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CREDIBILITY FOR A TOWER OF EXCESS LAYERS

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Agenda

1. Basic Credibility Concepts
2. Multivariate Credibility
3. The XOL Reinsurance Problem
4. The Recursive Form for XOL Pricing

Preliminaries

Criteria for an estimator of future losses:

- **Unbiased** = the expected value of the estimator is equal to the "true" expected loss

$$E(\hat{\mu}) = \mu$$

- **Minimum Variance** = on average the value produced by this estimator will be closer to the true expected loss than other estimates
- **Robust** = the estimator behaves well even if model assumptions are not exactly met; stable results even given outliers

Basic Credibility Concepts – Venter's *Credibility for Dummies*

Credibility theory that focuses on the goal of minimum variance is also known as "least squares" or "greatest accuracy" credibility.

The goal is simple to state: **We want to make use of all the available and relevant information, giving the proper weight to each piece of information.**

"Credibility theory is all about weighted averages."

-Gary Venter

Basic Credibility Concepts – Venter's *Credibility for Dummies*

A credibility-weighted (cw) average of two estimators is given as a linear weighted average:

$$\widehat{\mu_{cw}} = w \cdot \widehat{\mu_1} + (1 - w) \cdot \widehat{\mu_2}$$

The two estimators are unbiased and independent:

$$E(\widehat{\mu_1}) = E(\widehat{\mu_2}) = \mu$$

$$Cov(\widehat{\mu_1}, \widehat{\mu_2}) = 0$$

The variance of the credibility-weighted average is written as:

$$Var(\widehat{\mu_{cw}}) = w^2 \cdot Var(\widehat{\mu_1}) + (1 - w)^2 \cdot Var(\widehat{\mu_2})$$

Basic Credibility Concepts – Venter's *Credibility for Dummies*

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We can find the "best" credibility weight as the w that minimizes the variance of the credibility-weighted average.

$$\frac{\partial\, Var(\widehat{\mu_{cw}})}{\partial\, w_1} = 0$$

The result is that the "best" weight is inversely proportional to the variance of the estimator.

$$\widehat{w_1} = \frac{Var(\widehat{\mu_2})}{Var(\widehat{\mu_1}) + Var(\widehat{\mu_2})} = \frac{Var(\widehat{\mu_1})^{-1}}{Var(\widehat{\mu_1})^{-1} + Var(\widehat{\mu_2})^{-1}}$$

Basic Credibility Concepts

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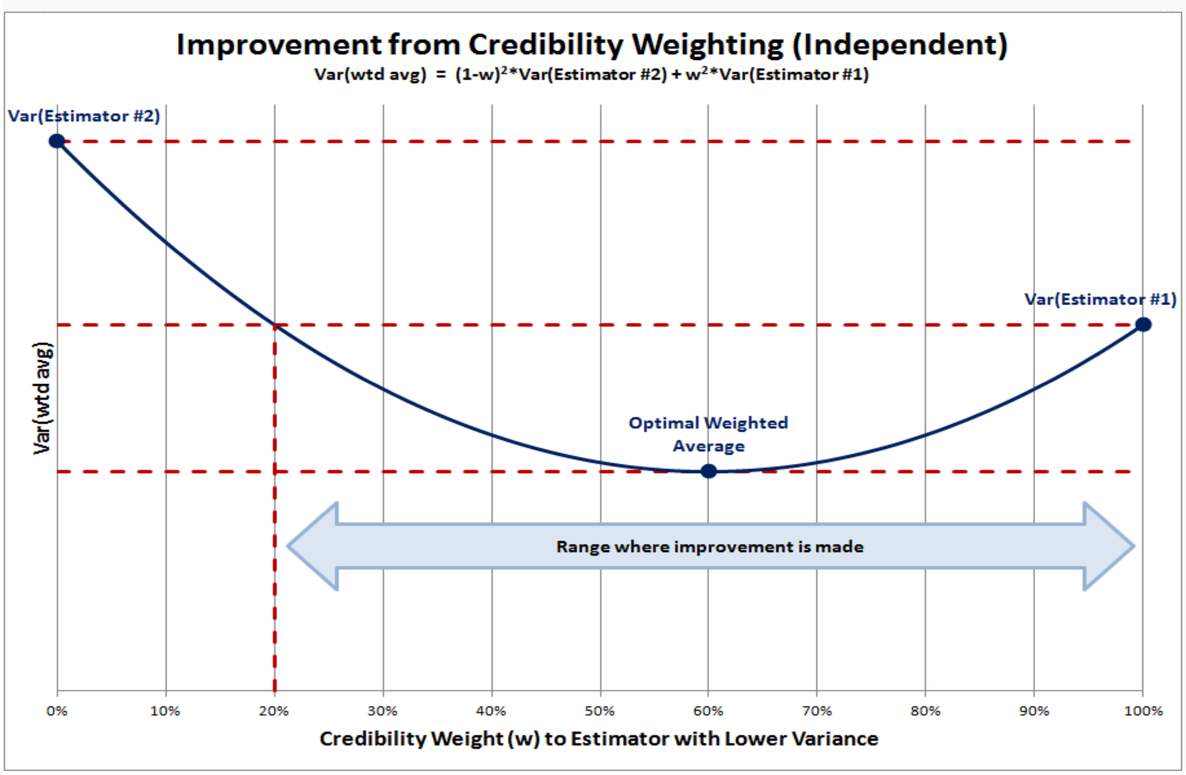


