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RETROSPECTIVE RATING: 1997 EXCESS LOSS FACTORS

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INTRODUCTION

This discussion will present some of the mathematical aspects of the effect of dispersion of loss development on excess ratios. It will be shown how the formulas developed in "Retrospective Rating: 1997 Excess Loss Factors" fit into this more general mathematical framework.

THE PROBLEM

Even if one included average loss development beyond fifth report in the estimation of excess ratios, there are at least two phenomena that would affect excess ratios that are not being considered. First, the different sizes of claims may have varying expected amounts of development. If larger claims had higher average development, this would raise the excess ratios for higher limits.

Secondly, there is a "dispersion" effect. Assume we have two claims of \$1 million each that are expected on average to develop by 10%. It makes a difference whether we assume we'll have two claims each at \$1.1 million or one claim at \$1 million and one claim at \$1.2 million. The ratio excess of \$1.1 million will differ in the two cases.[1]

It is assumed for simplicity that there is no development on average; alternatively, any average development has already been

[1] In the former case it is zero, since there are no dollars excess of \$1.1 million. In the latter case it is 0.1/2.2, since there are \$1.2–\$1.1 million = \$.1 million dollars excess of \$1.1 million, and total losses of \$2.2 million.

incorporated into the size of loss distribution. However, some individual claims will develop more than average while others will develop less than average. In total, the average development factor is assumed to be unity.

SIMPLE EXAMPLE

Assume we have a piece-wise linear size of accident distribution such that:[2]

$$
\begin{aligned}
F(0) &= 0 \\
F(100) &= .90 \\
F(1{,}000) &= .99 \\
F(5{,}000) &= 1.00.
\end{aligned}
$$

Any size of loss distribution can be approximated sufficiently well by such an "ogive."[3] For actual applications one would have many more intervals, but this example will illustrate the principles involved.

The probability density function is:

$$
f(x) = \begin{cases} .009 & 0 < x \leq 100 \\ .0001 & 100 < x \leq 1{,}000 \\ .0000025 & 1{,}000 < x \leq 5{,}000 \\ 0 & x > 5{,}000. \end{cases}
$$

One can compute the average size of claim as the sum of three integrals of $xf(x)$:

$$
\begin{aligned}
E[X] &= \int_0^{100} (.009)x\,dx + \int_{100}^{1000} (.0001)x\,dx \\
&\quad + \int_{1000}^{5000} (.0000025)x\,dx \\
&= 45 + 49.5 + 30 = 124.5.
\end{aligned}
$$

[2]Assume everything is in units of thousands of dollars. Thus, 5,000 actually corresponds to \$5 million.

[3]See Hogg and Klugman [3].

The excess ratio at limit L can be computed as:

$$R(L) = \int_L^\infty (x - L) f(x)\, dx / E[X].$$

In this case we can compute the numerator as a sum of three terms:

$$\begin{aligned}
&\int_L^\infty (x - L) f(x)\, dx \\
&\qquad = (\text{If } L < 100) \int_L^{100} (x - L)(.009)\, dx \\
&\qquad\quad + (\text{If } L < 1000) \int_{\max[100,L]}^{1000} (x - L)(.0001)\, dx \\
&\qquad\quad + (\text{If } L < 5000) \int_{\max[1000,L]}^{5000} (x - L)(.0000025)\, dx.
\end{aligned}$$

If $L < 100$, let

$$\begin{aligned}
R_1(L) &= \int_L^{100} (x - L)\, dx \Big/ \int_0^{100} x\, dx \\
&= \text{excess ratio at } L \text{ if losses are uniformly distributed} \\
&\quad\ \text{on the interval } [0, 100].
\end{aligned}$$

Note that $R_1(L) = 0$ if $L \geq 100$. Then the first term above is

$$R_1(L) \int_0^{100} .009 x\, dx = R_1(L) E_1[X],$$

where $E_1[X] = \int_0^{100} (.009) x\, dx$ is the contribution to the overall mean from claims in the first interval. Then, working similarly with the other two intervals:

$$\int_0^\infty (x - L) f(x)\, dx = R_1(L)E_1[X] + R_2(L)E_2[X] + R_3(L)E_3[X],$$

$$R(L) = \frac{R_1(L)E_1[X] + R_2(L)E_2[X] + R_3(L)E_3[X]}{E_1[X] + E_2[X] + E_3[X]}.$$

Thus, the overall excess ratio can be expressed as a weighted average of excess ratios each computed as if the losses were uniformly distributed on an interval. The weights are the contributions to the overall mean of the claims in each interval. In this case, the weights are 45, 49.5, and 30 or 45/124.5, 49.5/124.5, and 30/124.5.

For example, for a limit of 70, the individual excess ratios are:[4] .09, .8727, and .9767. The weighted average is

$$R(70) = \frac{(45)(.09) + (49.5)(.8727) + (30)(.9767)}{124.5} = .6149.$$

Further, if the losses were uniform from 100 to 1000 then the excess ratio would be:

$$\frac{1}{900}\int_{100}^{1000}(x-70)\,dx \bigg/ \frac{1}{900}\int_{100}^{1000} x\,dx = (550-70)/550$$
$$= 480/550 = .8727.$$

Table 1 shows the excess ratios for this simple example for several limits. As can be seen, in the absence of any loss development, the ratio excess of 5,000 is zero; there are no losses above 5,000.

SIMPLE DISPERSION

Assume for simplicity that each accident has an equal likelihood of developing in a manner such that it is divided[5] by either: .75, .833, 1, 1.25, or 1.5. Then the average development is

$$\frac{1}{5}\left(\frac{1}{.75} + \frac{1}{.833} + \frac{1}{1} + \frac{1}{1.25} + \frac{1}{1.5}\right) = 1.$$

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[4]For losses distributed uniformly on $[a,b]$, for $b > L > a$, $R(L) = (b-L)^2/(b^2-a^2)$; for $L < a$, $R(L) = 1 - 2L/(b+a)$; for $L > b$, $R(L) = 0$. For the interval [0,100] we have the first case. For the intervals [100,1,000] and [1,000,5,000] we have the second case.

[5]Loss development *divisors* are used in order to match the presentation in "Retrospective Rating: 1997 Excess Loss Factors." Loss development multipliers or factors could have been used equally well for the presentation.

TABLE 1

EXCESS RATIOS*

LIMIT	No Development	Simple Dispersion**	Gamma Dispersion***
50	.6888	.6813	.6945
100	.5582	.5597	.5684
500	.3012	.2932	.3076
1,000	.1606	.1632	.1721
2,000	.0904	.0856	.0930
3,000	.0402	.0394	.0459
4,000	.0100	.0156	.0190
5,000	.0000	.0045	.0069
6,000	.0000	.0005	.0024
7,000	.0000	.0000	.0008
8,000	.0000	.0000	.0003
9,000	.0000	.0000	.0001
10,000	.0000	.0000	.0000

*For simple piece-wise linear distribution with $F(0) = 0$, $F(100) = .9$, $F(1000) = .99$, $F(5000) = 1$.
**For five possibilities, see text. Mean development = 1; Variance of development = .060.
***For a gamma loss divisor with $\alpha = 16.67$, $\lambda = 15.67$, see text. Mean development = 1; Variance of development = .060.

