THE INTERACTION OF MAXIMUM PREMIUMS, MINIMUM PREMIUMS, AND ACCIDENT LIMITS IN RETROSPECTIVE RATING

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Abstract

This paper discusses the inaccuracies in workers compensation retrospective rating that resulted from the former method of separately calculating insurance charges from Table M and excess loss factors for loss limitations. These ideas have been previously presented by Glenn Meyers [1] and Ira Robbin [2]. However, this paper presents the ideas in a coherent fashion using Lee diagrams [3]. This should make these important ideas more accessible to CAS students while at the same time demonstrating the power of the techniques developed by Lee.

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1. RETROSPECTIVE RATING

As explained in Gillam and Snader [4], retrospective rating is an individual risk rating plan under which an insured's premium for a policy varies based on its experience during that policy period. For losses L, the retro premium, prior to the application of the minimum or maximum, is given by:

$$R = (b + ckE + cL)T,$$

where b is the basic premium, c is the loss conversion factor, k is the loss elimination ratio if an accident limit is selected,

E represents expected losses, and T is the tax multiplier. The retrospective premium varies between preselected minimum and maximum premiums. Usually, the plan is balanced to the guaranteed cost premium; i.e., the expected value of the retrospective premium should be equal to the standard premium less premium discounts. As explained in Gillam and Snader [4], this can be accomplished via the calculation of a net insurance charge using Table M.

Often the insurer and the insured agree to include in the retrospective rating plan an accident limit, which limits the dollars of loss that enter into the retrospective rating formula from any single accident. This stabilizes the insured's premium and protects the insured from the full impact of an extremely large accident. The imposition of an accident limit would also reduce the expected retrospective premium. Therefore, the insured must pay an additional amount for selecting an accident limit, so that the appropriate expected value of the retrospective premium is maintained. In the formula above, this impact was represented by the term $ckET$. Gillam [5] explains how excess loss factors (ELFs) can be used to quantify such an impact.

Skurnick [6] explains how Table L (which is based on the loss ratio distribution in the presence of an accident limitation) can be used to quantify the combined impact of the selection of minimum and maximum premiums together with the selection of an accident limit. Unfortunately, due to their interaction, the separate quantification of the effect of the former via Table M (which is based on the loss ratio distribution in the absence of an accident limitation) and of the latter via excess loss factors generally does *not* lead to the mathematically correct result (that is obtained in Skurnick via the use of Table L).

This paper will use Lee diagrams to explain this interaction and to illustrate how to quantify this error.

2. LEE DIAGRAMS

In Lee [3], a graphical technique is developed that is extremely useful for understanding retrospective rating.[1]

A key concept used in retrospective rating, as explained in Gillam and Snader [4], is the entry ratio. The entry ratio is defined as the observed loss ratio divided by the expected loss ratio. Equivalently, the entry ratio is the observed losses divided by the expected (unlimited) losses.

The Lee diagram has the entry ratio along the y-axis and probability along the x-axis. Figure 1 shows a relatively simple Lee diagram for retrospective rating without an accident limit. $F(x)$ is the cumulative distribution function for the (unlimited) entry ratios. Since the x-axis represents probability, as the entry ratio (y-value) increases, the distribution function approaches the vertical asymptote corresponding to a probability of unity. For an entry ratio of zero, the probability is zero in this example.[2] Generally, entry ratios are non-negative.[3]

Figure 1 is based on a simulation of 250 risks, each with an expected claim frequency of 100 accidents per year.[4] The

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[1]Lee uses the same techniques to illustrate applications to size of loss distributions as well as to retrospective rating.

