Risk Assessment Applications of Fuzzy Logic

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Acknowledgments

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> Shapiro. A. F., Koissi, M.-C. (2015) “Risk Assessment Applications of Fuzzy Logic,” Society of Actuaries, https://www.soa.org/Files/Research/Projects/2015-risk-assess-apps-fuzzy-logic.pdf

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Introduction

Fuzziness is frequently inherent in the decision-making process of risk assessment:

Data limitations and ambiguities, such as incomplete or unreliable data, and

Vague and subjective information owing to a reliance on human experts and their communication of linguistic variables.

This presentation focuses on conceptualizing risks when parameters are linguistic. Two topics are addressed:

A risk matrix based on linguistic variables – a fuzzy risk matrix application

Ordering risks when there is no consensus – a fuzzy Analytic Hierarchy Process application

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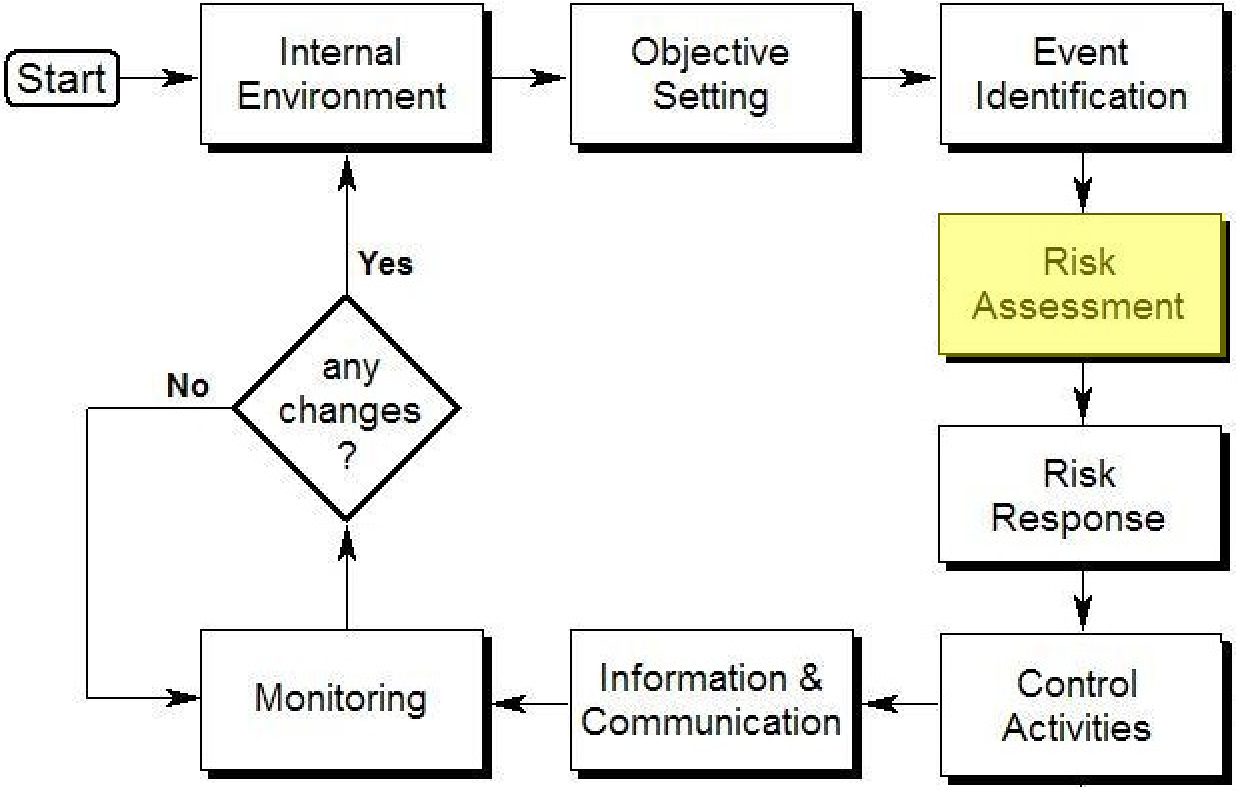
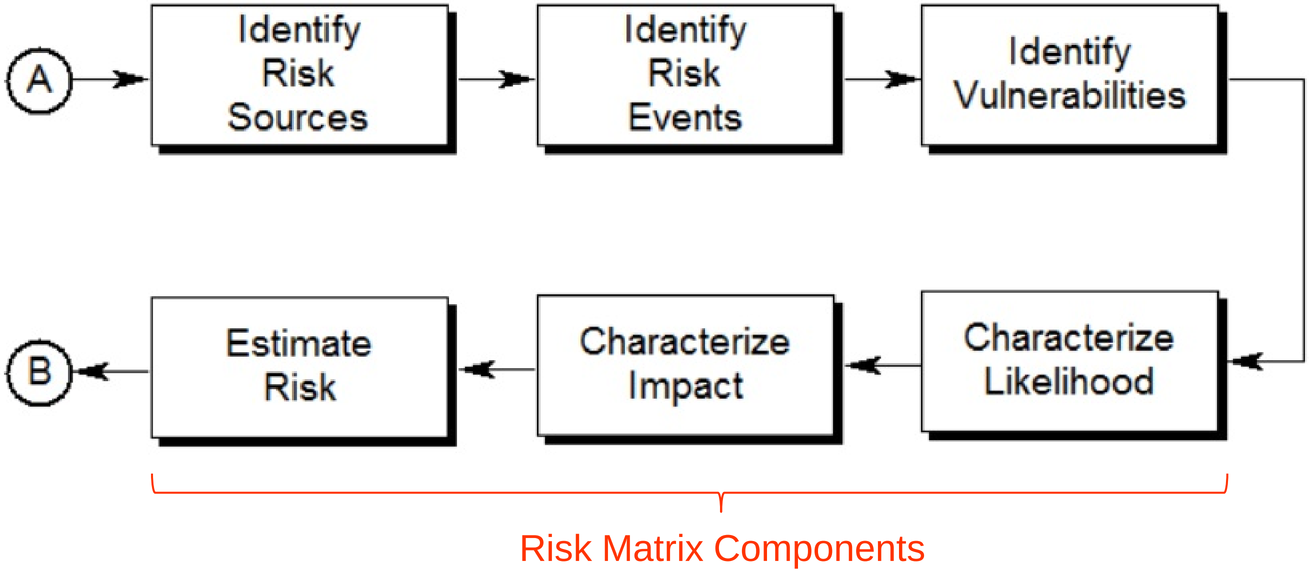
The COSO-ERM Model



