equals the total portfolio standard deviation ([2], p. 363).

[10]For example, Sqrt[9 + 16] < Sqrt[9] + Sqrt[16].

However, when the MV method is applied in the renewal of all the accounts in a portfolio, the sum of the individual risk loads will be more than the total portfolio variance times the multiplier. This is because the sum of the marginal variances (found by taking the difference in portfolio variance with and without each account in the portfolio) is greater than the total portfolio variance. **The covariance between any two risks in the portfolio is double counted**: when each account renews, it is allocated the full amount of its shared covariance with all the other accounts.

(8) A New Concept: Renewal Additivity

The renewal scenarios point out that these two methods are not what I call "**renewal additive**," defined as follows:

> For a given portfolio of accounts, a risk load method is **renewal additive** if the sum of the renewal risk loads calculated for each account equals the risk load calculated when the entire portfolio is treated as a single account.

Neither the MS nor the MV method is renewal additive: MS because the square root operator is sub-additive; MV because the covariance is double counted. So why should renewal additivity matter? Consider what happens when either of these non-renewal additive methods are used to renew the portfolio. The MV method would result in quoted renewal premiums the sum of whose risk loads would be greater than the "required" total risk load of (λ * total portfolio variance). One would in essence **overcharge** every account. The opposite is true for the MS case, where one would **undercharge** every account.

In order for the MS or MV methods to be renewal additive, one must assume an **entry order** for the accounts. Since the marginal impacts depend on the size of the existing portfolio, the entry order selected for an account could determine whether it is written or declined.

Renewal additivity reduces the renewal risk load calculation to an allocation of the total portfolio amount back to the individual accounts. An objective, systematic allocation methodology for renewals would be desirable. Examples of many such allocation methodologies can be found in the field of game theory.

(9) A New Approach from Game Theory

Two ASTIN papers by Jean Lemaire -- "An Application of Game Theory: Cost Allocation" [4], and "Cooperative Game Theory and Its Insurance Applications" [5] -- focus on general insurance applications of game theory. He also provides an extensive list of real world applications of game theory ([4], p. 77), including tax allocation among operating divisions of McDonnell-Douglas, maintenance costs of the Houston medical library, financing of large water resource development projects in Tennessee, construction costs of multi-purpose reservoirs in the U.S., and landing fees at Birmingham airport. Consider this example from [5]:

"The Treasurer of ASTIN (player 1) wishes to invest the amount of 1,800,000 Belgian francs on a short term (3 months) basis. In Belgium, the annual interest rate is a function of the sum invested.

Deposit (in Belgian Francs)	**Annual Interest Rate**
0 - 1,000,000	7.75%
1,000,000 - 3,000,000	10.25%
3,000,000 - 5,000,000	12.00%

The ASTIN Treasurer contacts the Treasurers of the International Actuarial Association (I.A.A. - player 2) and of the Brussels Association of Actuaries (A.A.Br. - player 3). I.A.A. agrees to deposit 900,000 francs in the common fund, A.A.Br. 300,000 francs. Hence the 3-million mark is reached and the interest rate will be 12%. How should the interests be split among the three associations?" ([5], p. 18)

Games such as this are referred to as "cooperative games with transferable utilities." They typically feature

(1) Participants (players) who have some benefits (or costs) to share (political power, savings, or money).

(2) The opportunity to share benefits (costs) results from cooperation of all participants or a sub-group of participants.

(3) Individuals are free to engage in negotiations, bargaining, coalition formation.

(4) Participants have conflicting objectives; each wants to secure the largest part of the benefits (smallest share of the costs) for himself. (see [5], p.20)

Cooperative games can be used as models for situations where participants must share or allocate an amount of money. Each player may want to maximize or minimize their allocation depending on the nature of the problem. If the group is deciding who pays what share of the total tax bill, each player will want to minimize their share. If the group is deciding how to split a pot of bonus money, each player will want to maximize their share.