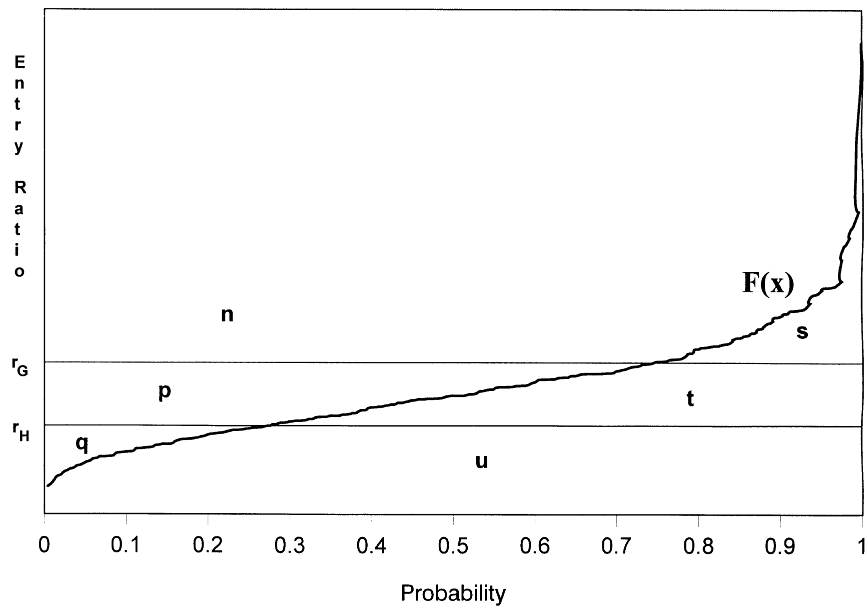
[2]For small risks, there is a significant probability of no losses in a year. For larger risks, such as those generally retrospectively rated, there is a small chance of no losses. For the example examined here, with 250 simulated risks with an average of 100 accidents each, the smallest observed entry ratio is .2177. Thus, in Figure 1, the cumulative distribution function $F(x)$ is zero for $x < .2177$. Since the entry ratios correspond to the vertical axis, the curve for $F(x)$ in Figure 1 hits the vertical axis at a height of about .2177.

[3]This follows from an assumption that actual losses are greater than or equal to zero and, therefore, entry ratios are greater than or equal to zero.

[4]In particular, the simulation employed a Poisson frequency assumption, based severity on random sampling from reported Massachusetts workers compensation claims, and assumed independence of frequency and severity. The Poisson frequency assumption was chosen for simplicity and may not reflect actual risks of this size. The results of this simulation are solely for illustrative purposes and many details of the behavior may not reflect actual insureds. However, we always expect $F(x)$ to be a non-decreasing function of x, even if, due to the limitations of the graphing software, it may not always appear to be so.

FIGURE 1

LEE DIAGRAM
RETROSPECTIVE RATING WITHOUT ACCIDENT LIMIT



Based on 250 simulated risks with an average of 100 accidents each.

diagram is intended solely for illustrative purposes; some details would differ depending on the particular risk process, but the general features would be retained. The same data was used as the basis for the diagram when an accident limit was imposed.

3. NOTATION

No Accident Limit

The notation used will, with minor exceptions, follow that in Skurnick [6]:

A = Actual (unlimited) losses for the risk for the policy period.

E = Expected (unlimited) losses.

r = A/E = (unlimited) entry ratio.

$f(r)$ = The probability density function of entry ratios.

$F(r)$ = The (cumulative) distribution function of entry ratios.

$\phi(r)$ = The Table M charge for entry ratio r
= The charge for entry ratio r, computed from F

$$= \int_r^\infty (s-r)f(s)\,ds = \int_r^\infty (1-F(s))\,ds.$$

$\psi(r)$ = The Table M savings for entry ratio r
= The savings for entry ratio r, computed from F

$$= \int_0^r (r-s)f(s)\,ds = \int_0^r F(s)\,ds.$$

$\{A\}$ = The losses that effectively enter the retrospective rating calculation with maximum premium G and minimum premium H

$$= \begin{cases} r_G E & \text{if} \quad A \geq r_G E \\ A & \text{if} \quad r_H E \leq A \leq r_G E \\ r_H E & \text{if} \quad A \leq r_H E. \end{cases}$$

$\{r\}$ = The entry ratio that effectively enters the retrospective rating calculation
= $\{A\}/E$.

G = Maximum premium.

r_G = Entry ratio corresponding to the maximum premium G. (The maximum premium G is attained when $A = r_G E$. Therefore, using the general formula for retrospective rating with $k = 0$, $r_G = G/cET - b/cE$.)

H = Minimum premium.

r_H = Entry ratio corresponding to the minimum premium H. (The minimum premium H is attained when $A = r_H E$. Therefore, using the general formula for retrospective rating with $k = 0$, $r_H = H/cET - b/cE$.)

As explained in Skurnick [6] and Gillam's review [7],

$$\mathrm{E}[\{r\}] = 1 + \psi_H - \phi_G.$$

With Accident Limit

A^* = The losses limited by the accident limit.
r^* = A^*/E = the limited entry ratio.
$f^*(r)$ = The density function for the limited entry ratios.
$F^*(r)$ = The distribution function for the limited entry ratios.
k = The loss elimination ratio[5] $= 1 - \mathrm{E}[A^*]/E$.
$\phi^*(r)$ = The Table L charge[6] for (limited) entry ratio r

$$= \int_r^\infty (s-r) f^*(s)\,ds + k$$

$$= \int_r^\infty (1 - F^*(s))\,ds + \int_0^\infty [F(s) - F^*(s)]\,ds.$$

$\psi^*(r)$ = The Table L savings for (limited) entry ratio r

$$= \int_0^r (r-s) f^*(s)\,ds$$

$$= \int_0^r F^*(s)\,ds.$$

4. LEE DIAGRAM, NO ACCIDENT LIMIT

In the case of no accident limit, the Lee diagram for retrospective rating (Figure 1) has horizontal lines corresponding to two entry ratios, r_G and r_H, related to a particular retrospective rating plan, and one distribution curve $F(x)$ for the (unlimited) entry ratios. This in general divides the diagram into six different non-overlapping areas, which have been labeled with small letters: n, p, q, s, t, and u.