COSO : Committee of Sponsoring Organizations of the Treadway Commission

A risk assessment flowchart*



*This figure is presented in a linear fashion in order to simplify the flowchart. In practice, of course, the order of identifying risk sources and risk events will depend on the circumstances, as will the order of characterizing the impact and the likelihood.

Risk Matrix

Severity					
5	MH 5	MH	MH	H	H 25
4	M	M	MH	MH	H
3	M	M 8	M 13	MH	MH
2	L	L 7	L 12	M	MH
1	L 1	L	L	M	MH
	1	2	3	4	5

Severity
1. negligible
2. low
3. moderate
4. high
5. catastrophic

Likelihood
1. very low
2. low
3. moderate
4. high
5. very high

Risk Categories
L: low
M: medium
MH: medium high
H: high

~ Wu et al (2013)

(Crisp) Rule 8

IF Likelihood is "Low" AND Severity is "Moderate"
THEN the risk is "medium"

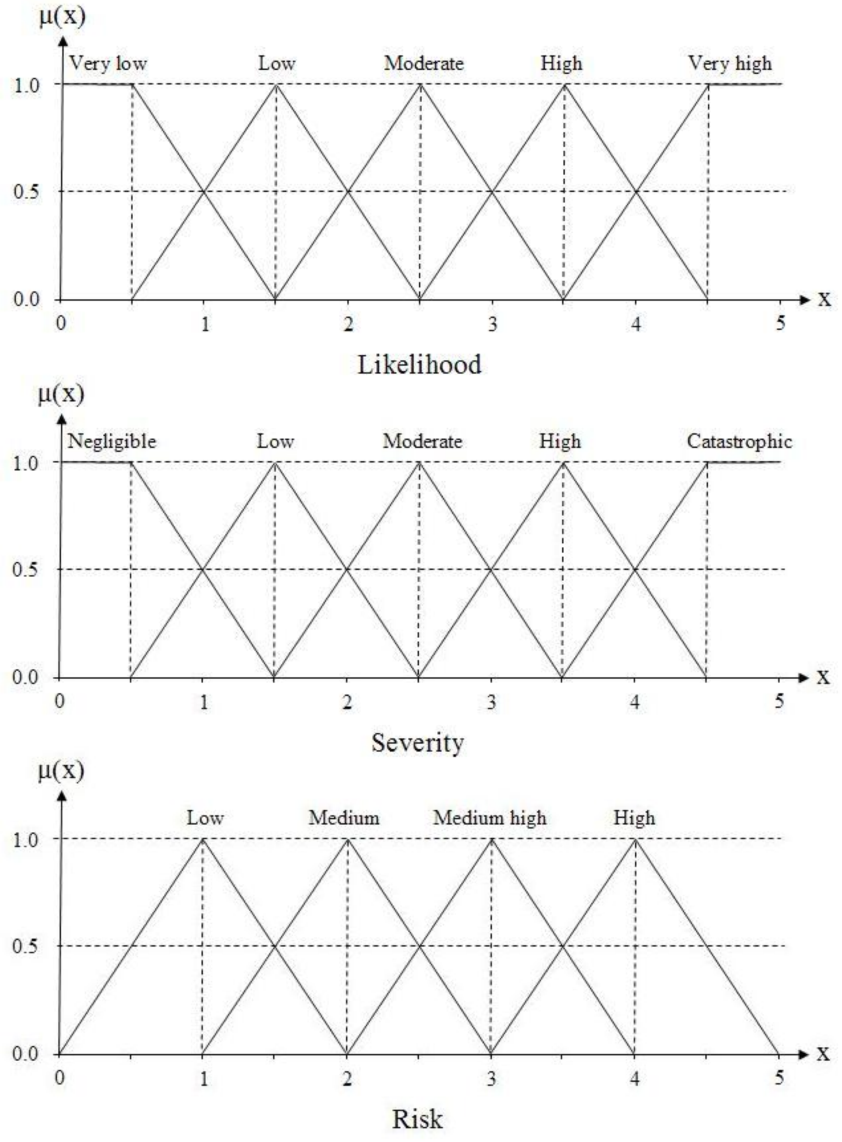
Severity
3. moderate

Severity					
5	MH 5	MH	MH	H	H 25
4	M	M	MH	MH	H
3	M	M 8	M 13	MH	MH
2	L	L 7	L 12	M	MH
1	L 1	L	L	M	MH
	1	2	3	4	5

Likelihood
2. low

Risk Categories
M: medium

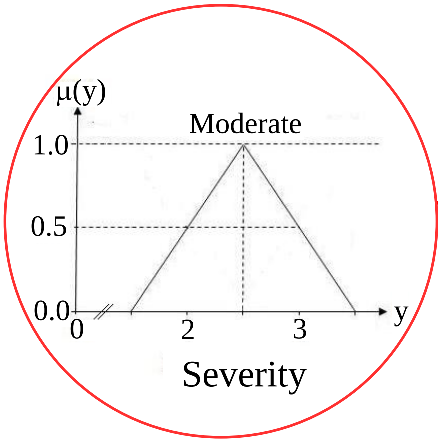
Likelihood, Severity and Risk MFs

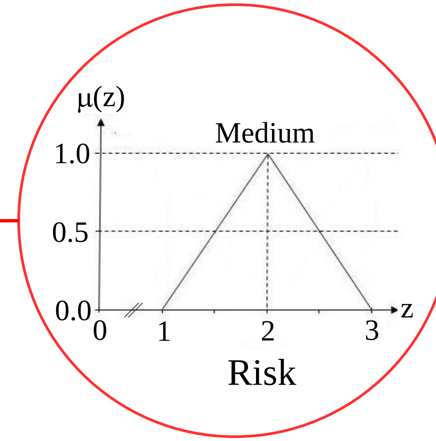


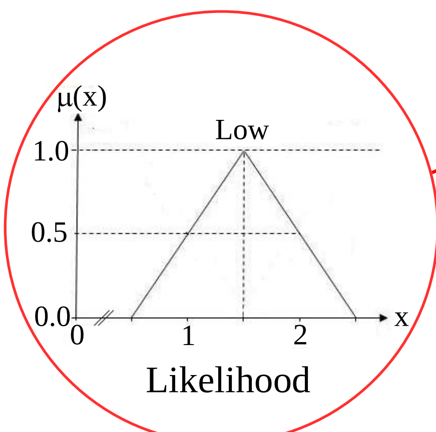
(Fuzzy) Rule 8

IF Likelihood is "Low" AND Severity is "Moderate"
THEN the risk is “Medium"

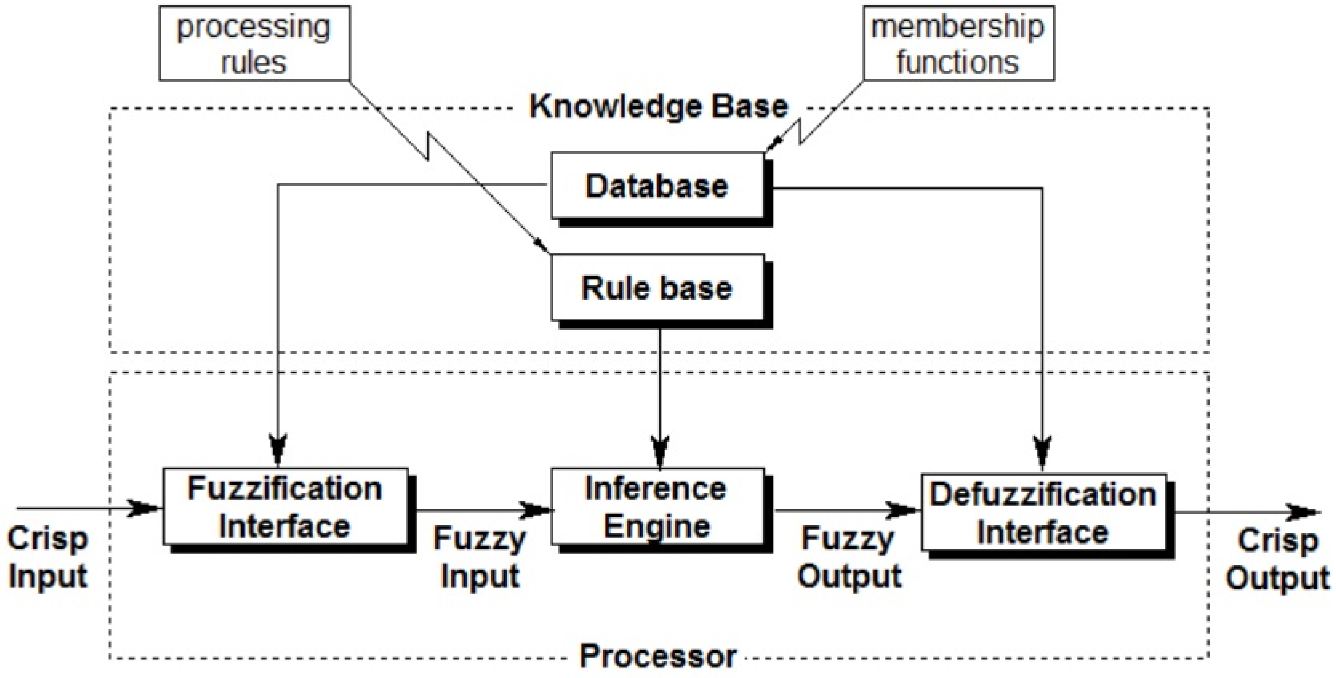
5	MH 5	MH	MH	H	H 25
4	M	M	MH	MH	H
3	M	M 8	M 13	MH	MH
2	L	L 7	L 12	M	MH
1	L 1	L	L	M	MH
	1	2	3	4	5