The total amount to be allocated is determined by the **characteristic function**, which associates a real number v(S) to each coalition (group) S of players. It can be either sub-additive or super-additive:

Sub-Additive	v(S) + v(T) >= v(S union T)	for every disjoint S and T
Super-Additive	v(S) + v(T) <= v(S union T)	for every disjoint S and T

In the actuarial association example above, the characteristic function would be the money earned by each coalition (combination) of associations. It is an example of a super-additive characteristic function where the players seek to maximize their allocation. An example of a sub-additive characteristic function would be the insurance premium for a risk purchasing group: the sum of the individual members' insurance premiums is more than the insurance premium for the risk purchasing group as a whole. These players would seek to minimize their allocations, since they want to be charged the lowest premium. (Equivalently, these players want to maximize their savings as a result of joining the group -- savings being the difference between their allocation from the group and their standalone premium.)

Allocation Rules

A player's marginal impact depends on its entry order; in the example, the "allocation [to the three associations] of course depends on the order of formation of the grand coalition" ([5], p. 27). In the interests of fairness and stability, a new member should probably receive an allocation amount somewhere between its standalone value and its full marginal impact on the coalition characteristic function -- but where in between? How much is fair? These questions must be answered simultaneously for all the players, balancing questions of stability, incentives to split from the group, bargaining power, and marginal impact to the coalition characteristic function value.

To help answer the allocation question, game theory has developed a set of standards or rules for allocations. Firstly, legitimate allocation methods must be **additive** -- the sum of the players' allocations must equal the total amount to be allocated. The MV and MS methods are not (renewal) additive: they either allocate too much (MV) or too little (MS) in the renewal case.

Secondly, a coalition should be **stable**, which roughly translates to fair. There must not be incentives for either a single player or a sub-group of players to split from the group and form a faction. These "rules of fairness" are referred to as the conditions of **individual** and **collective rationality** (see [4], p. 66 - 68) :

> **Individual rationality** means a player is no worse off for having joined the coalition.
>
> **Collective rationality** means no subgroup would be better off on their own.

These rules can be formalized into a set of acceptable ranges of allocations for each player. This set defines what is known as the **core** of the game. It consists of all allocations which satisfy these fairness and stability conditions.

Consider the Brussels Association of Actuaries (A.A.Br. - player 3) from the example. They have 300,000 francs, and on their own could earn 7.75%. If they join as the third player, they will push the coalition rate of return from 10.25% to 12.00%. How much should they earn? Certainly not less than 7.75% -- it is not individually rational for them to join. Conversely, they should not earn so much that the other two players end up earning less than 10.25% -- that would not be collectively rational for them. In that case, they would be better off forming their own faction. Similar exercises can be performed for the other two players. The resulting set of acceptable allocations defines the boundaries of the core (see [5], p. 26).

Translating to Property Cat Risk Load

Given this brief introduction, a reasonable first attempt at translating from the game theory context might be:

Table 9.1

Translation from Game Theory to

Property Cat Risk Load

Game Theory	**Property Cat Risk Load**
Player	Account
Coalition	Portfolio
Characteristic Function	Portfolio Variance or Standard Deviation

Because of the covariance component, portfolio variance is a super-additive characteristic function: the variance of a portfolio is greater than the sum of the individual account variances. Standard deviation, on the other hand, is a sub-additive characteristic function because of the sub-additivity of the square root operator: the standard deviation of a portfolio is less than the sum of the individual account standard deviations.

This means, from the game theory perspective at least, that the choice between variance and standard deviation *is material*: it determines whether the characteristic function is sub-additive or super-additive. This is a fundamental paradox of the game theory translation of the risk load problem, and will require further research to resolve.

Setting aside this paradox for the moment, however, the risk load problem fits remarkably well into the game theory framework. The "players" want to minimize their allocations of the portfolio total risk load. The allocation should fairly and objectively assign risk load to accounts in proportion to their contribution to the total. Using the current definition of marginal impact of a renewal account, however, an entry order would have to be assumed in order to make the allocation additive. The results of that allocation would be heavily dependent on the selected order, however.

How can one choose the entry order of a renewal? A well-known allocation method from game theory may provide the answer.

The Shapley Value

The **Shapley value** (named for Lloyd Shapley, one of the early leaders of the game theory field) is an allocation method which is:

(1) Additive;

(2) At the centroid of the core; and

(3) Order *independent*.

It equals the average of the marginal impacts taken over all possible **entrance permutations** -- the different orders in which a new member could have been added to the coalition[11] (i.e. a new account could have been added to a portfolio).