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[5] This is the portion of losses eliminated from the retrospective rating calculation. In other contexts, this would be referred to as the excess ratio, since it represents the portion of losses in excess of the accident limit.

[6] Note that the integral is similar to that for $\phi(r)$, except that it involves the density function for limited rather than unlimited entry ratios. Also note the extra term of the loss elimination ratio.

r_G is the entry ratio corresponding to the maximum premium for the particular retro plan. r_H is the entry ratio corresponding to the minimum premium. $r_G = L_G/E$ where E represents the expected (unlimited) losses and L_G represents those losses that correspond to the maximum premium G. As explained in Gillam and Snader [4], generally one selects the values of G and H and solves for the values of r_G and r_H. Herein, for simplicity it will be assumed that r_G and r_H are given.

The area under $F(x)$ is equal to the average (unlimited) entry ratio, which is 1.0 by definition. Therefore,

$$\text{Area } s + \text{Area } t + \text{Area } u = 1.$$

The insurance charge at r_G is the integral from r_G to infinity of $1 - F(x)$. Therefore, it is the area above r_G that is between $F(x)$ and the vertical line corresponding to Probability = 1. This area has been labeled s, and

$$\text{Area } s = \phi_G.$$

Similarly, the insurance charge at the minimum is the area above r_H and between $F(x)$ and 1. Thus,

$$\text{Area } s + \text{Area } t = \phi_H, \qquad \text{and}$$

$$\text{Area } t = \phi_H - \phi_G.$$

Also

$$\text{Area } u = 1 - \phi_H.$$

Similarly, one can get the savings in terms of areas on the diagram. The savings at the minimum are given by the integral from 0 to r_H of $F(x)$. This is the area between the vertical line at Probability = 0 and $F(x)$ that is below the horizontal line at r_H.

Thus

$$\text{Area } q = \psi_H.$$

Similarly,

$$\text{Area } p + \text{Area } q = \psi_G, \qquad \text{and}$$
$$\text{Area } p = \psi_G - \psi_H.$$

The net insurance charge is defined as the charge at the maximum minus the savings at the minimum:

$$\phi_G - \psi_H = \text{Area } s - \text{Area } q.$$

For small entry ratios, the insured pays the minimum premium, and therefore the insured pays the same premium as if it had an entry ratio of r_H. Similarly, for large entry ratios, the insured pays the same premium as if it had an entry ratio of r_G. Define the effective entry ratio as

$$\{r\} = \begin{cases} r_H & r \leq r_H \\ r & r_H \leq r \leq r_G \\ r_G & r_G \leq r. \end{cases}$$

$\{r\}$ measures how much the insured effectively pays for losses (other than indirectly through the net insurance charge).

Referring to the Lee diagram, $\mathrm{E}[\{r\}]$ is represented by the area below the line/curve going from left to right starting at r_H, going along the horizontal line until it meets $F(x)$, proceeding along $F(x)$ until it meets the horizontal line at r_G, and finally proceeding along the horizontal line at r_G.

Thus

$$\text{Area } q + \text{Area } t + \text{Area } u = \mathrm{E}[\{r\}].$$

In terms of entry ratios (and thus ignoring expenses and taxes) the insured pays $\mathrm{E}[\{r\}]$ + net insurance charge = Area q + Area t + Area u + Area s − Area q = Area s + Area t + Area u = 1, which balances to the guaranteed cost result. In other words, the expected premium ignoring expenses and taxes is expected losses.

TABLE 1

RETROSPECTIVE RATING PLAN WITH NO ACCIDENT LIMIT AS SHOWN IN FIGURE 1

Area	In Symbols	Size
n	N.A.	N.A.
p	$\psi_G - \psi_H$	.2549
q	ψ_H	.0469
s	ϕ_G	.1018
t	$\phi_H - \phi_G$	.2451
u	$1 - \phi_H$	.6531

Note: $r_G = 1.20$ and $r_H = .70$. Based on 250 simulated risks with an average of 100 accidents each. The average severity is about \$10,500. The coefficient of variation of the severity is about 4.7. The skewness of the severity is about 20.6.