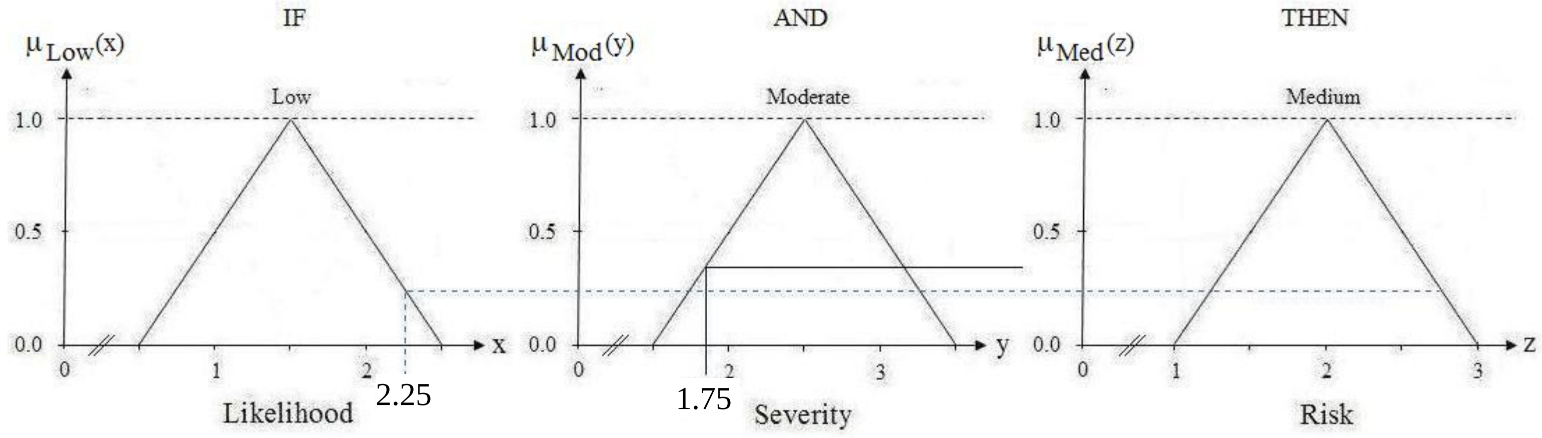


Fuzzy Inference System

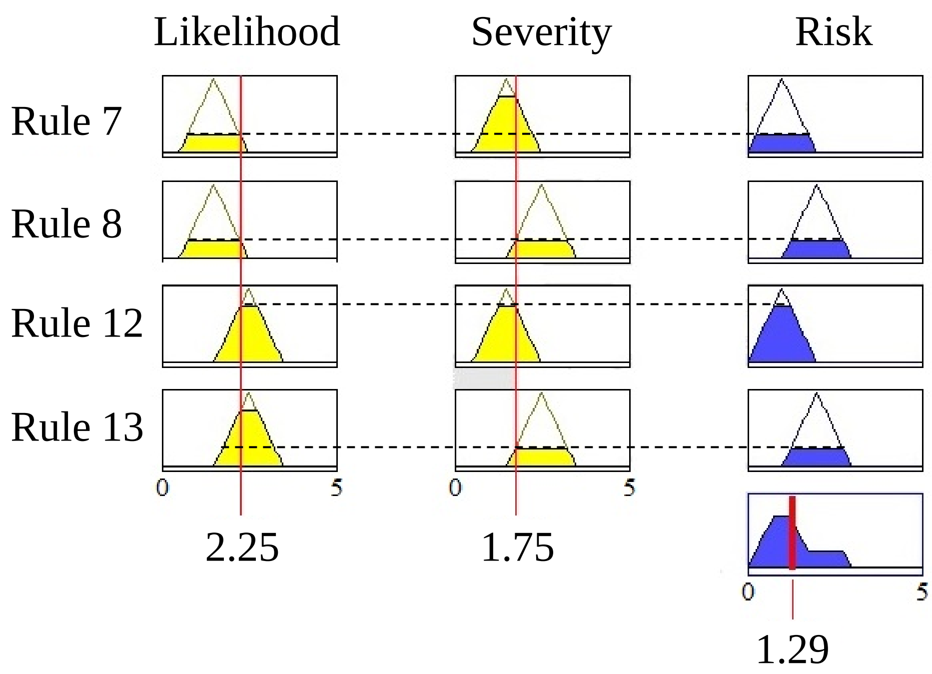


Fuzzy Rule 8

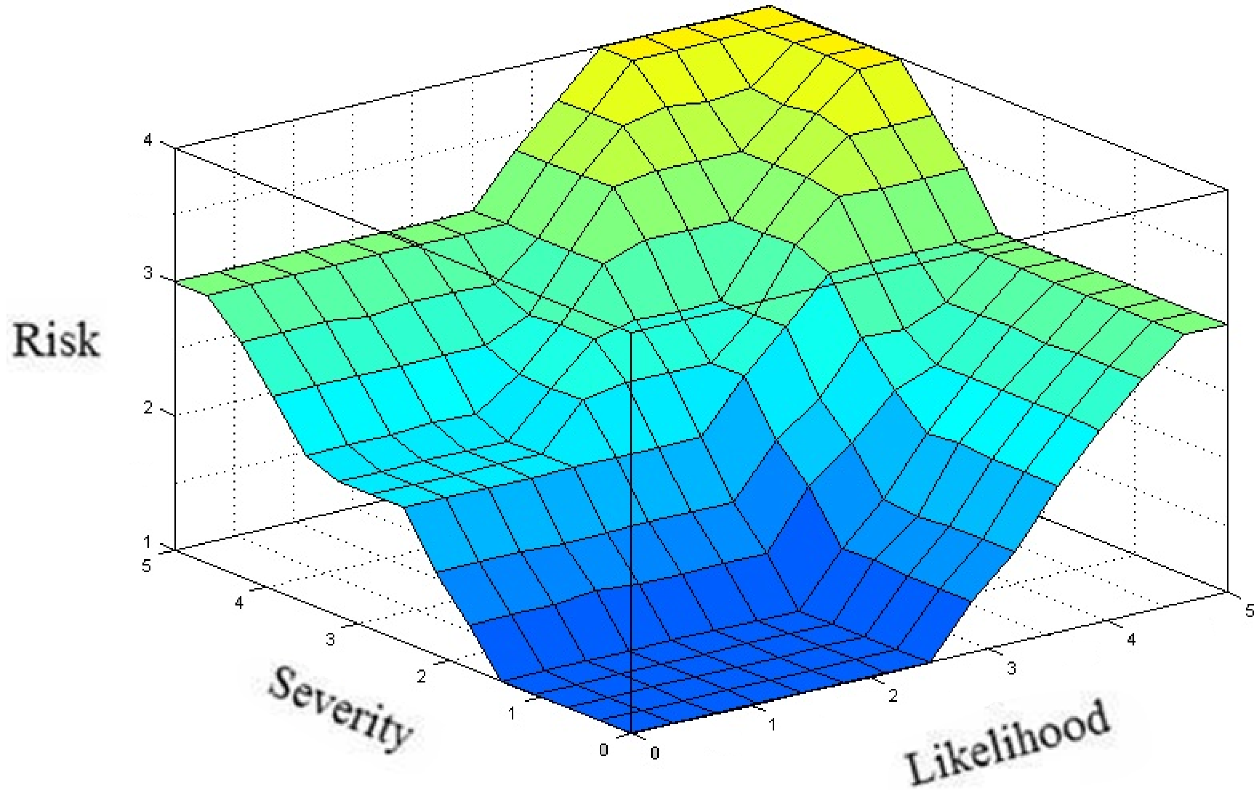
IF Likelihood is "Low" AND Severity is "Moderate"
THEN the risk is "medium"



Rule Viewer


Likelihood
Severity
Risk
Rule 7
Rule 8
Rule 12
Rule 13
0
5
0
5
2.25
1.75
0
5
1.29

Fuzzy Risk Matrix



Comments