A credibility-weighted (cw) average of multiple estimators:

$$\widehat{\mu_{cw}} = \sum_{i=1}^{n} w_i \cdot \widehat{\mu_i} \qquad \sum_{i=1}^{n} w_i = 1$$

If all estimators are assumed to be unbiased and independent:

$$Var(\widehat{\mu_{cw}}) = \sum_{i=1}^{n} w_i^2 \cdot Var(\widehat{\mu_i})$$

Assuming independence among the various estimators, the “best” weights are again inversely proportional to the individual variances.

$$\widehat{w_i} = \frac{Var(\widehat{\mu_i})^{-1}}{\sum_{j=1}^{n} Var\left(\widehat{\mu_j}\right)^{-1}}$$

Substituting these weights back into the variance equation produces the following:

$$Var(\widehat{\mu_{cw}} \,|\, \widehat{w_i}) = \frac{1}{\frac{1}{Var(\widehat{\mu_1})} + \frac{1}{Var(\widehat{\mu_2})} + \cdots + \frac{1}{Var(\widehat{\mu_n})}}$$

Multivariate Case

Where there is correlation between the estimators, we define a covariance matrix containing the covariance between every pair of estimators.

For the three variable case, we have:

$$\boldsymbol{\Sigma} = \begin{bmatrix} Var(\widehat{\mu_1}) & Cov(\widehat{\mu_1}, \widehat{\mu_2}) & Cov(\widehat{\mu_1}, \widehat{\mu_3}) \\ Cov(\widehat{\mu_2}, \widehat{\mu_1}) & Var(\widehat{\mu_2}) & Cov(\widehat{\mu_2}, \widehat{\mu_3}) \\ Cov(\widehat{\mu_3}, \widehat{\mu_1}) & Cov(\widehat{\mu_3}, \widehat{\mu_2}) & Var(\widehat{\mu_3}) \end{bmatrix}$$

Multivariate Case

The weights to be applied to the estimators are represented as a vector of numbers. $\overrightarrow{\boldsymbol{W}} = \langle w_1, w_2, \cdots, w_n \rangle^T$

The "best" value for the weights, constrained so that they sum to unity, is found by matrix operations.

$$\overrightarrow{\boldsymbol{W}} = \frac{\boldsymbol{\Sigma}^{-1} \cdot \mathbf{1}_{\mathrm{n}}}{\mathbf{1}_n^T \cdot \boldsymbol{\Sigma}^{-1} \cdot \mathbf{1}_{\mathrm{n}}}$$

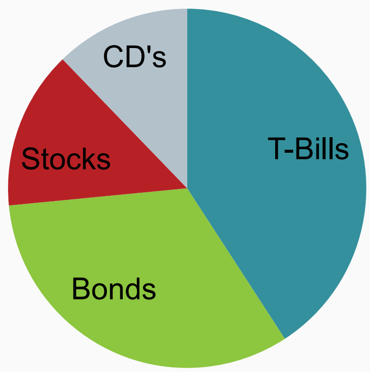
This is calculated by taking the inverse of the covariance matrix and then dividing each column total by the overall total.

Multivariate Case

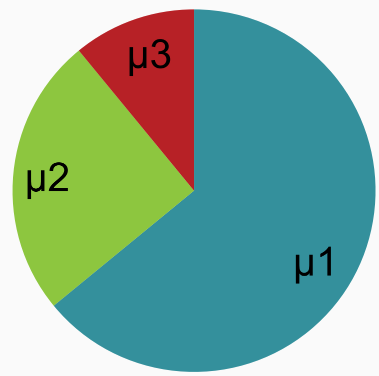
Interesting Tangent:

The math is equivalent to minimum variance portfolio optimization.