One should also note that the area under the horizontal line at r_G is equal to r_G, so:

$$\text{Area } p + \text{Area } q + \text{Area } t + \text{Area } u = r_G, \qquad \text{and}$$

$$\text{Area } q + \text{Area } u = r_H.$$

As pointed out in Lee [3], one can derive useful relationships easily using this diagram, for example, the fundamental relationship between charges and savings at a given entry ratio. Since Area $q = \psi_H$, and

$$\text{Area } u = 1 - (\text{Area } s + \text{Area } t) = 1 - \phi_H,$$

$$\text{therefore} \quad \psi_H + 1 - \phi_H = r_H.$$

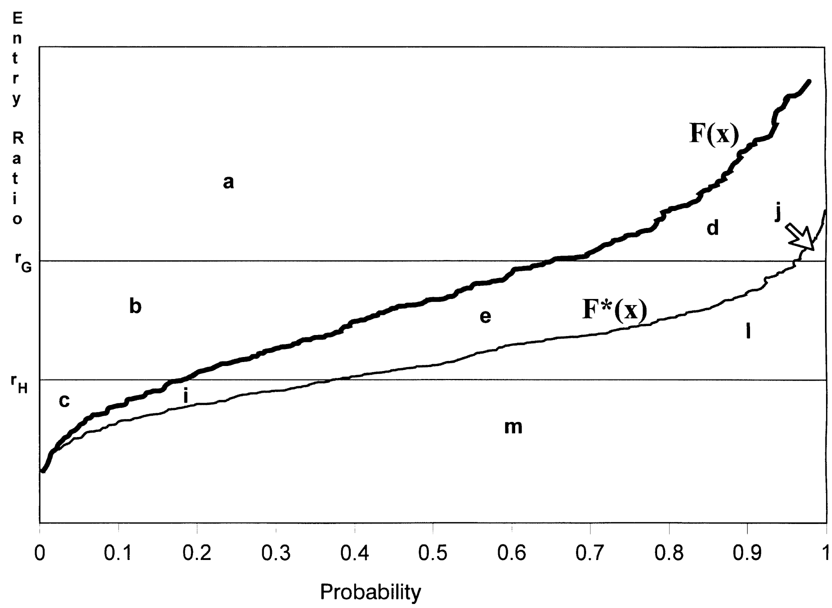
Table 1 summarizes this retrospective rating example.

5. LEE DIAGRAM, ACCIDENT LIMIT, TABLE L

The Lee diagram in Figure 2 relating to a specific accident limit has two distribution functions of entry ratios: $F(x)$ for unlimited losses and $F^*(x)$ for losses limited by the selected accident limit. For a given set of accidents, the unlimited losses are

FIGURE 2

LEE DIAGRAM, RETROSPECTIVE RATING WITH ACCIDENT LIMIT, TABLE L



Based on 250 simulated risks with an average of 100 accidents each, $100,000 Accident Limit, Loss Elimination Ratio of 0.314.

greater than or equal to the limited losses; the unlimited entry ratios are greater than or equal to the limited entry ratios. Thus $F(x)$ is above or equal to $F^*(x)$.

While $F(x)$ is usually above $F^*(x)$, for a sufficiently small entry ratio, $F(x)$ and $F^*(x)$ are identical. If the total unlimited losses for a risk in a year are less than the accident limit, we know none of the individual accidents can be affected by the accident limit. Thus, in this case, the limited and unlimited entry ratios are the same. In the particular example presented here, the average accident is about $10,500. For an expected frequency of 100, the expected losses are therefore about $1.05 million. A risk with total unlimited losses of \$100,000 or less is unaffected by a \$100,000 accident limit. Such a risk would have an entry ratio of 9.5% or less (\$100,000/\$1,050,000 = .095). Thus, for $x \leq .095$, $F(x) \quad F^*(x)$. In fact, for the 250 simulated risks in this case $F(x) \quad F^*(x)$ for $x \leq .30$.

In general, the curve for $F^*(x)$ branches off below the curve for $F(x)$ somewhere after the start. The higher the accident limit and/or the lower the expected losses the longer it takes for $F^*(x)$ to diverge. In this case, since the accident limit is small relative to the expected losses, $F^*(x)$ branches off relatively soon.

Since the average unlimited entry ratio is unity, the area under $F(x)$ is one, as it was for the previous Lee diagram.

The average limited entry ratio is $1 - k$, where k is the loss elimination ratio. (The average limited entry ratio = expected limited losses/expected unlimited losses = $1 - k$.) Thus the area under $F^*(x)$ is $1 - k$. (For an infinite limit, $k = 0$, which reduces to the unlimited case.) Therefore, the area between the $F(x)$ and $F^*(x)$ curves is always k. This result is very useful in working with the Lee diagram.