Thus, the total expected losses are unaffected. The variance of the development is .060.

We can compute excess ratios for each of the five possibilities and average the results together. If all the accidents were divided by 1.25; i.e., multiplied by .8, then a limit of 100 is equivalent to a limit of 125 without any development. So the excess ratio for 100 for the developed losses can be computed as $R(125)$ for the original distribution.[6]

Thus, to compute the excess ratio for the developed losses for a limit of 100:

$$\begin{aligned}\hat{R}(100) &= \tfrac{1}{5}(R(75) + R(83.3) + R(100) + R(125) + R(150)) \\ &= \tfrac{1}{5}(.6009 + .5817 + .5582 + .5384 + .5191) = .5597.\end{aligned}$$

[6] If each of the accidents are divided by 1.25, then the ratio excess of a limit of 100 declines from .5582 to .5384. Reducing the size of accidents reduces the excess ratio over any fixed limit.

Similarly, we can compute the excess ratio for the developed losses for a limit of 5,000:

$$\begin{aligned}\hat{R}(5000) &= \tfrac{1}{5}(R(3750) + R(4165) + R(5000)\\ &\qquad + R(6250) + R(7500))\\ &= \tfrac{1}{5}(.0157 + .0070 + 0 + 0 + 0) = .0045.\end{aligned}$$

So the dispersion effect has now produced some losses excess of 5,000, without affecting the total expected losses.

As can be seen in Table 1, the dispersion effect raises the excess ratios for higher limits and alters those for lower limits. While this example could be changed to include more than 5 possibilities, the essence of the dispersion effect has been captured. However, if the possibilities were more dispersed around the mean; i.e., if the variance of the development were greater, then the impact of the dispersion would be greater.

CONTINUOUS LOSS DIVISORS APPLIED TO A UNIFORM DISTRIBUTION ON AN INTERVAL

What if, rather than five possible loss divisors, one had a continuous probability distribution?

Assume:

1. Losses are distributed uniformly on the interval $[a,b]$.
2. Losses will develop with loss divisors r given by a distribution $H(r)$, with density $h(r)$.[7]

Then, as shown in Appendix A, the distribution function for the developed losses y, is given by:

$$F(y) = [y/(b-a)]\{E(R;b/y) - E(R;a/y)\},$$

[7] It is assumed that $\int_0^\infty (h(r)/r)\,dr$ is finite, so that the overall loss development is finite. In the case where H is a gamma distribution, this requirement means that the shape parameter s must be greater than one.

where $E[R;L]$ is the limited expected value of the distribution of loss divisors, at a limit L.

Appendix A also shows that the density function can be written in a number of forms, as summarized below:

$$\begin{aligned}
f(y) &= \frac{1}{b-a}\int_{a/y}^{b/y} r h(r)\,dr \\
&= \frac{1}{b-a}\{E[R;b/y]-(b/y)(1-H(b/y))-E[R;a/y] \\
&\qquad + (a/y)(1-H(a/y))\} \\
&= \frac{1}{b-a}\{E[R;b/y]-E[R;a/y]\} \\
&\quad + \frac{1}{y(b-a)}\{bH(b/y)-aH(a/y)\}-\frac{1}{y}.
\end{aligned}$$

Further, Appendix A describes how one can use the density function and distribution function to calculate the excess ratio of the developed losses, as follows:

$$\begin{aligned}
R(L) = \frac{1}{b^2-a^2}\bigg\{ & b^2\int_0^{b/L} h(r)/r\,dr - a^2\int_0^{a/L} h(r)/r\,dr \\
& + 2aLH(a/L) - 2bLH(b/L) \\
& + L^2\int_{a/L}^{b/L} r h(r)\,dr \bigg\} \bigg/ \int_0^{\infty} h(r)/r\,dr.
\end{aligned}$$

GAMMA DISPERSION APPLIED TO THE UNIFORM DISTRIBUTION

Assume that the loss divisor r is distributed according to a gamma distribution[8] with parameter s and l:

$$h(r) = \frac{l^s r^{s-1} e^{-lr}}{\Gamma(s)},$$

where $\Gamma(n) = (n-1)!$.

[8]Then the loss multipliers are distributed according to an inverse gamma. We assume $s > 1$, so that the overall loss development is finite.

Then, as shown in Appendix B, based on the general formula in Appendix A, if the losses are uniformly distributed on the interval $[a,b]$, after development the excess ratio for the limit L is given by:[9]

$$\begin{aligned} R(L) = {} & \frac{b^2}{b^2-a^2}\Gamma(s-1; lb/L) - \frac{a^2}{b^2-a^2}\Gamma(s-1; la/L) \\ & + \frac{2L(s-1)}{(b^2-a^2)l}\{a\Gamma(s; la/L) - b\Gamma(s; lb/L)\} \\ & + \frac{L^2(s-1)s}{(b^2-a^2)l^2}\{\Gamma(s+1; lb/L) - \Gamma(s+1; la/L)\}, \end{aligned}$$

where $\Gamma(s;y) = 1/\Gamma(s)\int_0^y t^{s-1}e^{-t}\,dt$ is the incomplete gamma function.

One can apply this "gamma dispersion" effect to a piece-wise linear size of accident distribution, such as in the prior example.

The mean development is the mean of an inverse gamma, $l/(s-1)$. For this discussion, the average development is unity, so we take $l = s-1$. The variance of the development is the variance of an inverse gamma, $l^2/\{(s-1)^2(s-2)\}$. For $l = s-1$, the variance is $1/s-2$. Thus, if one takes $s = 18.67$, (and $l = 17.67$) then the variance of the development is $1/16.67 = .060$, which matches that in the simple dispersion example. However, the gamma allows extreme possibilities (with a small probability), so one gets a somewhat different behavior than in the simple dispersion example.

As seen in Table 1, using the gamma dispersion for very high limits (7,000 and above) yields excess ratios that are now positive rather than zero. Gamma dispersion can have a particularly significant impact on very high limits, particularly if the variance is large.

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[9]These are the formulas developed and shown in "Retrospective Rating: 1997 Excess Loss Factors."