This portion of the talk presented a FL version of the RM.

The value-added by FL was that the verbal expressions and linguistic variables were easily accommodated into the RM model.

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Introduction

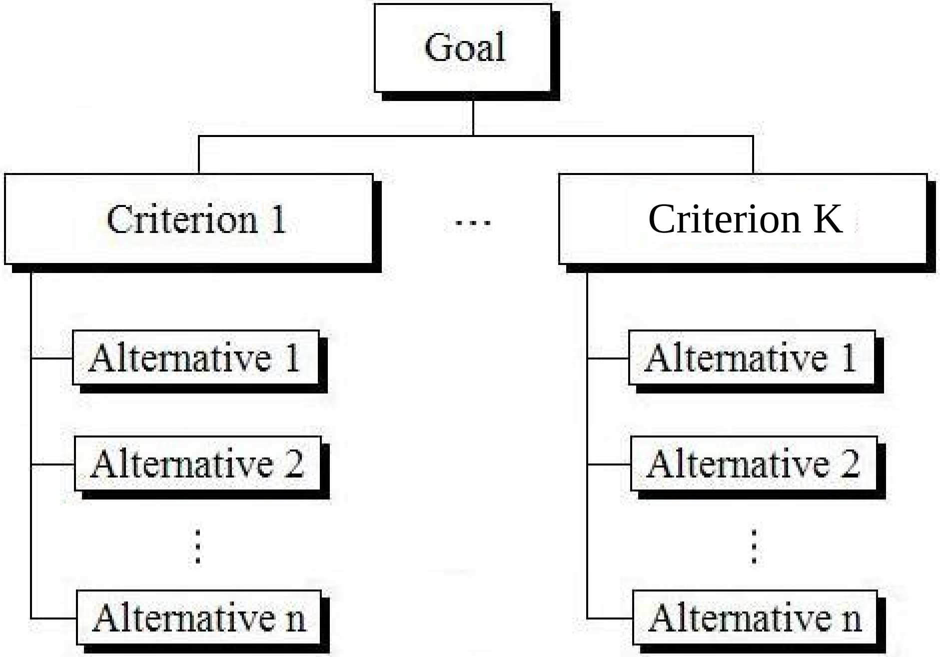
This section addresses the ordering risks when there is no consensus.