Using Figure 2 (with nine non-overlapping areas), as pointed out by Lee, one can derive many of the results in Skurnick [6], related to the use of Table L.[7]

$$\phi_G^* = k + \text{Area } j$$

$$= \text{Area } d + \text{Area } e + \text{Area } i + \text{Area } j.$$

$$\psi_H^* = \text{Area } c + \text{Area } i.$$

$$\text{Table L net insurance charge} = \phi_G^* - \psi_H^*$$

$$= \text{Area } d + \text{Area } e + \text{Area } j - \text{Area } c.$$

The expected value of the effective limited losses entering into the retrospective calculation is the area under the line

[7]Recall that the definition of the Table L charge ϕ^* is the sum of an integral (similar to the Table M charge) and the loss elimination ratio k.

starting at r_H, going horizontally until $F^*(x)$ is reached, along $F^*(x)$ until r_G is reached, and going horizontally until the vertical line Probability = 1 is reached. In other words, $\mathrm{E}[\{r^*\}] =$ Area c + Area i + Area l + Area m.

Using Table L, ignoring expenses and taxes, the insured pays on average the net Table L insurance charge plus $\mathrm{E}[\{r^*\}]$, which is

$$\text{Area } d + \text{Area } e + \text{Area } j - \text{Area } c + \text{Area } c$$
$$+ \text{Area } i + \text{Area } l + \text{Area } m$$
$$= \text{Area under } F(x) = 1.$$

Thus, the Table L plan balances to guaranteed cost; ignoring expenses and taxes, the insured pays for expected losses on average.

6. LEE DIAGRAM, ACCIDENT LIMIT, TABLE M

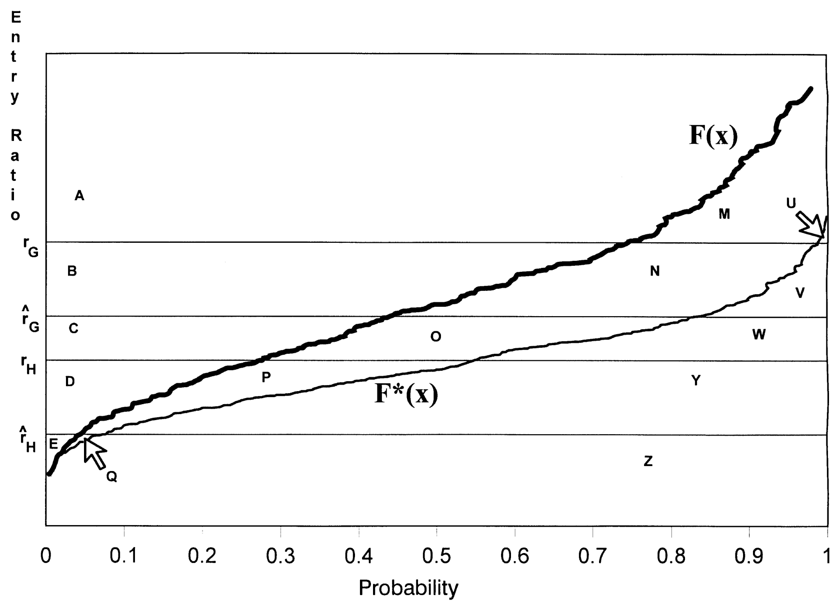
Figure 3 is a Lee diagram that is similar to Figure 2. There are horizontal lines corresponding to entry ratios r_G and r_H. However, there are also horizontal lines that correspond to additional entry ratios $\hat{r}_G$ and $\hat{r}_H$. We define $\hat{r}_G = \hat{L}_G/E$ where $\hat{L}_G$ is that level of (limited) losses such that including the charge for purchasing the accident limit but using the same basic premium as in the absence of the accident limit[8] we achieve the maximum premium.

It is assumed that the charge for the accident limit is the product of the loss conversion factor, tax multiplier, expected loss ratio, and loss elimination ratio. Thus we have ignored any risk loading that may be added to the excess loss factor. We assume the excess loss factor is kE.

[8]Using the same basic premium as in the absence of the accident limit results in an unbalanced plan, as will be discussed below.

FIGURE 3

LEE DIAGRAM, RETROSPECTIVE RATING WITH ACCIDENT LIMIT, SEPARATE USE OF TABLE M AND ELFS



Based on 250 simulated risks with an average of 100 accidents each, $100,000 Accident Limit, Loss Elimination Ratio of 0.314.

