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AGGREGATION OF CORRELATED RISK PORTFOLIOS: MODELS AND ALGORITHMS

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Abstract

In response to a request for proposal from the Committee on the Theory of Risk, Shaun Wang has written a paper that significantly advances, to quote the proposal, "the development of tools and models that improve the accuracy of the estimation of aggregate loss distributions for blocks of insurance risks."

Dr. Wang's charge was to "assume a book of business is the union of disjoint classes of business each of which has an aggregate distribution. …The classes of business are NOT independent. …The problem is how do you calculate the aggregate distribution for the whole book." Dr. Wang's paper covers a variety of dependency models and computational methods.

This discussion of his paper delves more deeply into a particular dependency model—correlation caused by parameter uncertainty—and then shows how his work applies to calculating the aggregate loss distribution for this case with one particular computational method—Fourier Inversion.

1. BACKGROUND

The collective risk model has long been one of the primary tools of actuarial science. One can view that model as a computer simulation where one first picks a random number of claims and then sums the random loss amounts for each claim. Simulating the distribution of losses for the collective risk model can (even today) be time consuming so, over the years, a number of mathematical methods have been developed to shorten the computing time. Klugman, Panjer, and Willmot [6, Ch. 4], provide an excellent description of the current computational methods.

The early uses of the collective risk model were mostly theoretical illustrations of the role of insurer surplus and profit margins. Such illustrations are still common today in insurance educational readings such as Bowers, Gerber, Jones, Hickman and Nesbitt [3, Ch. 13].

By the late 1970s, members of the Casualty Actuarial Society were beginning to use the collective risk model as input for real world insurance decisions. The early applications of the collective risk model included retrospective rating, e.g., Meyers [7], and aggregate stop loss reinsurance, which is described by Patrik [10]. Bear and Nemlick [2] provide further examples of the use of the collective risk model in the pricing of reinsurance contracts.

Some of these early efforts recognized the fact that the parameters of the collective risk model were unknown. Patrik and John [5] introduced parameter uncertainty by treating the parameters of the claim severity and claim count distributions as random variables. Heckman and Meyers [4] followed with an efficient computational algorithm that allows for some particular forms of parameter uncertainty in the collective risk model.

It is easy and instructive to consider the effect of parameter uncertainty on the variance of a distribution. Let X be a random variable that depends on a parameter θ. Then:

$$\mathrm{Var}[X] = \underbrace{\mathrm{E}_\theta[\mathrm{Var}[X \mid \theta]]}_{\text{Process Variance}} + \underbrace{\mathrm{Var}_\theta[\mathrm{E}[X \mid \theta]]}_{\text{Parameter Variance}}. \tag{1.1}$$

If there is no parameter uncertainty, the parameter variance will be zero. Introducing parameter uncertainty will increase the unconditional variance.

Suppose $X_1,\ldots,X_n$ are identically distributed random variables that depend on a parameter θ. Let $\mathrm{E}[X \mid \theta]$ and $\mathrm{Var}[X \mid \theta]$ be their common mean and variance given θ. Assume further that the X_i's are conditionally independent given θ. Then:

$$\mathrm{E}\left[\sum_{i=1}^{n} X_i \mid \theta\right] = n \cdot \mathrm{E}[X \mid \theta] \qquad \text{and}$$

$$\mathrm{Var}\left[\sum_{i=1}^{n} X_i \mid \theta\right] = n \cdot \mathrm{Var}[X \mid \theta].$$

Unconditionally:

$$\begin{aligned} \mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] &= \mathrm{E}_\theta\left[\mathrm{Var}\left[\sum_{i=1}^{n} X_i \mid \theta\right]\right] + \mathrm{Var}_\theta\left[\mathrm{E}\left[\sum_{i=1}^{n} X_i \mid \theta\right]\right] \\ &= \underbrace{n \cdot \mathrm{E}_\theta[\mathrm{Var}[X \mid \theta]]}_{\text{Process Variance}} + \underbrace{n^2 \cdot \mathrm{Var}_\theta[\mathrm{E}[X \mid \theta]]}_{\text{Parameter Variance}}. \end{aligned} \tag{1.2}$$

In most insurance situations, $\mathrm{E}_\theta[\mathrm{Var}[X \mid \theta]] \gg \mathrm{Var}_\theta[\mathrm{E}[X \mid \theta]]$, and we should expect the process variance to be dominant for small n. But as n increases, the parameter variance becomes increasingly important. This becomes apparent by looking at the coefficient of variation:

$$\begin{aligned} \mathrm{CV}\left[\sum_{i=1}^{n} X_i\right] &= \frac{\sqrt{n \cdot \mathrm{E}_\theta[\mathrm{Var}[X \mid \theta]] + n^2 \cdot \mathrm{Var}_\theta[\mathrm{E}[X \mid \theta]]}}{n \cdot \mathrm{E}[X]} \\ &\xrightarrow[n\to\infty]{} \frac{\sqrt{\mathrm{Var}_\theta[\mathrm{E}[X \mid \theta]]}}{\mathrm{E}[X]} > 0. \end{aligned} \tag{1.3}$$

More generally, we expect parameter uncertainty to play a minor role for small insureds and to play a major role for large insureds or for a reasonably sized insurance company.

In situations where parameter uncertainty affects several lines of insurance simultaneously, we expect high losses in one line to be associated with high losses in another line. Thus parameter uncertainty generates correlation. There are, of course, other generators of correlation. One example is in property insurance, where natural disasters cause damage to properties in close proximity.

Meyers and Schenker [9] provided some statistical methods of quantifying parameter uncertainty using observations spanning a period of years. However, any statistical method for quantifying parameter uncertainty requires considerable judgment because:

1. Data is scarce. You get one observation per insured per year.
2. The source of the historical variability in the parameters is often identifiable (at least after the fact). The user might not expect that source of variability to be present in future years. However, other sources of variability may arise.

2. DYNAMIC FINANCIAL ANALYSIS

The Casualty Actuarial Society coined the term "Dynamic Financial Analysis" (DFA) in the wake of the efforts to create a risk-based capital formula for insurers. To do DFA, one must often create an aggregate loss distribution for an entire insurance company. Now, for an insurance company, the primary source of parameter uncertainty is change over time. Thus parameter uncertainty will be a very important component in any collective risk model when it is applied to an entire insurance company.