Portfolio Asset Allocation



Credibility Weights



The XOL Reinsurance Problem

The experience rate is an estimator of the future loss. $\hat{\mu}_{exper}$

With variance based on:

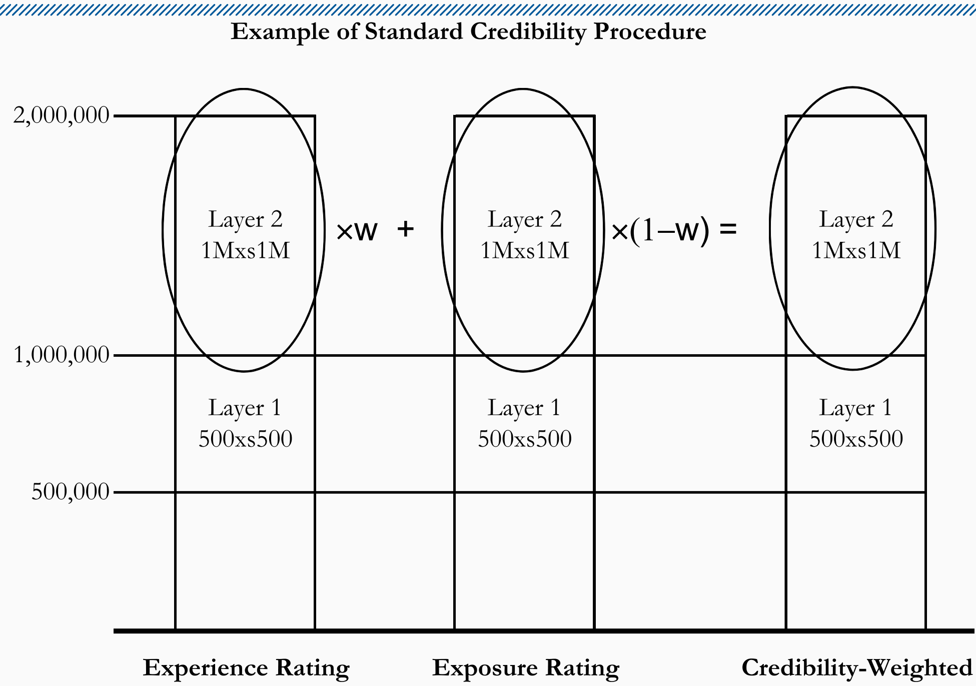
- Number of years and losses in the historical period
- Attachment Point and Limit of layer being priced
- Changing operations of the client company

The exposure rate is an estimator of the future loss. $\hat{\mu}_{expos}$

With variance based on:

- Volume of loss experience in the industry
- Relevance of industry experience to a specific client

Traditional Credibility Weighted Average



The XOL Reinsurance Problem

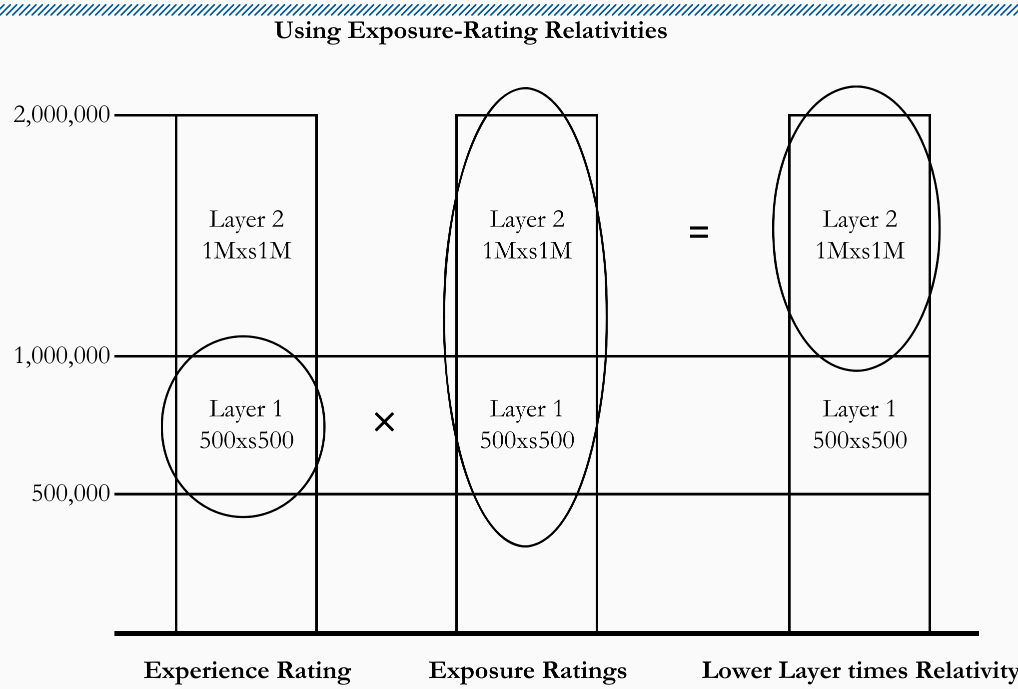
Our goal is to produce an unbiased, minimum variance estimator of the expected loss in the prospective period.