For (limited) losses L, the retro premium, prior to the application of the minimum or maximum, is given by

$$R = (b + ckE + cL)T,$$

where b is the basic premium, c is the loss conversion factor, and T is the tax multiplier. Thus we can solve for $\hat{L}_G$:

$$G = (b + ckE + c\hat{L}_G)T.$$

$$\hat{L}_G = \frac{G}{cT} - \frac{b}{c} - kE.$$

$$\hat{r}_G = \frac{\hat{L}_G}{E} = \frac{G}{cET} - \frac{b}{cE} - k.$$

Note that under the traditional (prior to insurance charge reflecting loss limitation[9]) Table M approach, the basic premium b is independent of the loss limit selected: the basic premium is calculated assuming no loss limit.[10] With no loss limit, $k = 0$ (there is no loss limitation charge), and therefore:

$$r_G = \frac{G}{cET} - \frac{b}{cE},$$

$$G = (b + cL_G)T,$$

$$L_G = \frac{G}{cT} - \frac{b}{c},$$

$$r_G = \frac{L_G}{E} = \frac{G}{cET} - \frac{b}{cE}.$$

Thus $\hat{r}_G = r_G - k$.

Similarly, with respect to the minimum rather than the maximum premium, $\hat{r}_H = r_H - k$. So, under the traditional Table M approach, the entry ratios $\hat{r}_G$ and $\hat{r}_H$ that actually achieve the maximum and minimum premiums are reduced by the loss elimination ratio k when a loss limit is selected, compared to the entry ratios r_G and r_H used in the calculation of the insurance charge that enters the basic. It is $\hat{r}_G$ and $\hat{r}_H$ that affect how often the maximum and minimum premiums are attained. Thus it is $\hat{r}_G$ and $\hat{r}_H$ rather than r_G and r_H that should be used to calculate the expected premiums when a loss limitation is selected.

The two distribution curves and four horizontal lines divide the Lee diagram (Figure 3) into a total of 15 different non-overlapping areas.[11] On Figure 3, they have been labeled using

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[9] See Robbin [2] for an explanation of ICRLL (insurance charge reflecting loss limitation).
[10] See Meyers [1].
[11] In particular situations, some of these 15 areas will be of zero size.

capital letters. As in the previous Lee diagrams, there are various relationships that always hold.

Since the area under the $F(x)$ curve is unity,

$$\text{Area } M + \text{Area } N + \text{Area } O + \text{Area } P + \text{Area } Q + \text{Area } U + \text{Area } V + \text{Area } W + \text{Area } Y + \text{Area } Z = 1.$$

The area under the $F^*(x)$ curve is $1-k$:

$$\text{Area } U + \text{Area } V + \text{Area } W + \text{Area } Y + \text{Area } Z = 1-k.$$

The area between the $F(x)$ curve and the $F^*(x)$ curve is k:

$$\text{Area } M + \text{Area } N + \text{Area } O + \text{Area } P + \text{Area } Q = k.$$

The Table M insurance charge for r_G is the area above the line r_G between $F(x)$ and the line Probability = 1:

$$\text{Area } M + \text{Area } U = \phi_G.$$

Similarly,

$$\text{Area } M + \text{Area } N + \text{Area } O + \text{Area } U + \text{Area } V + \text{Area } W = \phi_H$$

The Table M savings for r_H is the area below the line r_H between $F(x)$ and the line Probability = 0:

$$\text{Area } D + \text{Area } E = \psi_H, \qquad \text{and}$$

$$\text{Area } B + \text{Area } C + \text{Area } D + \text{Area } E = \psi_G.$$

The Table L insurance charge $\phi^*(r)$ is defined as the sum of k and an integral which corresponds to the area between $F^*(x)$ and Probability = 1, *above* the line corresponding to the chosen entry ratio. Thus,

$$\begin{aligned}\phi^*_G &= k + \text{Area } U \\ &= \text{Area } M + \text{Area } N + \text{Area } O + \text{Area } P \\ &\quad + \text{Area } Q + \text{Area } U,\end{aligned}$$

where the area between $F(x)$ and $F^*(x)$ is k, the loss elimination ratio.