Specifically, the section discusses the use of the Analytic Hierarchy Process (AHP) [Saaty (1980, 1999, 2008)], a theory of measurement through pair-wise comparisons that relies on judgment to derive priority scales.

During implementation of the AHP, one constructs hierarchies, then makes judgments or performs measurements on pairs of elements with respect to a criterion to derive preference scales, which are then synthesized throughout the structure to select the preferred alternative.

We begin with a brief overview of the AHP and its limitations when confronted with a fuzzy environment. This is followed with a discussion of FL modifications of the AHP.

A 3-level (K x n) hierarchy



Multiple Criteria Decision Making (MCDM)

You ~~can't~~ can compare apples and oranges

You can't compare apples and oranges, or can you?

Consider a hungry person who likes both apples and oranges and is offered a choice between an apple and an orange. Which one is that person more likely to choose?

That person will choose the apple or orange that yields the greater value across all their various attributes, according to her/his preferences.

The Analytic Hierarchy Process (AHP) can help make such choices.

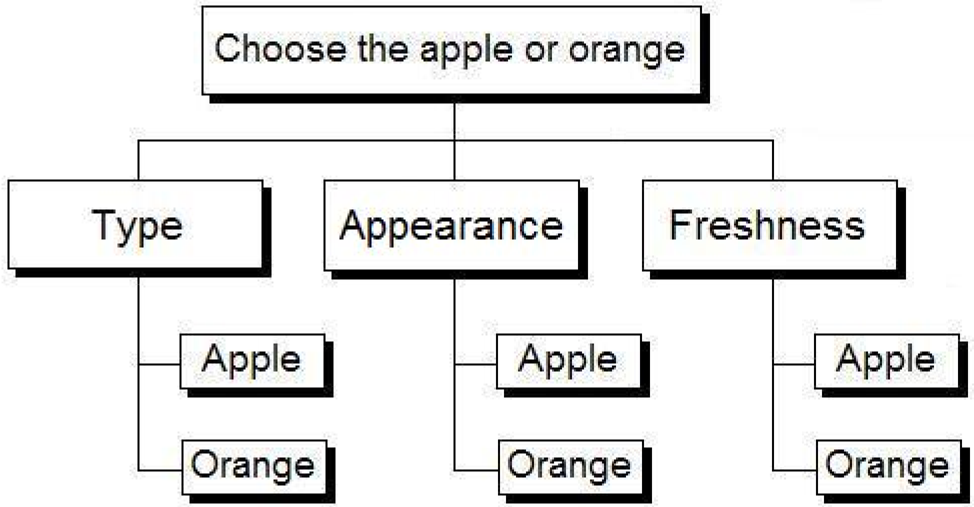
Saaty and Vargas (2012: 1)

Hierarchy example: apple or orange

Goal = G

Criteria = C_k

Alternatives = A_i



$$w_i^{A|G} = \sum_{k=1}^{3} w_k^{C|G} w_i^{A|C_k}$$

relative preference of C_k, with respect to G

relative preference of A_i, with respect to C_k

$$\sum_{k=1}^{3} w_k^{C|G} = \sum_{i=1}^{2} w_i^{A|C_k} = 1, \ \ w_k^{C|G}, \ w_i^{A|C_k} \geq 0$$

Developing $w_k^{C|G}$ and $w_i^{A|C_k}$ using the AHP

Start with a comparison matrix, W

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}, \text{ where } w_{ij} = \frac{w_i}{w_j}$$

Then, map the comparison matrix into the vector of weights

$$\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix} \rightarrow \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \text{vector of priorities}$$

Eigenvalue of a consistent matrix

Saaty advocated the eigenvalue method of determining vector of priorities (weights).

To recover (determine) the vector of weights, $(w_1, w_2, \ldots, w_n)$, given these ratios in W, we can take the matrix product of the matrix W with the vector $\underline{w}$ to obtain:

$$W\,\underline{w} = \begin{bmatrix} 1 & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} nw_1 \\ nw_2 \\ nw_3 \\ \vdots \\ nw_n \end{pmatrix} = n\,\underline{w}$$

That is:

$$W\,\underline{w} = n\,\underline{w}$$

Essentially, this reduces to the problem of finding the λ's (n's) that are the roots of $|W - \lambda\, I| = 0$.