Each excess ratio is computed as a weighted average of the excess ratios computed for losses uniformly distributed on each of the three assumed intervals. For example, for a limit of 2,000, the excess ratio for losses distributed uniformly from 1,000 to 5,000, with gamma dispersion with $s = 18.67$ and $l = 17.67$ is given by the formula from Appendix B:

$$
\begin{aligned}
R_3(2000)& \\
&= (1.04167)\Gamma(17.67;44.175) - (.04167)\Gamma(17.67;8.835) \\
&\quad + (.1667)\Gamma(18.67;8.835) - (.8333)\Gamma(18.67;44.175) \\
&\quad + (.1761)\Gamma(19.67;44.175) - (.1761)\Gamma(19.67;8.835) \\
&= (1.04167)(.9999980) - (.04167)(.0057148) \\
&\quad + (.1667)(.0011302) - (.8333)(.999987) \\
&\quad + (.1761)(.999949) - (.1761)(.0026) \\
&= .384.
\end{aligned}
$$

Similarly, for losses uniform from 100 to 1,000, $R_2(2000) = .00008$. For losses uniform from 0 to 100, $R_1(2000) = 10^{-19}$. Taking a weighted average, using weights of 45, 49.5, and 30, one obtains $R(2000) = .093$, as displayed in Table 1.

Note that the gamma distribution used in this example has a large value of s, the shape parameter. Therefore, the distribution of loss divisors is close to normal.[10] The distribution of loss divisors has a skewness of $2/\sqrt{s}$, which is only .49. The skewness of the distribution of loss multipliers is that of an inverse gamma: $4\sqrt{(s-2)}/(s-3) = 1.12$. If one were to take a different form of distribution with a larger skewness one would have a larger chance of extreme results. Therefore, in the case of very high limits, the excess ratios would be even larger.

[10]The distribution of loss multipliers is close to an inverse normal distribution.

DISTRIBUTION OF DEVELOPED LOSSES

The particular situations examined so far are a special case of a more general framework. As shown in Appendix C, if losses at latest report are distributed via $G(x)$ and the loss divisors r are distributed independently of x via density function $h(r)$,[11] then the distribution for the developed losses y is given by:

$$F(y) = \int_0^\infty G(yr)h(r)\,dr.$$

GAMMA LOSS DIVISORS APPLIED TO AN EXPONENTIAL DISTRIBUTION

For example, if $G(x)$ is an exponential distribution $G(x) = 1 - e^{-\lambda x}$ and the loss divisors are gamma distributed $h(r) = l^s r^{s-1} e^{-lr}/\Gamma(s)$, then

$$\begin{aligned} F(y) &= 1 - \frac{l^s}{\Gamma(s)} \int_0^\infty r^{s-1} e^{-lr} e^{-\lambda yr}\,dr \\ &= 1 - \frac{l^s}{\Gamma(s)} \frac{\Gamma(s)}{(l+\lambda y)^s} = 1 - \left(\frac{(l/\lambda)}{(l/\lambda)+y}\right)^s. \end{aligned}$$

Thus $F(y)$ has a Pareto distribution as per Hogg and Klugman [3], with shape parameter s and scale parameter l/λ. Thus, the excess ratio of the developed losses is that of a Pareto distribution:

$$R(L) = \left(\frac{(l/\lambda)}{(l/\lambda)+L}\right)^{s-1}.$$

MATHEMATICAL RELATION TO MIXED DISTRIBUTIONS

The calculation of the distribution of the developed losses is the same as that used to calculate the mixed distribution in the inverse gamma-exponential conjugate prior.[12] (An inverse gamma

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[11] With $\int_0^\infty (h(r)/r)\,dr$ finite.

[12] See Herzog [2], or Venter [4]. The mixed distribution in the case of an inverse gamma—Exponential conjugate prior is a Pareto distribution, as obtained above.

distributed multiplier is the same as a gamma distributed divisor.) In general, if the loss multipliers and the loss distribution form any of the well known pairs[13] of prior distributions of the scale parameters of the conditional distributions and conditional distributions, then the developed losses will be given by the mixed distribution. For example, as shown in Venter [4], a Weibull loss distribution and a transformed gamma loss divisor[14] would produce a Burr distribution of developed losses. Thus, there are a number of mathematically convenient examples that might approximate a particular real world application.

GAMMA LOSS DIVISORS APPLIED TO PARETO LOSSES

Since the Pareto distribution is often used to model losses (or at least the larger losses), it would be valuable to be able to apply the concept of loss divisors to the Pareto distribution.

As shown in Appendix C, one can develop the mathematics of applying gamma loss divisors to losses distributed by a Pareto distribution with parameters α and λ: $F(x) = 1 - (\lambda/(\lambda + x))^{\alpha}$. As derived in Appendix C, the excess ratio for the developed losses is given by:

$$R(L) = \left(\frac{Xl}{L}\right)^{s-1} U(s-1, s+1-\alpha, \lambda l/L),$$

where U is a confluent hypergeometric function.[15]

It is also shown in Appendix C that when the average development is unity[16] then the excess ratios of the developed losses can be approximated by replacing λ in the Pareto by $\lambda' = \lambda(s-1)/(s-(\alpha/2)-1)$.

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[13] Such as shown in Venter [4]. Venter displays a list of conjugate priors, but for the current application there is no requirement that it be a conjugate prior situation.

[14] An inverse transformed gamma loss multiplier.

[15] See Appendix D and *Handbook of Mathematical Functions* [1].

[16] Also, we need the shape parameter of the gamma, s, to be greater than $\alpha + 1$.

Table 2 and Figure 1 compare the excess ratios for a Pareto with $\alpha = 3.5$ and $\lambda = 1{,}000$, for the developed losses[17] with a gamma divisor with $s = 6$ and $l = 5$, and for an approximating Pareto with $\alpha = 3.5$ and $\lambda = 1{,}000(s-1)/(s-(\alpha/2)-1) = 1{,}538$. The excess ratios for the developed losses are larger than those for the undeveloped losses. The approximation using the rescaled Pareto yields excess ratios that are too high for the lower limits, but it does an excellent job of approximating the excess ratios for higher limits.

As shown in Appendix C, in the tail, the loss development[18] multiplies the excess ratios by a factor of approximately:

$$(s-1)^{\alpha-1}\Gamma(s-\alpha)/\Gamma(s-1) \approx \left((s-1)\Big/\left(s-\frac{\alpha}{2}-1\right)\right)^{\alpha-1}.$$

In this example, this factor is: $5^{2.5}\Gamma(2.5)/\Gamma(5) = 3.1$. Figure 2 shows how this adjustment factor varies as the shape parameters of the Pareto and gamma vary. As the shape parameter of the Pareto, α, gets smaller, the losses have a heavier tail and the impact of the dispersion increases. As the coefficient of variation of the gamma[19] increases, the impact of the dispersion increases.

In general, as the coefficient of variation of the loss divisors increases, the impact of the dispersion increases. As the coefficient of variation approaches zero, we approach the situation where each claim develops by the average amount and there is no effect of dispersion. Thus, in order to apply this technique, one of the key inputs would be the coefficient of variation of the loss divisors.