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[11] Mr. Lemaire [5] provides this more complete definition of the Shapley value (p. 29): "The Shapley value can be *interpreted* as the mathematical expectation of the admission value, when

For example, consider a portfolio of accounts (A) and (B) to which a new account (C) is added. Shown in Table 9.2 are the marginal variances for adding (C) in the 6 possible entrance permutations[12] ("ABC" in Column (1) below means A enters first, then B, then C) :

Table 9.2

Entry Permutations for Account C

(1)	**(2)**	**(3)**
Permutation	**C Enters...**	**Marginal Variance**
ABC	After (A) & (B)	Var [C] + 2*Cov [C, A] + 2*Cov [C, B]
ACB	After (A)	Var [C] + 2*Cov [C, A]
BAC	After (B) & (A)	Var [C] + 2*Cov [C, A] + 2*Cov [C, B]
BCA	After (B)	Var [C] + 2*Cov [C, B]
CAB	First	Var [C]

all orders of formation of the grand coalition are equiprobable. In computing the value, one can assume, for convenience, that all players enter the grand coalition one by one, each of them receiving the entire benefits he brings to the coalition formed just before him. All orders of formation of N are considered and intervene with the same weight 1/n! in the computation. The combinatorial coefficient results from the fact that there are (s-1)!(n-s)! ways for a player to be the last to enter coalition S: the (s-1) other players of S and the (n-s) players of N\S (*those players in N which are not in S - DM*) can be permuted without affecting i's position."

[12] The author is indebted to John Major for pointing out an error in the original version of this exhibit.

CBA	First	Var [C]

The Shapley value is the straight average of Column (3) Marginal Variance over the six permutations:

Shapley Value = { Sum [Column (3)] } / 6 [9.1]

= { 6*Var [C] +

6*Cov [C, A] +

6*Cov [C, B] } / 6

= Var [C] + Cov [C, A] + Cov [C, B]

Or, to generalize, given

L = existing portfolio

n = new account

Shapley Value = Var [n] + Cov [L, n]. [9.2]

Before seeing this result, there might have been concerns about the practicality of this approach -- how much computational time might be required to calculate all the possible entrance permutations for a portfolio of thousands of accounts? This simple reduction formula eliminates those concerns. The Shapley value is as simple to calculate as the marginal variance.

Comparing the Shapley value to the marginal variance formula from Section 4:

$$\text{Marginal Variance} = \text{Var}[\,n\,] + \mathbf{2} * \text{Cov}[\,L, n\,], \qquad [9.3]$$

the Shapley value only takes **1** times the covariance of the new account and the existing portfolio.

One can also calculate the Shapley value under the marginal standard deviation method. However, due to the complex nature of the mathematics -- differences of square roots of sums of products -- no simplifying reduction formula was immediately apparent[13].

Therefore, the remainder of the paper will focus on the MV method and the variance-based Shapley value. Life will be much easier (mathematically) working with the variances, and very little is lost by choosing variance. Citing Mr. Bault ([1], p. 82), from a risk load perspective, "both [variance and standard deviation] are simply special cases of a unifying covariance framework." In fact, Mr. Bault

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[13]Those wishing to employ standard deviation can use approximate methods to calculate the Shapley value. Two approaches suggested by John Major are (i) taking the average of marginal value if first in and last in; and (ii) employing Monte Carlo simulation to sample a subset of the

goes on to suggest "in most cases, the 'correct' answer is a marginal risk approach that incorporates covariance"[14].

(10) Sharing the Covariance

The risk load question, framed in a game-theoretical light, has now become:

> *How do accounts share their mutual covariance for purposes of calculating risk load?*

The Shapley method answers, "Accounts split their mutual covariance equally." At first glance this appears reasonable, but consider the following example.

Assume two accounts, (L) and (M). (M) has 100 times the losses of (L) for each event. Their total shared covariance is

$$2 * Cov(L, M) = 2 * \Sigma_i \{ L_i * M_i * p_i * (1 - p_i) \}$$

$$= 2 * \Sigma_i \{ L_i * 100L_i * p_i * (1 - p_i) \} \quad [10.1]$$

---

possible entrance permutations, presumably large enough to achieve satisfactory convergence while being much more computationally efficient.