Similarly,

$$\phi_H^* = \text{Area } M + \text{Area } N + \text{Area } O + \text{Area } P + \text{Area } Q$$
$$+ \text{Area } U + \text{Area } V + \text{Area } W, \qquad \text{and}$$
$$\phi_H^* - \phi_G^* = \text{Area } V + \text{Area } W.$$

The Table L savings $\psi^*(r)$ are defined as an integral that corresponds to the area between $F^*(x)$ and Probability = 0, *below* the line corresponding to the chosen entry ratio.

$$\psi_H^* = \text{Area } D + \text{Area } E + \text{Area } P + \text{Area } Q,$$
$$\psi_G^* = \text{Area } B + \text{Area } C + \text{Area } D + \text{Area } E + \text{Area } N$$
$$+ \text{Area } O + \text{Area } P + \text{Area } Q, \qquad \text{and}$$
$$\psi_G^* - \psi_H^* = \text{Area } B + \text{Area } C + \text{Area } N + \text{Area } O.$$

The net Table L insurance charge is

$$\phi_G^* - \psi_H^* = \text{Area } M + \text{Area } N + \text{Area } O + \text{Area } U$$
$$- (\text{Area } D + \text{Area } E).$$

The expected value of the effective limited losses entering the plan (based on Table L) is the area under the line starting at r_H, going horizontally until $F^*(x)$ is reached, along $F^*(x)$ until r_G is reached, and going horizontally until the vertical line corresponding to Probability = 1 is reached:

$$\mathrm{E}[\{r^*\}] = \text{Area } D + \text{Area } E + \text{Area } P + \text{Area } Q + \text{Area } V$$
$$+ \text{Area } W + \text{Area } Y + \text{Area } Z.$$

Using Table L, ignoring expenses and taxes, the insured pays on average the net Table L insurance charge plus $\mathrm{E}[\{r^*\}]$, which is

$$\begin{aligned}
&\text{Area } D + \text{Area } E + \text{Area } P + \text{Area } Q + \text{Area } V \\
&\qquad + \text{Area } W + \text{Area } Y + \text{Area } Z + \text{Area } M \\
&\qquad + \text{Area } N + \text{Area } O + \text{Area } U - (\text{Area } D + \text{Area } E) \\
&\quad = \text{Area under } F(x) = 1.
\end{aligned}$$

Thus, as was seen previously using Figure 2, the Table L plan balances to guaranteed cost; ignoring expenses and taxes, the insured pays for expected losses on average.

7. ERROR DUE TO INDEPENDENT USE OF TABLE M AND ELFs

As was seen above, the use of Table L produces a plan that balances to guaranteed cost. Ignoring expenses and taxes, the insured pays the expected losses on average.

In contrast, using Table M and ELFs independently, the retrospective rating plan will not, in general, balance to guaranteed cost. The net Table M insurance charge is

$$\phi_G - \psi_H = (\text{Area } M + \text{Area } U) - (\text{Area } D + \text{Area } E).$$

The expected value of the effective (limited) losses entering the plan is the area under the line starting at $\hat{r}_H$, going horizontally until $F^*(x)$ is reached, along $F^*(x)$ until $\hat{r}_G$ is reached, and going horizontally until the vertical line corresponding to Probability = 1 is reached. Note that we use $\hat{r}_H$ and $\hat{r}_G$, since these are the entry ratios at which the minimum and maximum premiums are attained when we add in the loss limitation charge. Thus, in this case,

$$\begin{aligned}
\mathrm{E}[\{r^*\}] = {}& \text{Area } E + \text{Area } Q + \text{Area } W \\
&+ \text{Area } Y + \text{Area } Z.
\end{aligned}$$

Thus, the average amount paid by the insured, including the loss limitation charge k, is

$$
\begin{aligned}
(\phi_G - \psi_H) + \mathrm{E}[\{r^*\}] + k \\
&= [(\text{Area } M + \text{Area } U) - (\text{Area } D + \text{Area } E)] \\
&\quad + [\text{Area } E + \text{Area } Q + \text{Area } W + \text{Area } Y + \text{Area } Z] \\
&\quad + [\text{Area } M + \text{Area } N + \text{Area } O + \text{Area } P + \text{Area } Q] \\
&= -\text{Area } D + 2(\text{Area } M) + \text{Area } N + \text{Area } O + \text{Area } P \\
&\quad + 2(\text{Area } Q) + \text{Area } U + \text{Area } W + \text{Area } Y + \text{Area } Z.
\end{aligned}
$$

One desires that, ignoring expenses and taxes, the insured pays on average for expected losses. This corresponds to the areas on the Lee diagram adding to unity, the area under $F(x)$ (Area M through Area Z).

In general, for the Table M case, the insured pays on average an amount different than unity. Comparing to unity (the area under $F(x)$), we find the error to be:

$$
\begin{aligned}
\text{Error} &= (\text{Table M Case}) - (\text{Area under } F(x)) \\
&= \text{Area } M + \text{Area } Q - \text{Area } D - \text{Area } V \\
&= (\text{Area } M - \text{Area } V) - (\text{Area } D - \text{Area } Q).
\end{aligned}
$$

The error consists of four separate areas on the Lee diagram, Figure 3. Area M and Area V involve the interaction of the maximum premium and the accident limit. Similarly, Area D and Area Q involve the interaction of the minimum premium and the accident limit.