As mentioned above, quantifying parameter uncertainty involves a fair amount of judgment. For example:

- Uncertain inflation will affect all claims simultaneously.

- Changes in the general economy can affect various lines of insurance in special ways. For example, directors and officers liability claims are more likely in a recession.
- Insurance companies write liability insurance at several different policy limits. We expect uncertainty in the claim frequency to affect policy limits in the same way.

The ultimate goal of DFA is to make financial decisions based on controlling the risk of an entire insurance company. DFA necessarily involves the more general concept of covariance, which can be driven by mechanisms other than parameter uncertainty. Practitioners familiar with the collective risk model should make the effort to express their knowledge in financial language. On the other hand, as we shall show, the collective risk model—with parameter uncertainty—can enrich the financial models.

3. PARAMETER UNCERTAINTY AND CORRELATION

For the h^{th} line of insurance let:

μ_h = Expected claim severity;

σ_h^2 = Variance of the claim severity distribution;

λ_h = Expected claim count; and

$\lambda_h + c_h \cdot \lambda_h^2$ = Variance of the claim count distribution.

Following Heckman and Meyers [4], we call c_h the contagion parameter. If the claim count distribution is:

Poisson, then $c_h = 0$;

negative binomial, then $c_h > 0$; and

binomial with n trials, then $c_h = -1/n$.

A good way to view the collective risk model is by a Monte Carlo simulation.

Simulation Algorithm #1
The Collective Risk Model Without Parameter Uncertainty

1. For lines of insurance 1 to n, select a random number of claims, K_h, for each line of insurance h.
2. For each line of insurance h, select random claim amounts Z_{hk}, for $k = 1,\ldots,K_h$. Each Z_{hk} has a common distribution $\{Z_h\}$.
3. Set $X_h = \sum_{k=1}^{K_h} Z_{hk}$.
4. Set $X = \sum_{h=1}^{n} X_h$.

The collective risk model describes the distribution of X. In this section we restrict ourselves to calculating the covariance structure of X. In the next section we will show how to calculate the entire distribution of X.

If we assume that K_h is independent of K_g for $g \neq h$, and that Z_h is independent of K_h, we have:

$$\begin{aligned}\mathrm{Var}[X_h] &= \mathrm{E}_{K_h}[\mathrm{Var}[X_h \mid K_h]] + \mathrm{Var}_{K_h}[\mathrm{E}[X_h \mid K_h]]\\ &= \lambda_h \cdot \sigma_h^2 + \mu_h^2 \cdot (\lambda_h + c_h \cdot \lambda_h^2). \qquad (3.1)\end{aligned}$$

Also

$$\mathrm{Cov}[X_g, X_h] = 0 \qquad \text{for} \quad g \neq h. \qquad (3.2)$$

We now introduce parameter uncertainty that affects the claim count distribution for several lines of insurance simultaneously. We partition the lines of insurance into covariance groups $\{G_i\}$. Our next version of the collective risk model is defined as follows.

Simulation Algorithm #2
The Collective Risk Model with Parameter Uncertainty in the Claim Count Distributions

1. For each covariance group i, select $\alpha_i > 0$ from a distribution with:

$$\mathrm{E}[\alpha_i] = 1 \qquad \text{and} \qquad \mathrm{Var}[\alpha_i] = g_i.$$

   g_i is called the covariance generator for the covariance group i.

2. For line of insurance h in covariance group i, select a random number of claims K_{hi} from a distribution with mean $\alpha_i \cdot \lambda_{hi}$.

3. For each line of insurance h in covariance group i, select random claim amounts Z_{hik} for $k = 1,\ldots,K_h$. Each Z_{hik} has a common distribution $\{Z_{hi}\}$.

4. Set $X_{hi} = \sum_{k=1}^{K_{hi}} Z_{hik}$.

5. Set $X_{\bullet i} = \sum_{h \in G_i} X_{hi}$.

6. Set $X = \sum_{i=1}^{n} X_{\bullet i}$.

We have:

$$\begin{aligned}\mathrm{Cov}[X_{di}, X_{hi}] = {} & \mathrm{E}_{\alpha_i}[\mathrm{Cov}[X_{di}, X_{hi} \mid \alpha_i]] \\ & + \mathrm{Cov}_{\alpha_i}[\mathrm{E}[X_{di} \mid \alpha_i], \mathrm{E}[X_{hi} \mid \alpha_i]].\end{aligned}$$

For $d \neq h$, X_{di} and X_{hi} are conditionally independent. Thus $\mathrm{Cov}[X_{di}, X_{hi} \mid \alpha_i] = 0$ and

$$\begin{aligned}\mathrm{Cov}[X_{di}, X_{hi}] &= \mathrm{Cov}_{\alpha_i}[\alpha_i \cdot \lambda_{di} \cdot \mu_{di}, \alpha_i \cdot \lambda_{hi} \cdot \mu_{hi}] \\ &= g_i \cdot \lambda_{di} \cdot \mu_{di} \cdot \lambda_{hi} \cdot \mu_{hi}. \qquad (3.3)\end{aligned}$$

Also,

$$\begin{aligned}\mathrm{Cov}[X_{hi},X_{hi}] &= \mathrm{Var}[X_{hi}]\\ &= \mathrm{E}_{\alpha_i}[\mathrm{Var}[X_{hi} \mid \alpha_i]] + \mathrm{Var}_{\alpha_i}[\mathrm{E}[X_{hi} \mid \alpha_i]]\\ &= \mathrm{E}_{\alpha_i}[\alpha_i \cdot \lambda_{hi} \cdot \sigma_{hi}^2 + \mu_{hi}^2 \cdot (\alpha_i \cdot \lambda_{hi} + \alpha_i^2 \cdot c_{hi} \cdot \lambda_{hi}^2)]\\ &\quad + \mathrm{Var}_{\alpha_i}[\alpha_i \cdot \lambda_{hi} \cdot \mu_{hi}]\\ &= \lambda_{hi} \cdot \sigma_{hi}^2 + \mu_{hi}^2 \cdot (\lambda_{hi} + (1+g_i) \cdot c_{hi} \cdot \lambda_{hi}^2) + g_i \cdot \lambda_{hi}^2 \cdot \mu_{hi}^2.\end{aligned} \tag{3.4}$$