The vector of priorities (weights)

Few have extended the eigenvalue method to fuzzy matrices [eg., Csutora & Buckley (2001)]

A major issue is that ax = b need not imply that $\tilde{a}\,\tilde{x} = \tilde{b}$. [Dubois (2011: 19)]

An alternate method discussed by Saaty, which we will use for our fuzzy AHP model, is the

Normalized geometric mean.

$$\hat{w}_i = \frac{\left(\prod_{j=1}^{n} \hat{w}_{ij}\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} \hat{w}_{ij}\right)^{1/n}}$$

Multiply the n elements in each row and take the nth root.
Normalize the resulting numbers.

A simple example using the geometric mean to compute the vector of priorities for the criteria of the apple & orange problem

Let the positive, reciprocal, criteria comparison matrix be

$$\begin{bmatrix} 1 & \hat{w}_{12}^{C|G} & \hat{w}_{13}^{C|G} \\ \hat{w}_{21}^{C|G} & 1 & \hat{w}_{23}^{C|G} \\ \hat{w}_{31}^{C|G} & \hat{w}_{32}^{C|G} & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & b \\ a^{-1} & 1 & c \\ b^{-1} & c^{-1} & 1 \end{bmatrix}$$

If the vector of priorities is

$$\hat{w}^{T} = \left(\hat{w}_1^{C|G}, \hat{w}_2^{C|G}, \hat{w}_3^{C|G} \right)$$

then (using the geometric mean approach)

$$\begin{bmatrix} 1 & a & b \\ a^{-1} & 1 & c \\ b^{-1} & c^{-1} & 1 \end{bmatrix} \Rightarrow \frac{\begin{matrix} a^{1/3} \; b^{1/3} \\ a^{-1/3} \; c^{1/3} \\ b^{-1/3} \, c^{-1/3} \end{matrix}}{T} \Rightarrow \begin{matrix} a^{1/3} \, b^{1/3} \; T^{-1} \\ a^{-1/3} \, c^{1/3} \, T^{-1} \\ b^{-1/3} \, c^{-1/3} \, T^{-1} \end{matrix} = \begin{bmatrix} \hat{w}_1^{C|G} \\ \hat{w}_2^{C|G} \\ \hat{w}_3^{C|G} \end{bmatrix}$$

where $T = a^{1/3} b^{1/3} + a^{-1/3} c^{1/3} + b^{-1/3} c^{-1/3}$.

Arguments for a fuzzy AHP (FAHP)

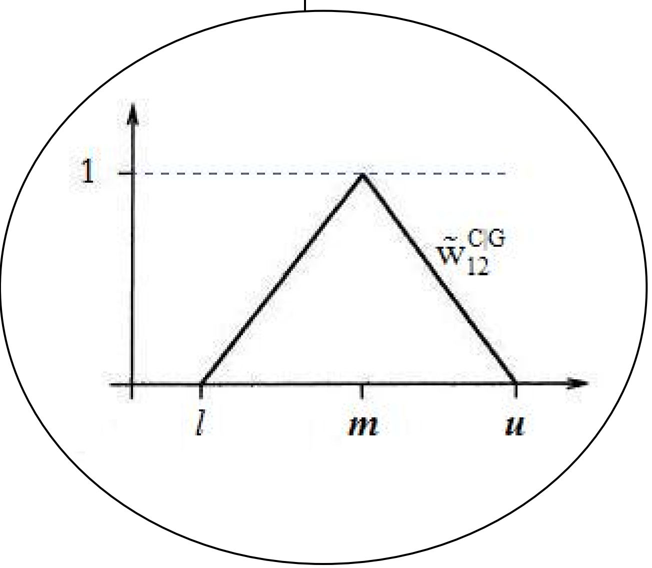
- It gives decision makers the opportunity to express their - essentially fuzzy - opinions in fuzzy numbers. [van Laarhoven and Pedrycz (1983)]

  •
  •
  •

- Asking for precise [pairwise comparison] … is debatable, because these coefficients are arguably imprecisely known. [Dubois (2011)]

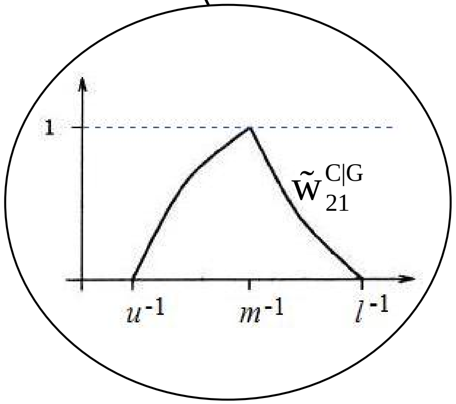
Adjusting the AHP for fuzziness in the criteria comparison matrix (upper triangular portion)

$$\begin{bmatrix} 1 & \tilde{w}_{12}^{C|G} & \cdots & \tilde{w}_{1K}^{C|G} \\ \tilde{w}_{21}^{C|G} & 1 & \cdots & \tilde{w}_{2K}^{C|G} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{w}_{K1}^{C|G} & \tilde{w}_{K2}^{C|G} & \cdots & 1 \end{bmatrix} \Rightarrow \begin{matrix} f(\tilde{C}_1) \\ f(\tilde{C}_2) \\ \vdots \\ f(\tilde{C}_K) \end{matrix} \Rightarrow \begin{bmatrix} \tilde{w}_{1}^{C|G} \\ \tilde{w}_{2}^{C|G} \\ \vdots \\ \tilde{w}_{K}^{C|G} \end{bmatrix}$$