CONCLUSIONS

The effect of the dispersion of loss development beyond the latest available report can be incorporated into the calculation of

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[17] Then $R(L) = (\lambda l/L)^{s-1}U(s-1, s+1-\alpha, \lambda l/L) = (L/5000)^{-5}U(5, 3.5, 5000/L)$.

[18] For gamma dispersion with $l = s-1$ so the average development is unity.

[19] The coefficient of variation is the standard deviation divided by the mean. For the gamma distribution with shape parameter s, the coefficient of variation is $1/\sqrt{s}$.

TABLE 2

EXCESS RATIOS

LIMIT	Undeveloped Losses Pareto ($\alpha = 3.5, \lambda = 1{,}000$)	Developed Losses*	Approximating Developed Pareto ($\alpha = 3.5, \lambda = 1{,}538$)
500	.3629	.3960	.4947
1,000	.1768	.2152	.2859
2,500	.0436	.0668	.0895
5,000	.0113	.0211	.0268
10,000	.0025	.0055	.0065
25,000	.00029	.00076	.00081
50,000	.00005	.00015	.00015
100,000	.000010	.000029	.000028

*Assuming gamma loss divisor, with $s = 6$ and $l = 5$. $R(L) = (5000/L)^{2.5}U(5, 1.5; 5000/L)$.

FIGURE 1

EXCESS RATIOS

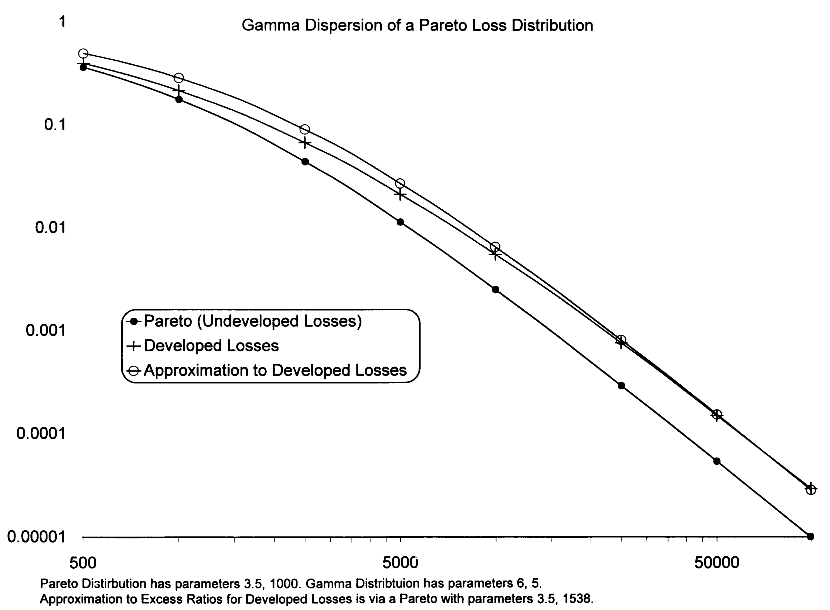
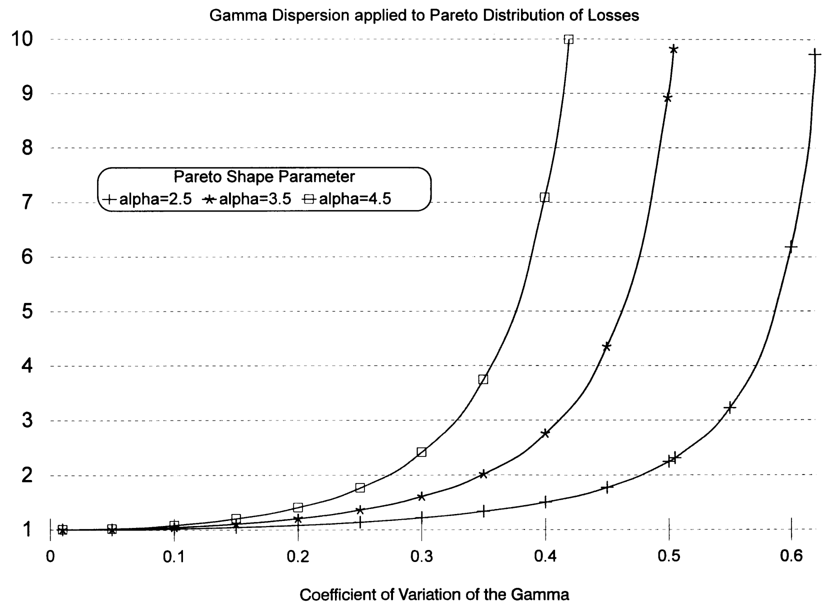


FIGURE 2

ADJUSTMENT FACTOR TO APPLY TO EXCESS RATIOS



excess ratios. In the case of loss dispersion which is (approximately) independent of size of loss, for many special cases one can calculate the distribution of the developed losses in closed form. In these cases, the excess ratios follow directly.

In other situations, one can approximate the loss distribution via a piece-wise linear distribution and then apply the effects of dispersion to each interval. Since on each interval the piece-wise linear approximation is a uniform distribution, one can apply the formulas developed in Appendix A. Then one can weight together the excess ratios for the developed losses from the individual intervals in order to get the excess ratio for all developed losses.

REFERENCES

[1] Abramowitz, Milton and Irene A. Stegun, Eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 1964, U.S. Department of Commerce.

[2] Herzog, Thomas N., *Introduction to Credibility Theory*, 1994, Mad River Books.

[3] Hogg, Robert V. and Stuart A. Klugman, *Loss Distributions*, John Wiley & Sons, 1984, New York.

[4] Venter, Gary G., "Credibility," Chapter 7 in *Foundations of Casualty Actuarial Science*, Casualty Actuarial Society, 1990, pp. 375–483.

APPENDIX A

LOSS DIVISORS APPLIED TO A UNIFORM DISTRIBUTION ON AN INTERVAL

Assume:

Losses are distributed uniformly on the interval $[a,b]$. Losses will develop with loss divisors r given by a distribution $H(r)$ and density $h(r)$.

Then:

The distribution function for the developed losses y, is given by:

$$F(y) = (y/b-a)\{E[R;b/y]-E[R;a/y]\},$$

where $E[R;L]$ is the limited expected value of the distribution of loss divisors, at a limit L.