And:

$$\mathrm{Cov}[X_{di},X_{hj}] = 0 \qquad \text{for} \quad i \neq j. \tag{3.5}$$

We now introduce parameter uncertainty in the severity distributions. Let β be a positive random variable with $\mathrm{E}[1/\beta] = 1$ and $\mathrm{Var}[1/\beta] = b$. Following Heckman and Meyers [4], we call b the mixing parameter. Let $X_{hi}^{\beta} = X_{hi}/\beta$ *for all* h *and* i. Then:

$$\begin{aligned}\mathrm{Cov}[X_{di}^{\beta},X_{hj}^{\beta}] &= \mathrm{E}_{\beta}[\mathrm{Cov}[X_{di}/\beta,X_{hj}/\beta]]\\ &\quad + \mathrm{Cov}_{\beta}[\mathrm{E}[X_{di}/\beta],\mathrm{E}[X_{hj}/\beta]]\\ &= \mathrm{Cov}[X_{di},X_{hj}] \cdot (1+b) + b \cdot \mathrm{E}[X_{di}] \cdot \mathrm{E}[X_{hj}].\end{aligned} \tag{3.6}$$

From Equations 3.3 to 3.6, we see that the first term of Equation 3.6 will be zero whenever $i \neq j$, and the second term will be positive whenever $b > 0$.

To calculate the coefficient of correlation, ρ_{XY}, between two separate lines of insurance with random losses X and Y, we use Equations 3.3 to 3.6 and the relationship:

$$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X] \cdot \mathrm{Var}[Y]}}. \tag{3.7}$$

We illustrate the effect of parameter uncertainty on correlation with an example. We use the illustrative claim severity distribu-

TABLE 3.1

CLAIM COUNT DISTRIBUTION PARAMETERS

Covariance Group	Covariance Generator	Line of Insurance	λ	c
#1	.01	GL-\$1M	Varies	0.00
		GL-\$5M	Varies	0.00
#2	.02	AL-\$1M	Varies	0.01
		AL-\$5M	Varies	0.01

tions for general liability and automobile liability given in Appendix A. Table 3.1 gives the covariance group and claim count distribution parameters. The examples use $b = 0.01$.

Table 3.2 gives the correlation matrices for the claim count distributions[1] and the aggregate loss distributions for each line of insurance with $\lambda = 10$, 100, and 100,000. Note that as λ increases the coefficients of correlation approach a limiting value. We can calculate that limiting value by dropping the terms with λ_{hi} (small compared with terms with λ_{hi}^2) in Equation 3.4. If $c = 0$, the limiting coefficients of correlation are 1.0.[2]

If we modify the claim severity distribution by a deductible, with p being the probability of exceeding the deductible, we must then change the λ parameter of a negative binomial claim count distribution by replacing λ with $p \cdot \lambda$. The contagion parameter c remains unchanged.[3] We can then apply Equations 3.3 to 3.7 to the modified claim count and claim severity distributions. Table 3.2 gives the resulting correlation matrices.

These examples show the practical utility of having correlation coefficients that are generated by a model. One should not

[1] We calculated claim count covariances from Equations 3.3 to 3.6 using $\mu_{hi} = 1$ and $\sigma_{hi} = 0$.

[2] Holding c as a constant while varying λ uses the interpretation of c as quantifying parameter uncertainty within a single line of insurance. See Heckman and Meyers [4] for details.

[3] This is proven on pp. 266–7 of Klugman et al. [6]. Note that, in our parameterization, $\lambda = r \cdot \beta$ and $c = 1/r$.

TABLE 3.2

ILLUSTRATED CORRELATION MATRICES

	Claim Count Correlations				Total Loss Correlations			
Expected Claim Count = 10								
	GL-$1M	GL-$5M	AL-$1M	AL-$5M	GL-$1M	GL-$5M	AL-$1M	AL-$5M
GL-$1M	1.00000	0.09091	0.00000	0.00000	1.00000	0.01361	0.00412	0.00354
GL-$5M	0.09091	1.00000	0.00000	0.00000	0.01361	1.00000	0.00355	0.00305
AL-$1M	0.00000	0.00000	1.00000	0.15361	0.00412	0.00355	1.00000	0.00560
AL-$5M	0.00000	0.00000	0.15361	1.00000	0.00354	0.00305	0.00560	1.00000
Expected Claim Count = 1,000								
	GL-$1M	GL-$5M	AL-$1M	AL-$5M	GL-$1M	GL-$5M	AL-$1M	AL-$5M
GL-$1M	1.00000	0.90909	0.00000	0.00000	1.00000	0.57819	0.18826	0.17271
GL-$5M	0.90909	1.00000	0.00000	0.00000	0.57819	1.00000	0.17671	0.16212
AL-$1M	0.00000	0.00000	1.00000	0.64103	0.18826	0.17671	1.00000	0.32042
AL-$5M	0.00000	0.00000	0.64103	1.00000	0.17271	0.16212	0.32042	1.00000
Expected Claim Count = 100,000								
	GL-$1M	GL-$5M	AL-$1M	AL-$5M	GL-$1M	GL-$5M	AL-$1M	AL-$5M
GL-$1M	1.00000	0.99900	0.00000	0.00000	1.00000	0.99272	0.34743	0.34674
GL-$5M	0.99900	1.00000	0.00000	0.00000	0.99272	1.00000	0.34705	0.34636
AL-$1M	0.00000	0.00000	1.00000	0.66203	0.34743	0.34705	1.00000	0.73582
AL-$5M	0.00000	0.00000	0.66203	1.00000	0.34674	0.34636	0.73582	1.00000
Limiting Correlations as the Expected Claim Count Approaches Infinity								
	GL-$1M	GL-$5M	AL-$1M	AL-$5M	GL-$1M	GL-$5M	AL-$1M	AL-$5M
GL-$1M	1.00000	1.00000	0.00000	0.00000	1.00000	1.00000	0.35048	0.35048
GL-$5M	1.00000	1.00000	0.00000	0.00000	1.00000	1.00000	0.35048	0.35048
AL-$1M	0.00000	0.00000	1.00000	0.66225	0.35048	0.35048	1.00000	0.74564
AL-$5M	0.00000	0.00000	0.66225	1.00000	0.35048	0.35048	0.74564	1.00000
Ground Up Expected Count = 1,000 with a $100,000 Deductible								
	GL-$1M	GL-$5M	AL-$1M	AL-$5M	GL-$1M	GL-$5M	AL-$1M	AL-$5M
GL-$1M	1.00000	0.43740	0.00000	0.00000	1.00000	0.38533	0.12445	0.11282
GL-$5M	0.43740	1.00000	0.00000	0.00000	0.38533	1.00000	0.11355	0.10294
AL-$1M	0.00000	0.00000	1.00000	0.21918	0.12445	0.11355	1.00000	0.20181
AL-$5M	0.00000	0.00000	0.21918	1.00000	0.11282	0.10294	0.20181	1.00000

use empirical correlation coefficients if they were applied to an insured with a different exposure, or if a deductible were imposed.