Adjusting the AHP for fuzziness in the criteria comparison matrix (lower triangular portion)

$$\begin{bmatrix} 1 & \tilde{w}_{12}^{C|G} & \cdots & \tilde{w}_{1K}^{C|G} \\ \tilde{w}_{21}^{C|G} & 1 & \cdots & \tilde{w}_{2K}^{C|G} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{w}_{K1}^{C|G} & \tilde{w}_{K2}^{C|G} & \cdots & 1 \end{bmatrix} \Rightarrow \begin{matrix} f(\tilde{C}_1) \\ f(\tilde{C}_2) \\ \vdots \\ f(\tilde{C}_K) \end{matrix} \Rightarrow \begin{bmatrix} \tilde{w}_{1}^{C|G} \\ \tilde{w}_{2}^{C|G} \\ \vdots \\ \tilde{w}_{K}^{C|G} \end{bmatrix}$$



Adjusting the AHP for fuzziness in the criteria comparison matrix (computing the order)

$$\tilde{>}$$

$$\begin{bmatrix} 1 & \tilde{w}_{12}^{C|G} & \cdots & \tilde{w}_{1K}^{C|G} \\ \tilde{w}_{21}^{C|G} & 1 & \cdots & \tilde{w}_{2K}^{C|G} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{w}_{K1}^{C|G} & \tilde{w}_{K2}^{C|G} & \cdots & 1 \end{bmatrix} \Rightarrow \begin{matrix} f(\tilde{C}_1) \\ f(\tilde{C}_2) \\ \vdots \\ f(\tilde{C}_K) \end{matrix} \Rightarrow \begin{bmatrix} \tilde{w}_1^{C|G} \\ \tilde{w}_2^{C|G} \\ \vdots \\ \tilde{w}_K^{C|G} \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{\tilde{w}}_1^{C|G} \\ \hat{\tilde{w}}_2^{C|G} \\ \vdots \\ \hat{\tilde{w}}_K^{C|G} \end{bmatrix}$$

An examples of FL adjustments to the AHP

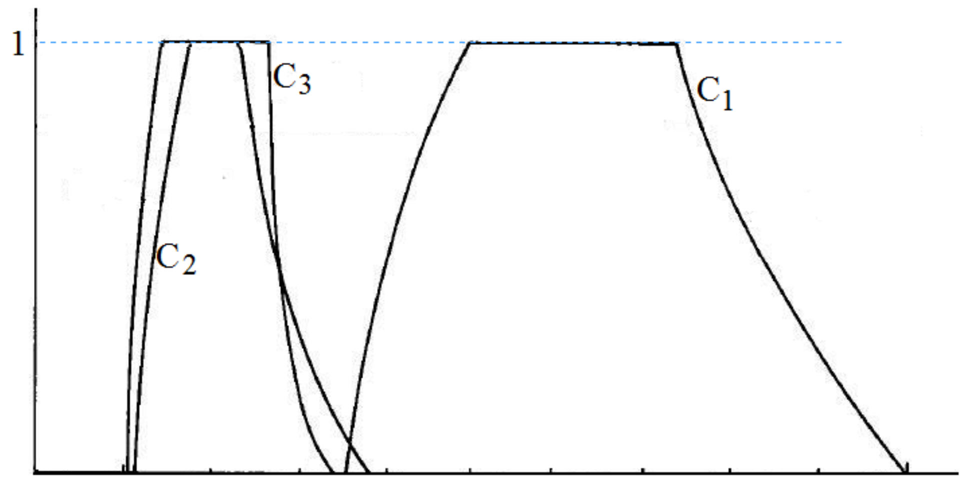
Use trapezoidal FNs to extend AHP to FAHP

Use the geometric mean method to derive fuzzy weights

Use fuzzy multiplication and the fuzzy n-th root

Buckley, J. J. (1985) “Fuzzy hierarchical analysis,” Fuzzy Sets and Systems 17, 233-247.

MFs for the fuzzy weights associated with the criteria



Comments on the FAHP

FL seems to have a legitimate role wrt the AHP

- There is subjective appraisal of ambiguity
- There is vagueness and imprecision
- There is lack of information

However, controversy abounds

Some argue for fuzzy weights
Others argue against them

Some argue for a more conceptually sound approach
Others argue for simpler models

Even the essence is challenged

"… the very question of determining how many more times a criterion is important than an other one is meaningless" Dubois (2011)

"Our concern here is with the validity of applying fuzzy thinking to decision making" Saaty and Tran (2007)

Closing comments

This presentation focused on conceptualizing and ordering risks where parameters are linguistic. The two topics addressed were

A risk matrix based on linguistic variables, and

Ordering risks when there is no consensus.

The broader issue, though, is how crisp risk assessment models, which have fuzzy components that are inadequately accommodated by the model, can be reformulated as fuzzy models.

The presentation will have met its goal if it gave useful insights into this issue.