

APPENDIX 1THE AXIOMATIC DEVELOPMENT

This appendix sets out the axiomatic development of Equation 1. It provides an overview only, as the references include much important material and the mathematical development is rather difficult to read.

Debreu (1959), in the classic Theory of Value: An Axiomatic Analysis of Economic Equilibrium, shows that an equilibrium structure exists for the economy. In this equilibrium, the prices and amounts of all goods and instruments of production are such that they maximize the value of the economy.

A general outline of the argument is shown in Figure 1.1 on the following page.

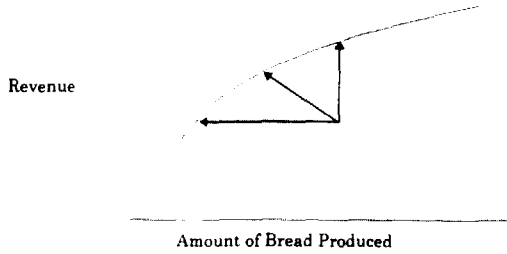
This conclusion rests on the following theorem, which Debreu proves in one of the most demanding proofs in economics. Debreu states the theorem in careful mathematical statements which I am merely paraphrasing here.

If a decision-maker can choose between any pair of alternatives offered, and if the choices of the decision-maker are consistent, then the decision-maker will make choices as if he or she had assigned numerical values to the alternatives and selected the alternative with the highest numerical value.

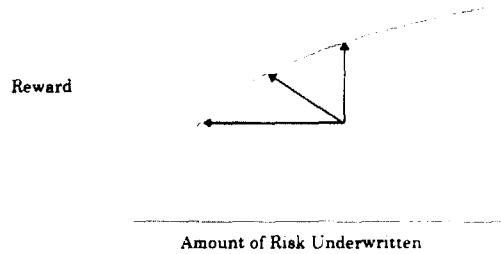
By consistent, Debreu means that if the decision-maker prefers A over B and prefers B over C, the decision-maker will prefer A over C. Also, if the decision-maker is indifferent between A and B and between B and C, he or she will be indifferent between A and C.

BORCH (1962) extended this to show that in a simple economy consisting only of the reinsurance of a particular policy, with the alternatives limited to various shares of the total to be reinsured, the economic players would share in the reinsurance risk in proportion to a unique parameter of each. Each player would take a share of the risk. Each player would get that same share of the reward, here called π . However measured, the ratio of π to risk undertaken would be the same for all the players.

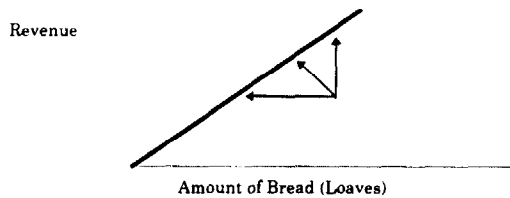
FIGURE 1.1. SUPPLY AND DEMAND



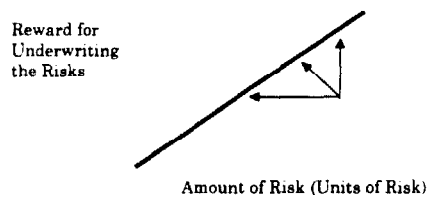
1A. PARETO-OPTIMAL FRONTIER. Each baker of bread will seek to bake the least possible number of loaves for a given reward, or seek to get the greatest total price for a given number of loaves. There is a limit, or frontier, to the amount that can be charged, however. The limit is defined by the *demand* for bread.



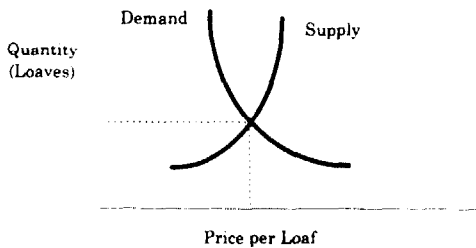
2A. PARETO-OPTIMAL FRONTIER FOR RISK. Each firm will seek to take the least possible risk for a given reward or to get the greatest possible reward for a given level of risk. Borch (1962) showed the equations for Pareto-optimal behavior which are consistent with the efficient market principle.



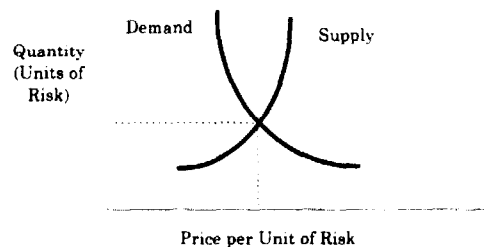
1B. CONSTANT PRICE PER UNIT. Each baker will receive a certain price per loaf of bread. No one baker can affect the market's price per loaf. No baker can charge more than the market price. The efficient market principle states that the price is as if each baker and each consumer uses all of the available information about the bread.