4. CALCULATING THE AGGREGATE LOSS DISTRIBUTION BY FOURIER INVERSION

In this section, we show how to use direct Fourier inversion to calculate the aggregate loss distribution described by Simulation Algorithm #2. We begin by summarizing the method of Heckman and Meyers [4] using the more compact notation of Klugman et al. [6, p. 316].[4]

Let Z be a random variable representing claim severity. Define the Fourier transform of Z as:

$$\phi_Z(t) \equiv \mathrm{E}[e^{itZ}].$$

A fundamental property of Fourier transforms is that:

$$\phi_{\underbrace{Z+\cdots+Z}_{\text{K Times}}}(t) = \phi_Z(t)^K,$$

where the Z's are independent.

Let K be a random variable representing claim count. Define the probability generating function (pgf) of a claim count distribution as:

$$P_K(t) \equiv \mathrm{E}[t^K].$$

Define the aggregate loss

$$X = \underbrace{Z + \cdots + Z}_{\text{K Times}}.$$

We then have:

$$\phi_X(t) = \mathrm{E}[(\phi_Z(t))^K] = P_K(\phi_Z(t)). \tag{4.1}$$

[4]Wang describes a similar process using the Fast Fourier Transform.

Let $X_1,\ldots,X_n$ be independent random variables of aggregate losses. Then:

$$\phi_{X_1+\cdots+X_n}(t) = \prod_{i=1}^{n} \phi_{X_i}(t). \tag{4.2}$$

Heckman and Meyers [4] provide a way to obtain the distribution of $X_1 + \cdots + X_n$ and the distribution[5] of $(X_1 + \cdots + X_n)/\beta$ given the Fourier transform $\phi_{X_1+\cdots+X_n}(t)$ and that β has a gamma distribution.

To summarize, Fourier inversion turns the time-consuming process of simulating the sum of random variables into the mathematically complex, but doable, process of multiplying the Fourier transforms of the random variables and then inverting this product. Until now, we have been assuming that the claim count distributions are independent and that the claim severity distribution is independent of the claim count.

To remove the assumption that the claim count distributions are independent, Wang uses the multivariate Fourier transform which is defined by:

$$\phi_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = \mathrm{E}[e^{i(t_1 X_1 + \cdots + t_n X_n)}]$$

and has the property that:

$$\phi_{X_1+\cdots+X_n}(t) = \phi_{X_1,\ldots,X_n}(t,\ldots,t). \tag{4.3}$$

When the lines of insurance are correlated, we can then apply the Heckman/Meyers Fourier inversion formula to Equation 4.3 to obtain the aggregate loss distribution.

We now use Equation 4.3 to calculate the Fourier transform for the aggregate loss distribution described by Simulation Algorithm #2—the collective risk model with parameter uncertainty

[5] See Equation 3.6 and the preceding paragraph.

in the claim count distributions.

$$\begin{aligned}
\phi_{X_{\bullet i}}(t) &= \phi_{X_{1i},\ldots,X_{n_i i}}(t,\ldots,t) \\
&\qquad \text{(from Equation 4.3)} \\
&= \mathrm{E}_{\alpha_i}[\phi_{X_{1i},\ldots,X_{n_i i}}(t,\ldots,t) \mid \alpha_i] \\
&= \mathrm{E}_{\alpha_i}\left[\prod_{h=1}^{n_i} \phi_{X_{hi}}(t) \mid \alpha_i\right] \\
&\qquad \text{(Equation 4.2 applies since the } X_{hi}\text{'s} \\
&\qquad \text{are } \textit{conditionally} \text{ independent.)} \\
&= \mathrm{E}_{\alpha_i}\left[\prod_{h=1}^{n_i} P_{K_{hi}}(\phi_{Z_{hi}}(t)) \mid \alpha_i\right]. \qquad (4.4) \\
&\qquad \text{(from Equation 4.1)}
\end{aligned}$$

Since the covariance groups are independent:

$$\phi_X(t) = \prod_{i=1}^{n} \phi_{X_{\bullet i}}(t). \qquad (4.5)$$

To complete the model description, we need to specify:

- the distribution of the a_i's;
- the pgf's $P_{K_{hi}}(t)$; and
- the Fourier transforms of the severity distributions $\phi_{Z_{hi}}(t)$.

We will use a three-point discrete distribution for a_i. Let:

$$\begin{aligned}
\alpha_{i1} &= 1 - \sqrt{3g_i} \qquad & \Pr\{\alpha_i = \alpha_{i1}\} &= 1/6 \\
\alpha_{i2} &= 1 \qquad & \Pr\{\alpha_i = \alpha_{i2}\} &= 2/3 \qquad (4.6) \\
\alpha_{i3} &= 1 + \sqrt{3g_i} \qquad & \Pr\{\alpha_i = \alpha_{i3}\} &= 1/6
\end{aligned}$$

This discrete distribution was motivated by an approximation to Equation 4.4 when a_i has a normal distribution. Equation 4.4

then becomes:

$$\mathrm{E}_{\alpha_i}\left[\prod_{h=1}^{n_i} P_{K_{hi}}(\phi_{Z_{hi}}(t)) \mid \alpha_i\right] = \frac{1}{\sqrt{2\pi g_i}}\int_{-\infty}^{\infty}\left[\prod_{h=1}^{n_i} P_{K_{hi}}(\phi_{Z_{hi}}(t)) \mid \alpha_i\right] \cdot e^{-(\alpha_i-1)^2/2g_i}\,d\alpha_i; \tag{4.7}$$

by using the Gauss–Hermite three-point quadrature formula:

$$\int_{-\infty}^{\infty} f(x)\cdot e^{-x^2}dx \approx \frac{\sqrt{\pi}}{6} f\left(-\sqrt{\frac{3}{2}}\right) + \frac{2\sqrt{\pi}}{3} f(0) + \frac{\sqrt{\pi}}{6} f\left(\sqrt{\frac{3}{2}}\right); \tag{4.8}$$

with the change of variables:

$$x = \frac{\alpha_i - 1}{\sqrt{2g_i}}.$$

One can use a higher-order formula, obtainable from many texts on numerical analysis. See, for example, Ralston [11].