Proof:

The developed losses y are the ratio of the undeveloped losses x and the loss divisor r; $y = x/r$ or $x = yr$. Thus since x is uniform on $[a,b]$,[20] the conditional distribution of y given r is:

$$F(y \mid r) = \begin{cases} 0 & yr \le a \\ \dfrac{ry-a}{b-a} & a \le yr \le b \\ 1 & yr \ge b. \end{cases}$$

The unconditional distribution of y can be computed by integrating the conditional distribution of y given r times the assumed density function of r:

[20]For the uniform distribution on $[a,b]$, $F(x) = 0$ if $x \le a$, $F(x) = (x-a)/(b-a)$ if $a \le x \le b$, and $F(x) = 1$ if $x \ge b$.

$$
\begin{aligned}
F(y) &= \int_{r=0}^{\infty} F(y \mid r) h(r)\,dr \\
&= \int_{a/y}^{b/y} \left(\frac{ry-a}{b-a}\right) h(r)\,dr + \int_{b/y}^{\infty} h(r)\,dr \\
&= \frac{y}{b-a}\int_{a/y}^{b/y} r h(r)\,dr + \frac{a}{b-a}\left(H\left(\frac{a}{y}\right) - H\left(\frac{b}{y}\right)\right) \\
&\quad + 1 - H(b/y) \\
&= \frac{y}{b-a}\left\{\int_0^{b/y} r h(r)\,dr - \int_0^{a/y} r h(r)\,dr + \frac{a}{y}H\left(\frac{a}{y}\right) - \frac{a}{y}\right. \\
&\qquad \left. + \frac{b}{y} - \frac{b}{y}H\left(\frac{b}{y}\right)\right\} \\
&= \frac{y}{b-a}\left\{\left[\int_0^{b/y} r h(r) + \left(\frac{b}{y}\right)(1 - H(b/y))\right]\right. \\
&\qquad \left. - \left[\int_0^{a/y} r h(r) + \left(\frac{a}{y}\right)(1 - H(a/y))\right]\right\} \\
&= \frac{y}{b-a}\left\{E\left[R;\frac{b}{y}\right] - E\left[R;\frac{a}{y}\right]\right\}.
\end{aligned}
$$

Similarly, we can get the density function $f(y)$. For conditional density at y given r is:

$$
f(y \mid r) = \begin{cases} 0 & yr \le a \\ \dfrac{r}{b-a} & a \le yr \le b \\ 0 & yr \ge b. \end{cases}
$$

The unconditional density at y can be computed by integrating the conditional density at y given r times the assumed density function of r:

$$\begin{aligned} f(y) &= \int_0^\infty f(y \mid r)h(r)\,dr \\ &= \int_{a/y}^{b/y} \frac{r}{b-a}h(r)\,dr \\ &= \frac{1}{b-a}\int_{a/y}^{b/y} rh(r)\,dr. \end{aligned}$$

We can put this type of integral in terms of limited expected values, since

$$\begin{aligned} E[R;r] &= \int_0^r rh(r)\,dr + r(1-H(r)) \\ f(y) &= \frac{1}{b-a}\{E[R;b/y] - (b/y)(1-H(b/y)) \\ &\qquad - E[R;a/y] + (a/y)(1-H(a/y))\} \\ &= \frac{1}{b-a}\{E[R;b/y] - E[R;a/y]\} \\ &\quad + \frac{1}{y(b-a)}\{bH(b/y) - aH(a/y)\} - \frac{1}{y}. \end{aligned}$$

One can use the density function and distribution function to calculate the excess ratio of the developed losses. The numerator of this excess ratio is the (developed) losses excess of L:

$$\int_L^\infty (y-L)f(y)\,dy = \int_L^\infty yf(y)\,dy - L(1-F(L)).$$

Since $f(y) = 1/(b-a)\int_{a/y}^{b/y} rh(r)\,dr$ we have

$$\int_L^\infty yf(y)\,dy = \frac{1}{b-a}\int_{y=L}^\infty y\int_{r=a/y}^{b/y} rh(r)\,dr\,dy.$$

Switching the order of integration:

$$\int_L^\infty y f(y)\,dy = \frac{1}{b-a}\int_{r=0}^{a/L}\int_{y=a/r}^{b/r} y r h(r)\,dy\,dr$$

$$+\frac{1}{b-a}\int_{r=a/L}^{b/L}\int_{y=L}^{b/r} y r h(r)\,dy\,dr$$

$$=\frac{1}{2(b-a)}\int_{r=0}^{a/L}\left(\frac{b^2}{r^2}-\frac{a^2}{r^2}\right) r h(r)\,dr$$

$$+\frac{1}{2(b-a)}\int_{r=a/L}^{b/L}\left(\frac{b^2}{r^2}-L^2\right) r h(r)\,dr$$

$$=\frac{b^2}{2(b-a)}\int_{r=0}^{b/L} h(r)/r\,dr-\frac{a^2}{2(b-a)}\int_{r=0}^{a/L} h(r)/r\,dr$$

$$-\frac{L^2}{2(b-a)}\int_{r=a/L}^{b/L} r h(r)\,dr.$$

In the course of deriving the form of the distribution function we had

$$F(y)=\frac{y}{b-a}\int_{a/y}^{b/y} r h(r)\,dr + 1 + \frac{a}{b-a}H\left(\frac{a}{y}\right)-\frac{b}{b-a}H\left(\frac{b}{y}\right).$$

Thus

$$1-F(L)=$$

$$\frac{b}{b-a}H(b/L)-\frac{a}{b-a}H(a/L)-\frac{L}{b-a}\int_{a/L}^{b/L} r h(r)\,dr.$$

Thus combining the terms, the numerator of the excess ratio is:

$$\begin{aligned}
&\int_L^\infty yf(y)\,dy - L(1-F(L))\\
&\quad= \frac{b^2}{2(b-a)}\int_0^{b/L} h(r)/r\,dr - \frac{a^2}{2(b-a)}\int_0^{a/L} h(r)/r\,dr\\
&\qquad+ \frac{aL}{(b-a)}H(a/L) - \frac{bL}{b-a}H(b/L)\\
&\qquad+ \frac{L^2}{2(b-a)}\int_{a/L}^{b/L} rh(r)\,dr.
\end{aligned}$$

The denominator of the excess ratio is:[21]

$$\begin{aligned}
\int_0^\infty yf(y)\,dy &= \lim_{L\to 0}\int_L^\infty yf(y)\,dy\\
&= \frac{b^2-a^2}{2(b-a)}\int_0^\infty h(r)/r\,dr\\
&= \frac{b+a}{2}\int_0^\infty h(r)/r\,dr.
\end{aligned}$$

Combining the numerator and denominator, the excess ratio (of the developed losses) at L is:

$$\begin{aligned}
R(L) = \frac{1}{b^2-a^2}\Bigg\{&b^2\int_0^{b/L} h(r)/r\,dr - a^2\int_0^{a/L} h(r)/r\,dr\\
&+ 2aLH(a/L) - 2bLH(b/L)\\
&+L^2\int_{a/L}^{b/L} rh(r)\,dr\Bigg\}\Bigg/\int_0^\infty h(r)/r\,dr.
\end{aligned}$$

[21] The denominator of the excess ratio is the mean of the developed losses. It is equal to the product of the mean undeveloped losses $(b+a)/2$, and the average loss development $\int_0^\infty h(r)/r\,dr$.