The Shapley value would equally divide this total covariance between (L) and (M), even though their relative contributions to the total are clearly not equal. There is no question that (L) should be assessed *some* share of the covariance. The issue is whether there is a more equitable share than simply half.

One could develop a generalized covariance sharing (GCS) method which uses a weight $W_i^L(L, X)$ to determine (L)'s share of the mutual covariance between itself and account (X) for event i:

$$\text{CovShare}_i^L (L, X) = W_i^L(L, X) * 2 * L_i * X_i * p_i * (1 - p_i) \quad [10.2]$$

Then (X)'s share of that mutual covariance would simply be

$$\text{CovShare}_i^X (L, X) = [1 - W_i^L (L, X)] * 2 * L_i * X_i * p_i * (1 - p_i) \quad [10.3]$$

The total covariance share allocation for (L) over all events would be

$$\text{CovShare}_{Tot}^L = \Sigma_Z \Sigma_i \{ \text{CovShare}_i^L(L, Z) \} \quad [10.4]$$

---

[14]Mr. Kreps [3] also incorporates covariance in his "Reluctance" R (p. 198), which has the formula R = [yz/(1+y)]/(2SC + σ)/(S' + S), where C is the correlation of the contract with the existing book. The Risk Load is then equal to Rσ.

[Σ_Z = *sum over every other account in the portfolio*]

The Shapley method is a generalized covariance sharing method with W_i^L (L, X) = 50% for all (L), (X), and i.

Returning to the example with (L) and (M), one could develop an example of a weighting scheme which assigns the shared covariance by event to each in proportion to their loss for that event. W_i^L (L, M), account (L)'s share of the mutual covariance between itself and account (M) for event i, equals

$$W_i^L (L, M) = [L_i / [L_i + M_i]] \quad [10.5]$$

$$= [L_i / [L_i + 100L_i]]$$

$$= (1 / 101)$$

= roughly 1% of their mutual covariance for event i

This shall be called the "Covariance Share" (CS) method.

(11) Applying the Shapley and CS Methods to the Example

Consider the Shapley and CS methods applied to the 2 Account example for both Build-up and Renewal.

(11.1) Portfolio Build-up

Table 14 shows the Build-up of accounts (X) and (Y) from Section 5, but for the Shapley and CS methods; pertinent values for the Shapley value are summarized here:

Table 11.1

Building Up (X) & (Y)*:** ***Shapley Value	Account (X)	Account (Y)	Account (X) + Account (Y)	Portfolio (X + Y)
(1) Change in Variance	19,619,900	1,828,509	21,448,409	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,353.02	$126.10	$1,479.11	$1,579.14

Pertinent values for the Covariance Share are summarized here:

Table 11.2

Building Up (X) & (Y)*:** ***Covariance Share	Account (X)	Account (Y)	Account (X) + Account (Y)	Portfolio (X + Y)
(1) Change in Variance	19,619,900	950,658	20,570,558	22,898,959

(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,353.02	$65.56	$1,418.57	$1,579.14

Both Shapley and CS produce the same risk load for (X) as the MV method on build-up - $1,353.02. This is because there is no covariance to share - (X) is the entire portfolio at this point. However, compare the results of the three variance-based methods for account (Y):

Table 11.3

Comparison of Build-up Risk Loads for Account (Y)	
Marginal Variance (MV) - Section 5.2	$226.13
Shapley Value	$126.10
Difference from MV	***$100.03***
Covariance Share (CS)	$65.56
Difference from MV	***$160.57***

Compared to MV, which charges account (Y) for the full increase in variance (Var [Y] + 2* Cov [X, Y]), the Shapley method only charges (Y) for Var [Y] + Cov [X, Y]. The same can be said for the CS method, although the share of the mutual covariance depends on each account's relative contribution by event, weighted and summed over all events. Now consider what happens to that ***difference from MV*** upon renewal.

