APPENDIX 2

IRR RULES ARE A SPECIAL CASE OF EQUATION 1

Assuming there is a single scenario, the economic value of an investment is

$$P[x] = -\frac{\pi}{s}\ln E\left[\exp\left(-s\frac{x}{\pi}\right)\right] \qquad (2.1)$$

where x is the present value of a cash flow that has a certain probability of occurring and the expectation operator E is the probability-weighted average. The present value of a cash flow is the amount of money required today to purchase a risk-free instrument with the same cash flow.

$$x_i = \sum_{t=0}^{\infty} v(t)x_i(t) \qquad (2.2)$$

Consider the following investment:

- A single scenario.
- An investment of I at time $t=0$.
- Cash flows at later times are expected to be inward. (More precisely, all $\mu(t)>0,\ t>0$, where $\mu(t)$ is the expected value of $x(t)$.)
- A constant risk-free rate of return, that is,

$$v(t) = v^t = \left(1 + R_f\right)^{-t}$$

- A risk premium proportional to the variance of the possible cash flows, assumed to be because the possible cash flows are normally distributed. Using **2.1**, the constant of proportionality is found to be $2\pi/s$. Then at time t

$$P[x|t] = \mu(t) - s\frac{\sigma^2(t)}{2\pi}$$

- The variance of the cash flows at any point in time is perceived to vary with the expected cash flow, and the ratio of the variance to the expected cash flow at any point in time is expected to increase over time (perhaps because it seems harder to forecast far into the future). That is,

$$\sigma^2(t) = c_1\mu(t)\left(1 - c_2^t\right) \qquad c_1 > 0 \quad 0 < c_2 < 1 \qquad (2.3)$$

The decision rule suggests the transaction should be accepted whenever the risk-adjusted present value of the uncertain cash flows is greater than the investment I. That is,

$$\sum_{t=0}^{\infty} v(t)P[x|t] > I \qquad (2.4)$$

Substituting the specific values for this example:

$$\sum_{t=0}^{\infty}\left(1 + R_f\right)^{-t}\left(\mu(t) - \left(s\frac{\sigma^2(t)}{2\pi}\right)\right) > I \qquad (2.5)$$

and

$$\sum_{t=0}^{\infty}\left(1 + R_f\right)^{-t}\left(\mu(t)\left(1 - s\frac{c_1}{2\pi}\left(1 - c_2^t\right)\right)\right) > I \qquad (2.6)$$

If $c_1 = 2\frac{\pi}{s}$, then

$$\sum_{t=0}^{\infty}\left(1 + R_f\right)^{-t}\mu(t)c_2^t > I \qquad (2.7)$$

Define a variable *IRR* such that

$$c_2 = \frac{1 + R_f}{1 + IRR} \qquad (2.8)$$

Then

$$\sum_{t=0}^{\infty}(1 + IRR)^{-t}\mu(t) > I \qquad (2.9)$$

That is, the decision-maker considering the hypothetical investment should adopt an *IRR* decision rule, and the minimum *IRR* is found from

$$\frac{\sigma^2(t)}{\mu(t)} = \left(\frac{2\pi}{s}\right)\left[1 - \left(\frac{1+R_f}{1+IRR}\right)^t\right] \qquad (2.10)$$

or

$$(1+IRR)^t > \frac{(1+R_f)^t}{\left(1 - s\dfrac{\sigma^2(t)}{2\pi\mu(t)}\right)} \qquad (2.11)$$

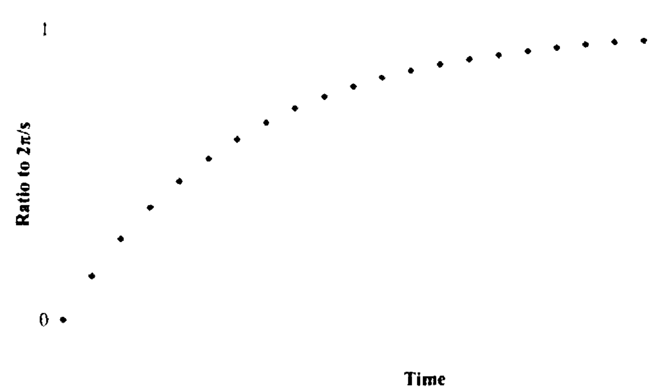
In this expression, the construction of the hypothetical investment is such that the value of *IRR* is the same for all points in time. The assumption is that the variance at each time t increases with the mean of the expected return at time t and the ratio of variance to mean increases with time asymptotically toward a maximum that is a function of the expected financial gain and the market's aversion toward risk. Recall that the cost of risk, π, is the difference between the present value of expected returns at the risk-free rate of interest and the present value at the rate *IRR* (which is the maximum one should pay for the investment).

Figure 2.1 shows the curve for the anticipated variance of future inward cash flows that is consistent with the IRR decision rule under the assumptions of this example. The value of *IRR* determines how quickly the curve rises towards its limit.

FIGURE 2.1

Assumption Leading to IRR Formula

Assumed Variance per Dollar of Expected Return



This set of assumptions is not as restrictive as it seems. Investments in bonds and capital equipment are characterized by an increase in risk over time, but not without limit. Also, in such investments, most of the risk in dollar terms is concentrated at the time the investment matures. Changes to the assumed level of risk at early periods will have only a small effect on the computed value of *IRR*. It is therefore not surprising that IRR rules work well for investments in bonds and many kinds of capital equipment.