Summary of Møller Paper on Pricing Transforms

The 2003 ASTIN Colloquium paper by Møller ("Stochastic orders in dynamic reinsurance markets," *ASTIN Colloquium Papers* http://www.astin2003.de/img/papers/moeller.pdf) summarizes much of the literature on probability-transform pricing for the compound Poisson process. The fundamental result he presents, based on Girsanov's Theorem – a basic element of arbitrage-pricing theory – describes a procedure for producing arbitrage-free transforms of frequency and severity distributions.

The starting point is selecting a function $\phi(y)$, where the loss size variable is Y, with the only restriction that $\phi(y) > -1$ for all positive losses y. The frequency parameter λ is transformed to $\lambda[1+E\phi(Y)]$. The severity density g(y) gets transformed to $g(y)[1+\phi(y)]/[1+ E\phi(Y)]$.

Møller introduces a ranking order for such transforms, based on specific pricing impacts. He provides several transforms, several of which are considered reasonable in terms of not being in the extremes of the ranking order. The transforms are calibrated by a parameter θ which is the loading in the primary rates, so that the primary loaded pure premium is $(1+ \theta)\lambda E(Y)$.

A transform Møller introduces, which he calls the minimum martingale measure, takes $\phi(y) = (y/EY)\theta/[1+CV^2]$, with $E\phi(Y) = \theta/[1+CV^2]$. The transformed frequency is $\lambda[1+\theta/(1+CV^2)]$, and the transformed severity density is $g(y)[1+CV^2+\theta y/EY]/[1+CV^2+\theta]$.

This transform can be simplified by setting $s = \theta/[1+CV^2+\theta]$. Then g(y) is transformed by application of the factor $1 - s + sy/EY$ and the transformed frequency becomes $\lambda/(1 - s)$. Fitting this transform to actual prices then comes down to finding s, which is a probably a small number, like 1%. Then the overall profit margin θ comes out as $(1+CV^2)s/(1 - s)$. Severity probabilities are reduced for losses below the mean, and increased for losses above it.

Another new transform Møller calls the minimum entropy martingale measure. It requires finding $\eta > 0$ so that $E[Y\exp(\eta Y)] = (1+ \theta)E(Y)$. This is not possible for some severity distributions, but if the severity has policy limits or is light tailed, like a mixed exponential, the expectation will exist. For any selected positive η, the transformed frequency is $\lambda E[\exp(\eta Y)]$ with severity $g(y)\exp(\eta y)/E[\exp(\eta Y)]$ and overall load $\theta = E[Y\exp(\eta Y)]/ E(Y) - 1$.

For a typically heavy-tailed severity with a policy cap, the minimum martingale measure is most intuitive, as the minimum entropy with its exponential expected value is highly sensitive to the cap. As an example of the minimum martingale measure, consider a book of business with 2500 expected claims, a Pareto severity $G(y) = 1 - (1+y/10{,}000)^{-1.2}$, a policy limit of 10,000,000, and a loading of θ = 20%. The severity mean and CV^2 are 37,443 and 43.11. This makes the λ load factor $(1+0.2/54.11) = 1.00453$. The factor on g(y) is $(54.11+y/187{,}215)/54.31$. This can be applied numerically to a discretization of the severity distribution. The maximum severity has to stay at 10,000,000, but its original probability mass is 0.025% which gets transformed to 0.055%. The severity mean is increased by 19.46%, which together with the frequency transform gets the overall 20% load.

The transformed probabilities can be used to price any type of contract on this business. For instance, a 4,000,000 xs. 1,000,000 contract ends up with a risk load of 62.3%, and a 5,000,000 xs. 5,000,000 gets 112.8%. The total amount of those loads is 13,730,500 – which is the risk load for the layer 9,000,000 xs. 1,000,000 calculated separately. This is 73.3% of the entire loading on the primary business – as most of the risk is attributed to the higher layers by this method.

It is also interesting to apply this example to a difficult test case for pricing methods, attributed to Thomas Mack, which is to price a buy-back of a franchise deductible. For example, for a deductible of 1000, this contract would pay the full loss if it is less than or equal to 1000, but nothing if it is greater. A number of pricing transforms were tried on such contracts in Venter's Discussion of Shaun Wang's "Implementation of Proportional Hazards Transforms in Ratemaking" (PCAS 1998). They all allow negative risk loads in some cases. For the book of business outlined above and a range of deductibles, minimum martingale pricing gives a (barely) positive risk load. In fact the severity-only risk loads are negative in all cases tested, but the frequency load, small as it is, is enough to compensate and make the total load positive. The combined frequency-severity increment $\lambda g(y)$ can be seen to transform by a factor of $1+\theta y/[(1+CV^2)EY]$, which is > 1 for any positive y. Thus any combination of losses will get a positive risk load. This will hold for the minimum entropy martingale as well.

The reason Mack's example is difficult is that transforms of severity have to produce a density that integrates to 1, so putting more probability on large losses must take it away from small losses. Thus contracts that cover only small losses tend to get negative loads. But as these examples show, that problem can be alleviated by making the percentage load on frequency greater than the largest reduction in severity probability.

Møller presents reasonable candidates for probability transforms for pricing insurance and reinsurance contracts. The price for a contract is the expected value of the contract outcomes under the transformed primary frequency and severity probabilities.

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