Appendix B of Klugman et al. [6] provides the pgf's for a wide variety of claim count distributions. We provide two examples here, translated into this paper's notation.

For the negative binomial claim count distribution:

$$P_{K_{hi}}(t) \mid \alpha_i = (1 - c_{hi}\cdot\lambda_{hi}\cdot\alpha_i\cdot(t-1))^{-1/c_{hi}}.$$

For the Poisson claim count distribution:

$$P_{K_{hi}}(t) \mid \alpha_i = e^{-\lambda_{hi}\cdot\alpha_i\cdot(t-1)}.$$

The Fourier transform of a claim severity distribution with probability density function $f(z)$:

$$\phi_Z(t) = \int_0^{\infty} e^{itx} f(x)\,dx.$$

This integral does not have a closed form for most of the commonly used claim severity distributions. Heckman and Meyers

[4] get around that difficulty by approximating the cumulative distribution function (cdf), $F(z)$, with a piecewise linear cdf, for which the integral does have a closed form.

To summarize this section, we have shown how to calculate the multivariate Fourier transform of the collective risk model with correlations generated by parameter uncertainty. We then used the direct Fourier inversion formulas of Heckman and Meyers to calculate the corresponding aggregate loss distribution.

Note that one could use the Fast Fourier Transform methods discussed by Wang.

5. AN ILLUSTRATIVE EXAMPLE

We now illustrate the effect of covariance on the aggregate loss distribution of the hypothetical XYZ Insurance Company. XYZ writes commercial lines exclusively—workers compensation, general liability, commercial auto and commercial property. Table 5.1 provides summary statistics for XYZ's book of business.

Following are some additional remarks about XYZ's loss distribution.

- We set the mixing parameter $b = 0.01$.
- The claim severity distributions are piecewise linear approximations to mixed exponential distributions. See Appendix A for details. Also, the standard deviations for the claim severity distributions reflect the mixing generated by the mixing parameter, b.
- The claim count distributions are all negative binomial.
- The correlations between the claim count distributions of the coverages in a given line are driven by the covariance generator listed with the first coverage of the line.

TABLE 5.1

XYZ SUMMARY LOSS STATISTICS
LINE/COVERAGE SUMMARY STATISTICS
AGGREGATE SUMMARY STATISTICS

Aggregate Mean	1,004,422,886
Aggregate Standard Deviation	156,034,063
Mixing Parameter	0.010000

Line Name/ Liability Limit	E[Count]	Std[Count]	E[Severity]	Std[Severity]	E[Tot.Loss]	Covariance Generator
WC-$5M Limit	80,000.00	8,005.00	5,339.89	52,927.43	427,191,200	
GL-$5M Limit	200.00	42.61	40,348.87	160,218.51	8,069,774	0.020000
GL-$2M Limit	800.00	163.27	39,892.11	152,516.66	31,913,688	
GL-$1M Limit	2,200.00	444.68	36,966.16	124,853.59	81,325,552	
GL-$.5M Limit	1,250.00	253.72	31,085.63	87,532.67	38,857,038	
AL-$5M Limit	350.00	53.03	12,809.55	99,730.27	4,483,342	0.010000
AL-$2M Limit	1,350.00	194.89	12,626.84	94,724.36	17,046,234	
AL-$1M Limit	3,700.00	528.08	11,456.65	76,434.03	42,389,605	
AL-$.5M Limit	2,300.00	329.59	9,131.21	50,896.52	21,001,783	
APhD	1,100.00	159.44	4,360.00	6,331.53	4,796,000	
CP-$50M Limit	2,000.00	667.83	10,999.77	224,488.75	21,999,540	0.100000
CP-$10M Limit	8,000.00	2,666.83	6,999.95	45,887.29	55,999,600	
CP-$5M Limit	18,500.00	6,165.08	6,499.98	24,515.84	120,249,630	
CP-$2M Limit	10,000.00	3,333.17	6,199.99	13,467.32	61,999,900	
CP-$1M Limit	11,000.00	3,666.33	6,100.00	11,066.55	67,100,000	

FIGURE 5.1

XYZ AGGREGATE LOSS DISTRIBUTION

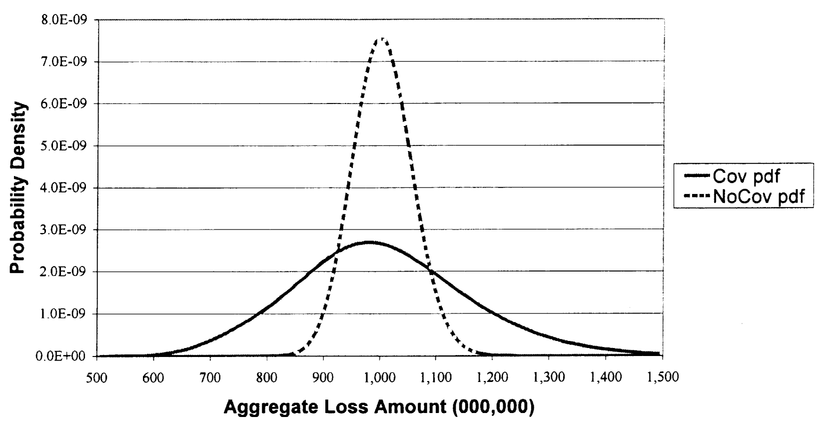

TABLE 5.2

COMPARISON OF AGGREGATE LOSS DISTRIBUTIONS† WITH AND WITHOUT THE COVARIANCE GENERATORS AND THE MIXING PARAMETER

	WO/Covariance	W/Covariance
Aggregate Mean	1,004,422,886	1,004,422,886
Aggregate Std. Dev.	52,698,873	156,034,063