(11.3) <u>Renewal</u>

Table 15 shows the renewal of (X) and (Y) for the Shapley and CS methods; pertinent values for the Shapley method are summarized here:

Table 11.4

Renewing (X) & (Y): ***Shapley Value***	Account (X)	Account (Y)	Account (X) + Account (Y)	Portfolio (X + Y)
(1) Change in Variance	21,070,450	1,828,509	22,898,959	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,453.05	$126.10	$1,579.14	$1,579.14
(4) Build-up Risk Load	$1,353.02	$126.10	$1,479.11	$1,579.14
(5) Difference	$100.03	$0	$100.03	$0

Pertinent values for the Covariance Share method are summarized here:

Table 11.5

Renewing (X) & (Y): ***Covariance Share***	Account (X)	Account (Y)	Account (X) + Account (Y)	Portfolio (X + Y)
(1) Change in Variance	21,948,301	950,658	22,898,959	22,898,959
(2) Risk Load Multiplier	0.000069	0.000069	-	0.000069
(3) Risk Load = (1) * (2)	$1,513.59	$65.56	$1,579.14	$1,579.14
(4) Build-up Risk Load	$1,353.02	$65.56	$1,418.57	$1,579.14
(5) Difference	$160.57	$0	$160.57	$0

With both the Shapley and CS methods, the sum of the risk loads for Account (X) and Account (Y) equals the risk load for Account (X + Y), namely $1,579.14. This means both new methods are **renewal additive**.

To see what happened to ***difference from MV***, compare the risk loads calculated at renewal for (X) with those at build-up:

Table 11.6

Build-up vs Renewal Risk Loads for Account (X)	**Shapley**	**Cov Share**
Renewal	$1,453.05	$1,513.59
Build-up	$1,353.02	$1,353.02
Additional Renewal Risk Load over Build-up	***$100.03***	***$160.57***
Difference from MV	***$100.03***	***$160.57***

The difference from MV during build-up is simply the portion of (X)'s risk load attributable to its share of covariance with (Y). It was missed during build-up because it was unknown -- account (Y) had not been written.

(12) Conclusion

This paper introduced two new approaches to determination of renewal risk load that address concerns with renewal additivity and point out the issue of covariance sharing between accounts. The ideal solution in practice might involve using a marginal method for the pricing of new accounts, and a renewal additive method for renewals.

This paper also represents a first step in addressing the perplexing question of order dependency. As mentioned in the introduction, order dependency is an issue which stretches beyond the confines of actuarial pricing to the finance community at large. It will likely take a joint effort between finance professionals and actuaries to reach a satisfactory solution.

Finally, this paper brings important information from game theory to the Proceedings. Game theory is a rich field for actuaries to find new ideas on cost allocation, fairness and order dependency. Many sticky social issues (taxation, voting rights, utility costs) have been resolved using ideas from game theory. Further research could be done on several questions raised during the review of this paper, including the relative bargaining power of accounts, portfolio departure rules, lack of account information, and the unresolved paradox of the sub-additive MS characteristic function versus the super-additive MV characteristic function.

References

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[5] Lemaire, Jean, "Cooperative Game Theory and its Insurance Applications," *ASTIN Bulletin*, Vol. 21, No. 1, 1991, pp. 17-40.

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Table 12 - Build a Portfolio of 2 Accounts

			Loss for Account		
Event i	P(i)	1-P(i)	(X)	(Y)	Portfolio (X + Y)
1	2.0%	98.0%	25,000	200	25,200
2	1.0%	99.0%	15,000	500	15,500
3	3.0%	97.0%	10,000	3,000	13,000
4	3.0%	97.0%	8,000	1,000	9,000
5	1.0%	99.0%	5,000	2,000	7,000
6	2.0%	98.0%	2,500	1,500	4,000

E[L]	1,290	179	1,469
Var[L]	**19,619,900**	**377,959**	**22,898,959**
Std Dev[L]	4,429	615	4,785

Covariances	**(X)**	**(Y)**
(X)	**19,619,900**	1,450,550
(Y)	1,450,550	**377,959**

		(X)	**(Y)**	**(X) + (Y)**
Change in Std Deviation		4,429	356	4,785
Risk Load (Std Dev)		1,461.71	117.43	1,579.14
Multiplier :	**0.33**	**Risk Load for (X + Y) :**		1,579.14