APPENDIX B

GAMMA LOSS DIVISORS APPLIED TO LOSSES UNIFORM ON AN INTERVAL

For the situation discussed in Appendix A but for the specific case where the distribution of the loss divisors, $h(r)$, is a gamma distribution with parameters s and l:

$$\begin{aligned}
\int_0^x h(r) r\,dr &= \int_0^x l^s e^{-lr} r^s/\Gamma(s)\,dr = (l^s/\Gamma(s))\int_0^x e^{-lr} r^s\,dr \\
&= (l^s/\Gamma(s))(\Gamma(s+1)/l^{s+1})\Gamma(s+1;lx) \\
&= (s/l)\Gamma(s+1;lx) \\
H(x) &= \int_0^x h(r)\,dr = \int_0^x l^s e^{-lr} r^{s-1}/\Gamma(s)\,dr = \Gamma(s;lx) \\
\int_0^x h(r)/r\,dr &= \int_0^x l^s e^{-lr} r^{s-2}/\Gamma(s)\,dr \\
&= (l^s/\Gamma(s))(\Gamma(s-1)/l^{s-1})\Gamma(s-1;lx) \\
&= \frac{l}{s-1}\Gamma(s-1;lx) \\
\int_0^\infty h(r)/r\,dr &= (l/s-1)\Gamma(s-1;\infty) = l(s-1).
\end{aligned}$$

Thus, using the formula from Appendix A, the excess ratio of the developed losses for limit L is in this case:

$$\begin{aligned}
R(L) = {} & \frac{b^2}{b^2-a^2}\Gamma(s-1;lb/L) - \frac{a^2}{b^2-a^2}\Gamma(s-1;la/L) \\
& + \frac{2L(s-1)}{(b^2-a^2)l}\{a\Gamma(s;la/L) - b\Gamma(s;lb/L)\} \\
& + \frac{L^2(s-1)s}{(b^2-a^2)l^2}\{\Gamma(s+1;lb/L) - \Gamma(s+1;la/L)\}.
\end{aligned}$$

For the loss divisors given by a gamma distribution with parameters s and l, we can plug in the limited expected value for the gamma distribution in terms of the incomplete gamma function:[22]

$$E[R;r] = \frac{s}{l}\Gamma(s+1;lr) + r[1-\Gamma(s;lr)].$$

Thus using the formula derived in Appendix A:

$$\begin{aligned} F(y) &= \frac{y}{b-a}\left\{E\left[R;\frac{b}{y}\right] - E\left[R;\frac{a}{y}\right]\right\} \\ &= \frac{ys}{l(b-a)}\left\{\Gamma\left(s+1;\frac{lb}{y}\right) - \Gamma\left(s+1;\frac{la}{y}\right)\right\} \\ &\quad + 1 + \frac{a}{b-a}\Gamma\left(s;\frac{la}{y}\right) - \frac{b}{b-a}\Gamma\left(s;\frac{lb}{y}\right). \end{aligned}$$

Also using the formula derived in Appendix A, the probability density function is given by:

$$\begin{aligned} f(y) &= \frac{1}{b-a}\int_{a/y}^{b/y} rh(r)\,dr \\ &= \frac{s}{(b-a)l}\{\Gamma(s+1;lb/y) - \Gamma(s+1;la/y)\}. \end{aligned}$$

[22]See Hogg and Klugman [3, page 226].

APPENDIX C

GAMMA LOSS DIVISORS APPLIED TO A PARETO DISTRIBUTION

Assume:

Losses are distributed (at latest report) on $(0,\infty)$ via a distribution function $G(x)$. Losses will develop with loss divisors r given by a density function $h(r)$.[23] (The distribution of r is independent of x.)

Then:

The distribution function for the developed losses y, is given by:

$$F(y) = \int_0^\infty G(yr)h(r)\,dr.$$

Proof:

The developed losses y are the ratio of the undeveloped losses x and the loss divisor r; $y = x/r$ or $x = yr$.

Given a value for r, the developed losses are less than y if the undeveloped losses are less than yr. Thus:

$$F(y \mid r) = G(yr).$$

Integrating over all possible values of r we have

$$F(y) = \int_0^\infty F(y \mid r)h(r)\,dr = \int_0^\infty G(yr)h(r)\,dr.$$

In the specific case where r follows a gamma distribution with parameters s and l and the undeveloped losses follow a Pareto distribution with parameters α and λ:

$$G(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^\alpha,$$

$$h(r) = l^s r^{s-1} e^{-lr}/\Gamma(s).$$

[23] It is assumed that $\int_0^\infty (h(r)/r)\,dr$ is finite, so that the average loss development is finite.

Then the distribution function for the developed losses is

$$
\begin{aligned}
F(y) &= \int_0^\infty h(r)G(yr)\,dr \\
&= \int_0^\infty \left(1-\left(\frac{\lambda}{\lambda+yr}\right)^{\alpha}\right) l^s r^{s-1} e^{-lr}/(\Gamma(s))\,dr \\
&= \int_0^\infty l^s r^{s-1} e^{-lr}/\Gamma(s)\,dr - \frac{\lambda^\alpha l^s}{\Gamma(s)}\int_0^\infty r^{s-1}e^{-lr}(\lambda+yr)^{-\alpha}\,dr.
\end{aligned}
$$

The first integral is unity,[24] while the second integral can be put in terms of confluent hypergeometric functions.[25]

Let $q = (y/\lambda)r$, then the second integral becomes

$$
\begin{aligned}
&\frac{\lambda^{s-\alpha}}{y^s}\int_{q=0}^\infty q^{s-1}e^{-\lambda l q/y}(1+q)^{-\alpha}\,dq \\
&\qquad = \frac{\lambda^{s-\alpha}}{y^s}\Gamma(s)U(s,s+1-\alpha;\lambda l/y)
\end{aligned}
$$

where U is a confluent hypergeometric function such that[26]

$$
U(a,b;z) = (1/\Gamma(a))\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}\,dt.
$$

Thus the distribution function of the developed losses is:

$$
\begin{aligned}
F(y) &= 1 - \frac{\lambda^\alpha l^s}{\Gamma(s)}\frac{\lambda^{s-\alpha}}{y^s}\Gamma(s)U(s,s+1-\alpha;\lambda l/y) \\
&= 1 - \left(\frac{\lambda l}{y}\right)^s U(s,s+1-\alpha;\lambda l/y).
\end{aligned}
$$

Similarly one can compute the density function of the developed losses. Differentiating the distribution function one gets:

$$
f(y) = \int_0^\infty r g(yr)h(r)\,dr.
$$

[24]It is the cumulative distribution function of the gamma distribution at infinity.

[25]See Appendix D and *Handbook of Mathematical Functions* [1].

[26]See Equation 13.2.5 in *Handbook of Mathematical Functions* [1].