2B. CONSTANT PRICE PER UNIT OF RISK. Van Slyke (1995) extended Borch's work, showing that it implies that each investor or underwriter is paid a market price per unit of risk. We assume no one investor or underwriter can affect the capital market's price per unit of risk.



1C. EQUILIBRIUM OF SUPPLY AND DEMAND. For the economy as a whole, as the price of bread increases, the demand decreases. As the price increases, the supply increases. The demand schedule is related to the frontier in Fig. 1A. Prices move toward the equilibrium level even though every individual strives to be at its own point on the Pareto-optimum frontier.



2C. EQUILIBRIUM OF SUPPLY AND DEMAND FOR RISKY TRANSACTIONS. Capital plays many roles. For the economy as a whole, the equilibrium price for capital in its role in underwriting risk in on the supply schedule from the equations shown above. Regardless of the demand schedule, these equations hold in equilibrium.

Gerber (1979) shows the train of thought in a very readable way. At page 70, he lists five properties that a principle of premium calculation should have. These are:

1. Non-negative safety loading
2. No rip-off
3. Consistency
4. Additivity
5. Iterativity

He then supplements these (p. 73) with prohibitions against charging nothing for the cost of risk and against charging the full value of the loss for a chance of a loss (with the chance less than 100%). Gerber shows that only an exponential charge for risk such as that in Borch's result is consistent with these axioms.

Borch and Gerber considered the static case: risk exists at the time of the choice, and is resolved immediately upon the completion of the transaction. They ignored the time value of money. VAN SLYKE (1995) extended this work to include the time value of money. The result is a clear distinction between the effect of currency prices and the effect of risk aversion.

To see that such a distinction is reasonable, consider two risks with very different payment patterns, property catastrophe cover and an investment in bonds. The losses under property catastrophe cover are realized at roughly the time the premium is received. This is the timing considered by Borch and Gerber. They showed that the cost of capital is in proportion to the share of the risk that is reinsured. This means that if a benchmark risk is established, every part of every treaty would involve a certain number of units of risk, and every unit of risk would command the same cost.

Bonds, on the other hand, involve payments over many years. Most of the risk arises from the possibility of default on all or part of the principle. Consider a set of bonds with a range of different chances of default. As the chance of default decreases, the price of the bond approaches that of a bond issued by the Federal government. The yield on such bonds is called the risk-free rate of interest. That is, in the limit, as the risk of default approaches zero, the discount factor for the bond's cash flows approaches the risk-free rate of return. Therefore, the risk-free rate of return must appear in the equations of the value of investments and insurance risks, and appear at the limit.

There are three levels of summation in the equations above. Note that a series of payments that are *certain* to be paid or received over time are precisely as valuable in the capital markets as the sum of their present values. Therefore, amounts that are to be combined across time using risk-free present-value factors must be expressed as the value they would have if they were certain to occur, which is their economic value. Therefore, when adding amounts over time, if there is uncertainty in the outcomes at a certain time for a certain scenario, the economic value of the possible outcomes must be found by application of the exponential adjustments indicated by Borch. Only these economic values can be offset against one another over time; to do otherwise is to combine apples and oranges. This is the reason for the innermost summation. Finally, the present values,

or economic values, of all of the possible outcomes under the various scenarios entail a risk because of the uncertainty about which scenario will be realized. The cost of capital for that risk must be recognized by an exponential adjustment as indicated by Borch. This is the reason for the outermost summation.

The Efficient Market Principle states that in the aggregate, prices reflect all of the information available to all of the players. The exponential charge for risk in the equations above preserves all of the information in the probability distribution of outcomes. The use of prices for risk-free Treasury securities preserves all of the information about the time value of money. The Efficient Market Principle implies that the price of capital will be the same in all capital markets. That is, there will be a unique value of s that reflects the capital markets as a whole. Just as the price of bread varies from one transaction to the next depending on differences in shipping, spoilage, and so on, the price of capital will vary from transaction to transaction. But in the aggregate, just as the price of bread is set by the markets for bread, other commodities, and factors of production, so is the price of capital set by the capital markets.

And this is why the cost of risk per unit of risk is the same for industrial bonds and the \$2 billion excess of \$4 billion layer of the California Earthquake Authority, as shown in Figure 2 above.