The traditional credibility weighting can bring us part of the way, but it does not make use of all the available information. Namely, the experience in lower layers is ignored.

An additional estimate can be produced using exposure-rating relativities applied to a lower layer (e.g., 500,000 xs 500,000).

$$\hat{\mu}_{rel} = \hat{\mu}_{exper\ \_500x500} \cdot \left\{ \frac{\hat{\mu}_{expos\ \_1Mx1M}}{\hat{\mu}_{expos\ \_500x500}} \right\}$$

Estimating Higher Layer based on Exposure-Rating Relativities Applied to Lower Layer



Calculating Variances - For Numerical Example

For a numerical example in the paper, we estimate variances for the three methods:

- **Experience Rate** – based on loss volume in historical period (ignores uncertainty for changing exposures, etc)
- **Exposure Rate** – based on uncertainty in Pareto distribution used for size-of-loss and on uncertainty in overall frequency
- **Relativity Method** – based on Pareto distribution in size-of-loss curve and on variance of experience rate for lower layer

Note: The covariances between the methods are set by the structure of the model and do not have to be separately estimated.

Calculating Variances - Covariance Matrix

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	Exposure	Experience	Relativity
Covariance Matrix:	1.573E+11	0	3.790E+10
	0	1.716E+11	7.322E+10
	3.790E+10	7.322E+10	8.788E+10
Inverse:	7.580E-12	2.165E-12	-5.073E-12
	2.165E-12	9.663E-12	-8.986E-12
	-5.073E-12	-8.986E-12	2.105E-11
Row Total:	4.672E-12	2.843E-12	6.996E-12
Weights:	32.2%	19.6%	48.2%

Total Variance: 6.891E+10

Recursive Credibility Form

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The result is a three-factor credibility formula.

$$\hat{\mu}_{cw} = w_1 \cdot \hat{\mu}_{expos\ \_1Mx1M} + w_2 \cdot \hat{\mu}_{exper\ \_1Mx1M} + w_3 \cdot \hat{\mu}_{rel}$$

We can rearrange this expression into a recursive form:

$$\hat{\mu}_{cw\,500x500} = \left(\frac{w_1}{w_1 + w_3}\right) \cdot \hat{\mu}_{expo\ s_{500x500}} + \left(\frac{w_3}{w_1 + w_3}\right) \cdot \hat{\mu}_{expe\ r_{500x500}}$$

$$\hat{\mu}_{cw\,1Mx1M} = (w_1 + w_3) \cdot \hat{\mu}_{cw\,500x500} \cdot \left\{\frac{\mu_{expo\ s_{1Mx1M}}}{\mu_{expo\ s_{500x500}}}\right\} + w_2 \cdot \hat{\mu}_{exper\ 1Mx1M}$$

Recursive Credibility Form
Numerical Example

Alternative Recursive Form

	Experience Rating		Exposure Rating			Credibility-Weighted	
	Loss Cost	Cred%	Loss Cost	Relativity	Cred%	Loss Cost	Cred%
500 xs 500	5,000,000	60.0%	4,000,000	1.000	40.0%	**4,600,000**	100.0%
1M xs 1M	4,000,000		3,000,000	0.750			

	Experience Rating		Complement of Credibility			Credibility-Weighted	
	Loss Cost	Cred%	Loss Cost	Relativity	Cred%	Loss Cost	Cred%
500 xs 500	5,000,000		**4,600,000**	1.000			
1M xs 1M	4,000,000	19.6%	3,450,000	0.750	80.4%	3,557,800	100.0%

Numbers for illustration only

Recursive Credibility Form

The recursive form of this credibility formula is already commonly used; known as "decay method" or "layer comparison."

And the end of all our exploring

Will be to arrive where we started

And know the place for the first time

- T.S. Eliot

THANK YOU VERY MUCH FOR YOUR ATTENTION.

Dave Clark