In the specific case where h is gamma and g is Pareto it turns out that the density of the developed losses is:

$$f(y) = \frac{s\alpha l^s \lambda^s}{y^{s+1}} U(s+1, s+1-\alpha; l\lambda/y).$$

This can be obtained either by substituting the specific form of y and h into the above integral or by differentiating $F(y)$, and making use of the facts that[27]

$$\frac{d}{dz} U(a,b;z) = -aU(a+1,b+1;z),$$

$$U(a-1,b;z) - zU(a,b+1;z) = (a-b)U(a,b;z),$$

$$\begin{aligned}
f(y) = \frac{d}{dy}F(y) &= \frac{d}{dy}\left(1 - \frac{\lambda^s l^s}{y^s} U\left(s, s+1-\alpha; \frac{\lambda l}{y}\right)\right) \\
&= \frac{\lambda^s l^s s}{y^{s+1}} U\left(s, s+1-\alpha; \frac{\lambda l}{y}\right) \\
&\quad - \frac{\lambda^s l^s}{y^s}\left(\frac{\lambda l}{y^2}\right) U'\left(s, s+1-\alpha; \frac{\lambda l}{y}\right) \\
&= \frac{s\lambda^s l^s}{y^{s+1}}\{U(s, s+1-\alpha; \lambda l/y) \\
&\qquad - (\lambda l/y) U(s+1, s+2-\alpha; \lambda l/y)\} \\
&= \frac{s\lambda^s l^s}{y^{s+1}}\{(s+1) - (s+1-\alpha)\} U(s+1, s+1-\alpha; \lambda l/y) \\
&= \frac{s\alpha\lambda^s l^s}{y^{s+1}} U(s+1, s+1-\alpha; \lambda l/y).
\end{aligned}$$

One can use the density function and distribution function to calculate the excess ratio of the developed losses. The numerator of this excess ratio is the total (developed) losses excess of L:

$$\int_L^\infty (y-L) f(y)\, dy = \int_L^\infty y f(y)\, dy - L(1 - F(L)),$$

[27] See Equations 13.4.21 and 13.4.18 in *Handbook of Mathematical Functions* [1].

$$\int_L^\infty y f(y)\,dy = s\alpha\lambda^s l^s \int_L^\infty y^{-s} U(s+1, s+1-\alpha; \lambda l/y)\,dy.$$

Letting $z = \lambda l/y$ this integral becomes

$$-s\alpha \int_0^{\lambda l/L} z^s U(s+1, s+1-\alpha; z)\frac{\lambda l}{-z^2}dz$$

$$= \lambda l s\alpha \int_0^{\lambda l/L} z^{s-2} U(s+1, s+1-\alpha; z)\,dz.$$

Using the theorem from Appendix D:

$$\int_L^\infty y f(y)\,dy = \frac{\lambda l s\alpha z^{s-1}}{s\alpha}\left[U(s, s+1-\alpha; z) + \frac{U(s-a, s+1-\alpha; z)}{(s-1)(\alpha-1)}\right]_{Z=0}^{\lambda l/L}$$

$$= \lambda l \left(\frac{\lambda l}{L}\right)^{s-1}\left(U\left(s, s+1-\alpha; \frac{\lambda l}{L}\right) + \frac{U(s-1, s+1-\alpha; \lambda l/L)}{(s-1)(\alpha-1)}\right).$$

Now

$$L(1-F(L)) = \frac{\lambda^s l^s}{L^{s-1}} U(s, s+1-\alpha; \lambda l/L).$$

Thus the numerator of the excess ratio is

$$\int_L^\infty y f(y)\,dy + L(1-F(L))$$

$$= \frac{\lambda^s l^s}{(s-1)(\alpha-1)L^{s-1}} U(s-1, s+1-\alpha; \lambda l/L).$$

The denominator of the excess ratio is the total (developed) losses or the mean of the undeveloped losses times the mean loss development. The former is the mean of the Pareto or $\lambda/(\alpha-1)$. The latter is the mean of the inverse gamma or $l/(s-1)$.

Thus the excess ratio is:

$$R(L) = \left(\frac{\lambda l}{L}\right)^{s-1} U(s-1, s+1-\alpha, \lambda l/L).$$

Note that this compares to the excess ratio for the undeveloped losses (given by a Pareto) of $(\lambda/\lambda + L)^{\alpha-1}$. For z small:[28]

$$U(a,b,z) \approx z^{1-b}\Gamma(b-1)/\Gamma(a) \quad b > 2.$$

Thus for large limits L, and $s > \alpha + 1$

$$\begin{aligned} R(L) &= \left(\frac{\lambda l}{L}\right)^{s-1} U(s-1, s+1-\alpha, \lambda l/L) \\ &\approx \left(\frac{\lambda l}{L}\right)^{s-1} \frac{\Gamma(s-\alpha)}{\Gamma(s-1)} \left(\frac{\lambda l}{L}\right)^{\alpha-s} \\ &= \left(\frac{\lambda l}{L}\right)^{\alpha-1} \Gamma(s-\alpha)/\Gamma(s-1). \end{aligned}$$

For the Pareto for large limits

$$R(L) = (\lambda/(\lambda + L))^{\alpha-1} \approx (\lambda/L)^{\alpha-1}.$$

Thus the ratio of the excess ratios for the developed and the undeveloped losses is approximately: $l^{\alpha-1}\Gamma(s-\alpha)/\Gamma(s-1)$. If the mean development is unity, then $l = s-1$. Then this ratio is:

$$\begin{aligned} &(s-1)^{\alpha-1}/\{(s-\alpha-1)\cdots(s-2)\} \\ &\approx \left((s-1)\Big/\left(s - \frac{\alpha}{2} - 1\right)\right)^{\alpha-1}. \end{aligned}$$

Since for a Pareto for large limits $R(L) \approx \lambda^{\alpha-1}/L^{\alpha-1}$ if one adjusts λ by multiplying by a factor of $(s-1)/(s-(\alpha/2)-1)$, then one will approximately multiply the excess ratios by the desired adjustment factor.

[28]See Equation 13.5.6, *Handbook of Mathematical Functions* [1].

APPENDIX D

CONFLUENT HYPERGEOMETRIC FUNCTIONS[29]

There are a number of related functions referred to as confluent hypergeometric functions. They can be usefully thought of as generalizations of the beta and gamma functions. They can be thought of as two parameter distributions. Let:

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1}\, dt,$$

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1}\, dt.$$

Then M can be computed using the following power series in z:

$$M(a,b;z) = 1 + \frac{az}{b} + \frac{a(a+1)z^2}{b(b+1)(2!} + \frac{a(a+1)(a+2)z^3}{b(b+1)(b+2)(3!)} + \cdots.$$

U can be computed as a combination of two values of M:

$$U(a,b;z) = \frac{\pi}{\sin \pi b}\left(\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - \frac{z^{1-b}M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right).$$

U is related to the incomplete gamma function:

$$U(1-a,1-a;x) = e^x \Gamma(a;x).$$

Among the facts used in Appendix C are:

$$\frac{d}{dz}U(a,b;z) = -aU(a+1,b+1;z),$$

$$U(a-1,b;z) - zU(a,b+1;z) = (a-b)U(a,b;z).$$

For z small and $b > 2$, $U(a,b;z) \approx z^{1-b}\Gamma(b-1)/\Gamma(a)$.