Aggregate	Cumulative Probability		Limited Pure Premium Ratio	
Loss	WO/Covariance	W/Covariance	WO/Covariance	W/Covariance
500,000,000	0.00000	0.00000	0.49780	0.49780
600,000,000	0.00000	0.00070	0.59736	0.59734
700,000,000	0.00000	0.01617	0.69692	0.69634
800,000,000	0.00001	0.08782	0.79648	0.79136
900,000,000	0.01954	0.25528	0.89570	0.87477
1,000,000,000	0.47643	0.51146	0.97685	0.93653
1,100,000,000	0.96097	0.74683	0.99909	0.97282
1,200,000,000	0.99970	0.89181	1.00000	0.99004
1,300,000,000	1.00000	0.96115	1.00000	0.99688
1,400,000,000	1.00000	0.98831	1.00000	0.99916
1,500,000,000	1.00000	0.99703	1.00000	0.99981
1,600,000,000	1.00000	0.99935	1.00000	0.99996
1,700,000,000	1.00000	0.99987	1.00000	0.99999
1,800,000,000	1.00000	0.99998	1.00000	1.00000
1,900,000,000	1.00000	1.00000	1.00000	1.00000
2,000,000,000	1.00000	1.00000	1.00000	1.00000

†The cumulative probability is the probability that the aggregate loss amount is less than the stated loss amount. The limited pure premium is the expected aggregate loss when limited to the stated loss amount. The limited pure premium ratio is the limited pure premium divided by the expected aggregate loss.

Appendix B gives the correlation matrices generated by mixing the claim count and claim severity distributions.

Table 5.2 and Figure 5.1 illustrate the significant effect that correlations have on the aggregate loss distribution of XYZ Insurance Company.

6. CONCLUSION

We congratulate Dr. Wang for his fine work in introducing dependency into the collective risk model. This discussion has attempted to expand the applicability of his work and illustrate its importance in Dynamic Financial Analysis.

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APPENDIX A

THE CLAIM SEVERITY DISTRIBUTIONS

The Heckman/Meyers algorithm requires that the cumulative distribution functions for the claim severity distributions be piecewise linear. Users of the algorithm usually have an analytic model for claim severity, so some approximation is necessary. This appendix gives the analytic models used in this paper and their piecewise linear approximations. The claim severity distributions are merely illustrative and the reader should note that we did not derive the claim severity distributions from any proprietary data available to us.

This paper uses the mixed exponential claim severity model for all lines of insurance. The cumulative distribution function (cdf) is given by:

$$F(x) = 1 - \sum_{i=1}^{4} w_i \cdot e^{-x/b_i}. \tag{A.1}$$

The limited average severity (LAS) is given by:

$$L(x) = \sum_{i=1}^{4} w_i \cdot b_i \cdot (1 - e^{-x/b_i}). \tag{A.2}$$

A piecewise linear cdf approximates each mixed exponential cdf. For the specified values $x_0, x_2, \ldots, x_{2n}$, the piecewise linear cdf has the same value as its corresponding mixed exponential cdf, and the piecewise linear LAS has the same value as its corresponding mixed exponential LAS. We accomplish this matching of the LAS values by setting:

$$x_{2n-1} = \frac{L(x_{2n}) - L(x_{2n-2}) - x_{2n} \cdot (1 - F(x_{2n})) + x_{2n-2} \cdot (1 - F(x_{2n-2}))}{F(x_{2n}) - F(x_{2n-2})} \tag{A.3}$$

TABLE A.1

MIXED EXPONENTIAL PARAMETERS

Line Names	b_1	b_2	b_3	b_4	w_1	w_2	w_3	w_4
WC	1,000	10,000	100,000	500,000	0.940	0.040	0.015	0.005
GL	1,000	10,000	100,000	500,000	0.350	0.500	0.100	0.050
AL	1,000	2,500	10,000	500,000	0.360	0.500	0.120	0.020
APhD	1,000	5,000	10,000	15,000	0.360	0.500	0.120	0.020
CP-$50M Limit	2,000	5,000	20,000	5,000,000	0.360	0.500	0.139	0.001
CP-$10M Limit	2,000	5,000	20,000	1,000,000	0.360	0.500	0.139	0.001
CP-$5M Limit	2,000	5,000	20,000	500,000	0.360	0.500	0.139	0.001
CP-$2M Limit	2,000	5,000	20,000	200,000	0.360	0.500	0.139	0.001
CP-$1M Limit	2,000	5,000	20,000	100,000	0.360	0.500	0.139	0.001

and

$$F(x_{2n-1}) = F(x_{2n}) - (F(x_{2n}) - F(x_{2n-2}))\frac{x_{2n-1} - x_{2n-2}}{x_{2n} - x_{2n-2}}. \tag{A.4}$$

Table A.1 gives the parameters of the mixed exponential distributions used in this paper. Table A.2 gives the piecewise linear approximations for two of these distributions. The values $x_0, x_2, \ldots$ are the same for all of the piecewise linear distributions used in this paper.