Change in Variance		19,619,900	**3,279,059**	22,898,959
Risk Load (Variance)		1,353.02	226.13	1,579.14
Multiplier :	**0.000069**	**Risk Load for (X + Y) :**		1,579.14

Table 13 - Renew the Portfolio of 2 Accounts

Event i	P(i)	1-P(i)	Loss for Account (X)	(Y)	Portfolio (X + Y)
1	2.0%	98.0%	25,000	200	25,200
2	1.0%	99.0%	15,000	500	15,500
3	3.0%	97.0%	10,000	3,000	13,000
4	3.0%	97.0%	8,000	1,000	9,000
5	1.0%	99.0%	5,000	2,000	7,000
6	2.0%	98.0%	2,500	1,500	4,000

	(X)	(Y)	Portfolio (X + Y)
E[L]	1,290	179	1,469
Var[L]	**19,619,900**	**377,959**	**22,898,959**
Std Dev[L]	4,429	615	4,785

Covar	**(X)**	**(Y)**
(X)	**19,619,900**	1,450,550
(Y)	1,450,550	**377,959**

		(X)	(Y)	(X)+(Y)
Change in Std Deviation		4,171	356	4,526
Risk Load (Std Dev)		**1,376.27**	117.43	1,493.70
0.33	*Build Up Risk Load*	1,461.71	117.43	1,579.14
	Difference	**(85.45)**		**(85.45)**

		(X)	(Y)	(X)+(Y)
Change in Variance		22,521,000	3,279,059	25,800,059
Risk Load (Variance)		**1,553.08**	226.13	1,779.21
0.000069	*Build Up Risk Load*	1,353.02	226.13	1,579.14
	Difference	**200.06**		**200.06**

Table 14 - Build a Portfolio of 2 Accounts - Alternatives

Event i	P(i)	1-P(i)	Covariance Share $ (X)	Covariance Share $ (Y)	Total
1	2.0%	98.0%	9,920,635	79,365	
2	1.0%	99.0%	14,516,129	483,871	
3	3.0%	97.0%	46,153,846	13,846,154	
4	3.0%	97.0%	14,222,222	1,777,778	
5	1.0%	99.0%	14,285,714	5,714,286	
6	2.0%	98.0%	4,687,500	2,812,500	
			2,328,401	572,699	2,901,100

Chg in Variance		(X)	(Y)
If added 1st		19,619,900	377,959
If added 2nd	after 1		3,279,059
	after 2	22,521,000	
Average	*(Shapley Value)*	21,070,450	1,828,509

		(X)	(Y)	Total
Shapley Value		19,619,900	1,828,509	21,448,409
Risk Load (Shapley)		**1,353.02**	**126.10**	**1,479.11**
0.000069				**1,579.14**
		Deferred Risk Load		**100.03**

		(X)	(Y)	Total
Covariance Share		19,619,900	950,658	20,570,558
Risk Load (Cov Share)		**1,353.02**	**65.56**	**1,418.57**
0.000069				**1,579.14**
		Deferred Risk Load		**160.57**

Table 15 - Renew the Portfolio of 2 Accounts - Alternatives

			Covariance Share $	
Event i	**P(i)**	**1-P(i)**	**(X)**	**(Y)**
1	2.0%	98.0%	9,920,635	79,365
2	1.0%	99.0%	14,516,129	483,871
3	3.0%	97.0%	46,153,846	13,846,154
4	3.0%	97.0%	14,222,222	1,777,778
5	1.0%	99.0%	14,285,714	5,714,286
6	2.0%	98.0%	4,687,500	2,812,500

		Total
2,328,401	572,699	2,901,100

Chg in Variance		**(X)**	**(Y)**
If added 1st		19,619,900	377,959
If added 2nd	after 1		3,279,059
	after 2	22,521,000	
Average	*(Shapley Value)*	21,070,450	1,828,509

Shapley Value		21,070,450	1,828,509	22,898,959
Risk Load (Shapley)		1,453.05	126.10	**1,579.14**
0.000069		**Risk Load for Portfolio (X + Y)**		**1,579.14**

Covariance Share		21,948,301	950,658	22,898,959
Risk Load (Cov Share)		1,513.59	65.56	**1,579.14**
0.000069		**Risk Load for Portfolio (X + Y)**		**1,579.14**