[29] See *Handbook of Mathematical Functions* [1].

THEOREM

$$\int z^{a-3}U(a,b,z)\,dz$$
$$= -\frac{z^{a-2}}{(a-1)(b-a)}\frac{U(a-1,b,z)-U(a-2,b,z)}{\{(a-2)(b+1-a)\}}.$$

Given:

$$\frac{dU(a,b,z)}{dz} = -aU(a+1,b+1,z), \qquad \text{and}$$
$$zU(a,b+1,z)-U(a-1,b,z) = (b-a)U(a,b,z).$$

Proof:

Let

$$\nu = z^{a-2}(U(a-1,b,z)-U(a-2,b,z)/\{(a-2)(b+1-a)\})$$
$$\frac{d\nu}{dz} = (a-2)\nu/z + z^{a-2}\frac{-(a-1)U(a,b+1,z)+U(a-1,b+1,z)}{(b+1-a)}$$
$$= z^{a-3}\{(a-2)U(a-1,b,z)-U(a-2,b,z)/(b+1-a)$$
$$-(a-1)zU(a,b+1,z)$$
$$+zU(\alpha-1,b+1,z)/(b+1-\alpha)\}$$
$$= z^{a-3}\{(zU(\alpha-1,b+1,z)-U(\alpha-2,b,z))/(b+1-\alpha)$$
$$-(a-1)(zU(a,b+1,z)-U(\alpha-1,b,z))-U(\alpha-1,b,z)\}$$
$$= z^{a-3}(U(a-1,b,z)-(a-1)(b-a)U(a,b,z)-U(a-1,b,z))$$
$$= -z^{a-3}(a-1)(b-a)U(a,b,z). \qquad \text{Q.E.D.}$$

Errata for Discussion by Howard Mahler of "Retrospective Rating: 1997 Excess Loss Factors"

At the bottom of page 320, the equation for $\hat{R}(100)$ is incorrect.

This example of simple dispersion is an example of a mixture with five pieces.

The excess ratio of the mixture is a weighted average of individual excess ratios, with the weights the product of the means and the probabilities for each piece of the mixture.[1]

If the probability of each piece of a mixture is p_i, $\sum p_i = 1$, the mean of each piece of the mixture is m_i, and R_i is the excess ratio for each piece of the mixture, then $\hat{R}(L) = \sum p_i \, m_i \, R_i(L)$.

If each loss is divided by for example .75, then after development, the excess ratio at L is the same as the original excess ratio at .75 L.[2]
$R_i(L)$ is the excess ratio when the losses have all been divided by r_i.
Thus $R_i(L) = R(r_i \, L)$.

In the example on page 320, each mean is proportional to 1/divisor = $1/r_i$, and each probability is the same at 1/5. Thus the weights are: $(1/5)(1/r_i)$.

The sum of the weights is: $\sum (1/5)(1/r_i) = (1/5)(1/.75 + 1/.833 + 1/1 + 1/1.25 + 1/1.5) = 1$.[3]

Thus $\hat{R}(L) = \sum (1/5)(1/r_i) \, R(r_i \, L) = (1/5) \sum R(r_i \, L) / r_i$.

Therefore, the corrected equation at the bottom of page 320 is:

$$\hat{R}(100) = (1/5)\{R(75)/.75 + R(83.3)/.833 + R(100)/1 + R(125)/1.25 + R(150)/1.50\}$$
$$= (1/5)\{.6009/.75 + .5817/.833 + .5582/1 + .5384/1.25 + .5191/1.50\} = .5669.$$

Similarly, the corrected equation at the top of page 321 is:

$$\hat{R}(5000) = (1/5)\{R(3750)/.75 + R(4165)/.833 + R(5000)/1 + R(6250)/1.25 + R(7500)/1.50\}$$
$$= (1/5)\{.0157/.75 + .0070/.833 + 0/1 + 0/1.25 + 0/1.50\} = .0059.$$

---

[1] See page 154 of "Workers Compensation Excess Ratios: An Alternate Method of Estimation" by Mahler.
[2] If each loss is multiplied by 1/.75 = 1.333, this is mathematically the same as uniform inflation of 33.3%. Thus we can get the excess ratio after development, by taking the original excess ratio at the deflated value of L/1.333 = .75 L. Increasing the sizes of loss, increases the excess ratio over a fixed limit.
[3] Mahler chose these loss divisors so that the total expected losses are unaffected.

At page 324, some of the numerical values shown in the computation of $R_3(2000)$ are mixed up, although the final value is correct at 0.384 as shown.
It should have read:

$$R_3(2000) = (1.04167)(0.9999980) - (0.04167)(0.0057148) + (0.1667)(0.0026029) - (0.8333)(0.999995) + (0.1761)(0.999987) - (0.1761)(0.0011302) = 0.384.$$

Also, in Table 1 the excess ratios were computed for Gamma loss divisors with shape parameter 16.67 and inverse scale parameter 15.67. However, the text at page 323 refers to Gamma loss divisors with shape parameter s = 18.67 and inverse scale parameter l = 17.67; this distribution of loss divisors corresponds to a mean loss development of 1 and a variance of loss development of 0.060, matching the simple dispersion example.

Using the intended Gamma parameters of s = 18.67 and l = 17.67 changes the excess ratios in Table 1 slightly, although the pattern remains the same.

The values in the simple dispersion column of Table 1 at page 320 are revised in a similar manner to that for 5000.
The values in Gamma dispersion column of Table 1 at page 320 are revised based on a shape parameter of s = 18.67 and inverse scale parameter of l = 17.67.

Corrected Table 1
Excess Ratios

LIMIT	No Development	Simple Dispersion	Gamma Dispersion
50	.6888	.6949	.6939
100	.5582	.5669	.5673
500	.3012	.3080	.3069
1,000	.1606	.1705	.1709
2,000	.0904	.0931	.0927
3,000	.0402	.0462	.0453
4,000	.0100	.0194	.0182
5,000	.0000	.0059	.0062
6,000	.0000	.0007	.0020
7,000	.0000	.0000	.0006
8,000	.0000	.0000	.0002
9,000	.0000	.0000	.0001
10,000	.0000	.0000	.0000

As can be seen in corrected Table 1, the simple dispersion effect raises the excess ratios, especially at the higher limits.[4]

At page 326, the formula near the bottom of page should have λ in place of X:

$R(L) = (\lambda\, l/L)^{s-1}\, U(s-1, s+1-\alpha, \lambda\, l/L).$

[4] It can be demonstrated that when dispersion has no overall effect, loss dispersion either increases an excess ratio or keeps it the same. In most practical applications, the excess ratio will be increased by loss dispersion.