TABLE A.2

PIECEWISE LINEAR APPROXIMATIONS TO MIXED EXPONENTIAL DISTRIBUTIONS

WC-$5M Limit	w's	Means	GL-$5M Limit	w's	Means
Exp #1	0.940	1,000	Exp #1	0.350	1,000
Exp #2	0.040	10,000	Exp #2	0.500	10,000
Exp #3	0.015	100,000	Exp #3	0.100	100,000
Exp #4	0.005	500,000	Exp #4	0.050	500,000
Loss Amount	cdf	LAS	Loss Amount	cdf	LAS
0.00	0.000000	0.00	0.00	0.000000	0.00
49.15	0.045700	48.02	49.21	0.019500	48.73
100.00	0.089867	95.43	100.00	0.038392	98.05
149.19	0.131200	139.18	149.37	0.056200	145.08
200.00	0.171217	182.31	200.00	0.073565	192.43
342.56	0.276533	292.95	343.62	0.120000	322.15
500.00	0.371892	399.35	500.00	0.162648	456.43
729.42	0.494340	529.40	733.42	0.219840	645.21
1,000.00	0.598159	652.18	1,000.00	0.269918	846.51
1,419.20	0.727210	793.58	1,443.94	0.339720	1,155.13
2,000.00	0.820353	924.97	2,000.00	0.395447	1,506.79
2,883.28	0.911960	1,043.19	3,256.69	0.485113	2,210.19
5,000.00	0.950186	1,189.09	5,000.00	0.549751	3,051.45
6,797.29	0.960808	1,269.07	7,275.66	0.618840	3,997.45
10,000.00	0.966769	1,385.05	10,000.00	0.676551	4,957.25
14,264.10	0.972925	1,513.63	14,236.37	0.749097	6,173.83
20,000.00	0.977502	1,655.80	20,000.00	0.802420	7,466.28
30,790.44	0.983013	1,868.83	30,030.69	0.861207	9,153.31
50,000.00	0.986108	2,165.42	50,000.00	0.890736	11,630.07
72,261.57	0.988482	2,448.25	71,743.39	0.908547	13,812.20
100,000.00	0.990386	2,741.34	100,000.00	0.922253	16,202.71
142,933.77	0.992801	3,102.25	143,357.97	0.939641	19,196.72
200,000.00	0.994618	3,461.20	200,000.00	0.952951	22,238.65
306,605.45	0.996837	3,916.67	311,738.74	0.970510	26,514.86
500,000.00	0.998060	4,410.19	500,000.00	0.980932	31,085.63
700,063.34	0.998817	4,722.62	702,893.51	0.988239	34,213.12
1,000,000.00	0.999323	5,001.59	1,000,000.00	0.993229	36,966.16
1,343,154.66	0.999707	5,168.02	1,343,292.63	0.997074	38,630.66
2,000,000.00	0.999908	5,294.21	2,000,000.00	0.999084	39,892.11
2,493,216.63	0.999985	5,320.55	2,492,457.58	0.999848	40,155.10
5,000,000.00	1.000000	5,339.89	5,000,000.00	0.999998	40,348.87

APPENDIX B

Correlation Matrix for Claim Counts

	WC-$5M Limit	GL-$5M Limit	GL-$2M Limit	GL-$1M Limit	GL-$0.5M Limit	AL-$5M Limit	AL-$2M Limit	AL-$1M Limit	AL-$0.5M Limit	APhD	CP-$50M Limit	CP-$10M Limit	CP-$5M Limit	CP-$2M Limit	CP-$1M Limit
WC-$5M Limit	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GL-$5M Limit	0.0000	1.0000	0.4599	0.4644	0.4624	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GL-$2M Limit	0.0000	0.4599	1.0000	0.4848	0.4828	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GL-$1M Limit	0.0000	0.4644	0.4848	1.0000	0.4875	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GL-$0.5M Limit	0.0000	0.4624	0.4828	0.4875	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
AL-$5M Limit	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.4572	0.4624	0.4606	0.4553	0.0000	0.0000	0.0000	0.0000	0.0000
AL-$2M Limit	0.0000	0.0000	0.0000	0.0000	0.0000	0.4572	1.0000	0.4853	0.4834	0.4779	0.0000	0.0000	0.0000	0.0000	0.0000
AL-$1M Limit	0.0000	0.0000	0.0000	0.0000	0.0000	0.4624	0.4853	1.0000	0.4889	0.4834	0.0000	0.0000	0.0000	0.0000	0.0000
AL-$0.5M Limit	0.0000	0.0000	0.0000	0.0000	0.0000	0.4606	0.4834	0.4889	1.0000	0.4815	0.0000	0.0000	0.0000	0.0000	0.0000
APhD	0.0000	0.0000	0.0000	0.0000	0.0000	0.4553	0.4779	0.4834	0.4815	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
CP-$50M Limit	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.8984	0.8987	0.8985	0.8985
CP-$10M Limit	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.8984	1.0000	0.9002	0.9000	0.9000
CP-$5M Limit	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.8987	0.9002	1.0000	0.9003	0.9003
CP-$2M Limit	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.8985	0.9000	0.9003	1.0000	0.9001
CP-$1M Limit	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.8985	0.9000	0.9003	0.9001	1.0000

APPENDIX B

CORRELATION MATRIX FOR AGGREGATE LOSSES

	WC-$5M Limit	GL-$5M Limit	GL-$2M Limit	GL-$1M Limit	GL-$0.5M Limit	AL-$5M Limit	AL-$2M Limit	AL-$1M Limit	AL-$0.5M Limit	APhD	CP-$50M Limit	CP-$10M Limit	CP-$5M Limit	CP-$2M Limit	CP-$1M Limit
WC-$5M Limit	1.0000	0.1859	0.2577	0.2879	0.2841	0.1500	0.2530	0.3311	0.3248	0.3758	0.1186	0.1915	0.1952	0.1954	0.1955
GL-$5M Limit	0.1859	1.0000	0.3090	0.3452	0.3406	0.0596	0.1004	0.1314	0.1290	0.1492	0.0471	0.0760	0.0775	0.0776	0.0776
GL-$2M Limit	0.2577	0.3090	1.0000	0.4784	0.4721	0.0825	0.1392	0.1822	0.1787	0.2068	0.0653	0.1054	0.1074	0.1075	0.1076
GL-$1M Limit	0.2879	0.3452	0.4784	1.0000	0.5275	0.0922	0.1555	0.2035	0.1997	0.2310	0.0729	0.1177	0.1200	0.1201	0.1202
GL-$0.5M Limit	0.2841	0.3406	0.4721	0.5275	1.0000	0.0910	0.1535	0.2009	0.1971	0.2280	0.0719	0.1162	0.1184	0.1185	0.1186
AL-$5M Limit	0.1500	0.0596	0.0825	0.0922	0.0910	1.0000	0.1629	0.2132	0.2091	0.2420	0.0380	0.0614	0.0625	0.0626	0.0626
AL-$2M Limit	0.2530	0.1004	0.1392	0.1555	0.1535	0.1629	1.0000	0.3595	0.3528	0.4081	0.0641	0.1035	0.1054	0.1056	0.1056
AL-$1M Limit	0.3311	0.1314	0.1822	0.2035	0.2009	0.2132	0.3595	1.0000	0.4616	0.5341	0.0838	0.1354	0.1380	0.1381	0.1382
AL-$0.5M Limit	0.3248	0.1290	0.1787	0.1997	0.1971	0.2091	0.3528	0.4616	1.0000	0.5240	0.0823	0.1329	0.1354	0.1355	0.1356
APhD	0.3758	0.1492	0.2068	0.2310	0.2280	0.2420	0.4081	0.5341	0.5240	1.0000	0.0952	0.1537	0.1566	0.1568	0.1569
CP-$50M Limit	0.1186	0.0471	0.0653	0.0729	0.0719	0.0380	0.0641	0.0838	0.0823	0.0952	1.0000	0.5384	0.5486	0.5492	0.5496
CP-$10M Limit	0.1915	0.0760	0.1054	0.1177	0.1162	0.0614	0.1035	0.1354	0.1329	0.1537	0.5384	1.0000	0.8860	0.8869	0.8876
CP-$5M Limit	0.1952	0.0775	0.1074	0.1200	0.1184	0.0625	0.1054	0.1380	0.1354	0.1566	0.5486	0.8860	1.0000	0.9038	0.9045
CP-$2M Limit	0.1954	0.0776	0.1075	0.1201	0.1185	0.0626	0.1056	0.1381	0.1355	0.1568	0.5492	0.8869	0.9038	1.0000	0.9054
CP-$1M Limit	0.1955	0.0776	0.1076	0.1202	0.1186	0.0626	0.1056	0.1382	0.1356	0.1569	0.5496	0.8876	0.9045	0.9054	1.0000

APPENDIX B

Covariance Matrix for Aggregate Losses

	WC-$5M Limit	GL-$5M Limit	GL-$2M Limit	GL-$1M Limit	GL-$0.5M Limit	AL-$5M Limit	AL-$2M Limit	AL-$1M Limit	AL-$0.5M Limit	APhD	CP-$50M Limit	CP-$10M Limit	CP-$5M Limit	CP-$2M Limit	CP-$1M Limit
WC-$5M Limit	3.897E+15	3.447E+13	1.363E+14	3.474E+14	1.660E+14	1.915E+13	7.282E+13	1.811E+14	8.972E+13	2.049E+13	9.398E+13	2.392E+14	5.137E+14	2.649E+14	2.866E+14
GL-$5M Limit	3.447E+13	8.823E+12	7.778E+12	1.982E+13	9.470E+12	3.618E+11	1.376E+12	3.421E+12	1.695E+12	3.870E+11	1.775E+12	4.519E+12	9.704E+12	5.003E+12	5.415E+12
GL-$2M Limit	1.363E+14	7.778E+12	7.182E+13	7.838E+13	3.745E+13	1.431E+12	5.440E+12	1.353E+13	6.702E+12	1.531E+12	7.021E+12	1.787E+13	3.838E+13	1.979E+13	2.141E+13
GL-$1M Limit	3.474E+14	1.982E+13	7.838E+13	3.737E+14	9.543E+13	3.646E+12	1.386E+13	3.447E+13	1.708E+13	3.900E+12	1.789E+13	4.554E+13	9.779E+13	5.042E+13	5.457E+13
GL-$0.5M Limit	1.660E+14	9.470E+12	3.745E+13	9.543E+13	8.760E+13	1.742E+12	6.624E+12	1.647E+13	8.161E+12	1.864E+12	8.548E+12	2.176E+13	4.673E+13	2.409E+13	2.607E+13
AL-$5M Limit	1.915E+13	3.618E+11	1.431E+12	3.646E+12	1.742E+12	4.183E+12	1.536E+12	3.820E+12	1.893E+12	4.322E+11	9.863E+11	2.511E+12	5.391E+12	2.780E+12	3.008E+12
AL-$2M Limit	7.282E+13	1.376E+12	5.440E+12	1.386E+13	6.624E+12	1.536E+12	2.126E+13	1.452E+13	7.196E+12	1.643E+12	3.750E+12	9.546E+12	2.050E+13	1.057E+13	1.144E+13
AL-$1M Limit	1.811E+14	3.421E+12	1.353E+13	3.447E+13	1.647E+13	3.820E+12	1.452E+13	7.677E+13	1.789E+13	4.086E+12	9.326E+12	2.374E+13	5.097E+13	2.628E+13	2.844E+13
AL-$0.5M Limit	8.972E+13	1.695E+12	6.702E+12	1.708E+13	8.161E+12	1.893E+12	7.196E+12	1.789E+13	1.958E+13	2.025E+12	4.620E+12	1.176E+13	2.525E+13	1.302E+13	1.409E+13
APhD	2.049E+13	3.870E+11	1.531E+12	3.900E+12	1.864E+12	4.322E+11	1.643E+12	4.086E+12	2.025E+12	7.626E+11	1.055E+12	2.686E+12	5.767E+12	2.974E+12	3.218E+12
CP-$50M Limit	9.398E+13	1.775E+12	7.021E+12	1.789E+13	8.548E+12	9.863E+11	3.750E+12	9.326E+12	4.620E+12	1.055E+12	1.611E+14	1.367E+14	2.936E+14	1.514E+14	1.639E+14
CP-$10M Limit	2.392E+14	4.519E+12	1.787E+13	4.554E+13	2.176E+13	2.511E+12	9.546E+12	2.374E+13	1.176E+13	2.686E+12	1.367E+14	4.003E+14	7.475E+14	3.854E+14	4.171E+14
CP-$5M Limit	5.137E+14	9.704E+12	3.838E+13	9.779E+13	4.673E+13	5.391E+12	2.050E+13	5.097E+13	2.525E+13	5.767E+12	2.936E+14	7.475E+14	1.778E+15	8.276E+14	8.956E+14
CP-$2M Limit	2.649E+14	5.003E+12	1.979E+13	5.042E+13	2.409E+13	2.780E+12	1.057E+13	2.628E+13	1.302E+13	2.974E+12	1.514E+14	3.854E+14	8.276E+14	4.716E+14	4.618E+14
CP-$1M Limit	2.866E+14	5.415E+12	2.141E+13	5.457E+13	2.607E+13	3.008E+12	1.144E+13	2.844E+13	1.409E+13	3.218E+12	1.639E+14	4.171E+14	8.956E+14	4.618E+14	5.516E+14