8. ERROR TERMS INVOLVING THE MAXIMUM

Area M enters into the error term due to some double counting in the separate calculation of the losses eliminated from the retrospective rating plan due to the maximum premium and the accident limit. When the insured has one or more large accidents and a large (unlimited) loss ratio, some of the same dollars will be eliminated by both the maximum premium and the accident limit.

For example, take an insured with \$1.3 million in small accidents and a single \$2 million accident. With expected losses of \$1 million, the unlimited entry ratio is 3.3. With a \$100,000 accident limit, the limited entry ratio is 1.4. With a maximum entry ratio of 1.2, at most \$1.2 million of losses enter the retro calculation. Thus the maximum premium has reduced the losses entering the retro by \$3.3 − \$1.2 = \$2.1 million. The accident limit has reduced the losses by \$3.3 − \$1.4 = \$1.9 million. The total reduction seen by the insured is only \$2.1 million, *not* the sum of the two separately calculated effects. It is such examples of double counting that explain why Area M appears in the error as an overcharge to the insured.

Area V enters into the error term with a minus sign. It is there because the maximum entry ratio r_G used in the calculation of the Table M insurance charge assuming no accident limit charge is not the entry ratio at which a retro with an accident limit charge achieves the maximum. With the accident limit charge, it is easier to hit the maximum. Therefore, the maximum has more effect than we had calculated. Thus we have undercharged the insured.

One can rewrite the terms in the error involving the maximum:

$$\begin{aligned}\text{Area } M - \text{Area } V &= (\text{Area } M + \text{Area } U) - (\text{Area } U + \text{Area } V)\\ &= \phi_G - (\phi^*_{\hat{G}} - k),\end{aligned}$$

where the notation $\phi^*_{\hat{G}} = \phi^*(\hat{r}_G)$ has been used. In the particular example here, as shown in Table 2, Area M − Area V = .1007 − .0227 = .0780. In general, we expect this difference $\phi_G - (\phi^*_{\hat{G}} - k)$ to be positive, representing an overcharge to the insured. In other words, we expect $\phi_G > \phi^*_{\hat{G}} - k$.

TABLE 2

RETROSPECTIVE RATING PLAN WITH $100,000 ACCIDENT LIMIT AS SHOWN IN FIGURE 3

Area	In Symbols	Size
A	N.A.	N.A.
B	$\psi_G - \psi_{\hat{G}}$	.1881
C	$\psi_{\hat{G}} - \psi_H$	.0668
D	$\psi_H - \psi_{\hat{H}}$	.0439
E	$\psi_{\hat{H}}$	.0030
M	$\phi_G - \phi^*_G$	.1007
N	$(\phi_{\hat{G}} - \phi_G) - (\phi^*_{\hat{G}} - \phi^*_G)$	.1032
O	$(\phi_H - \phi_{\hat{G}}) - (\phi^*_H - \phi^*_{\hat{G}})$	.0620
P	$(\phi_{\hat{H}} - \phi_H) - (\phi^*_{\hat{H}} - \phi^*_H)$	.0467
Q	$k + \phi^*_{\hat{H}} - \phi_{\hat{H}}$	.0012
U	ϕ^*_G	.0011
V	$\phi^*_{\hat{G}} - \phi^*_G$	.0227
W	$\phi^*_H - \phi^*_{\hat{G}}$	.0572
Y	$\phi^*_{\hat{H}} - \phi^*_H$	.2234
Z	$1 - (k + \phi^*_{\hat{H}})$	.3818

Note: $r_G = 1.20$, $r_H = .70$, $k = .314$, $r_{\hat{G}} = .886$, and $r_{\hat{H}} = .386$. Same simulated data as described in Table 1.

This can be seen by comparing the two terms. The former is the integral, from r_G to ∞, of the amount by which the (unlimited) entry ratio exceeds r_G. The latter is a similar integral, but starts at $\hat{r}_G = r_G - k$, and the integrand is the amount by which the limited entry ratio exceeds $\hat{r}_G$.

On average, the difference between unlimited and limited entry ratios is k, the loss elimination ratio. However, larger-than-average entry ratios are more likely to be associated with larger accidents and vice versa.[12] Thus the imposition of the accident

[12]This is the case in real world situations. One can construct mathematical situations where this is not true.

limit will reduce large entry ratios on average by more than k. Therefore we expect fewer risks to have limited entry ratios that exceed $\hat{r}_G = r_G - k$ than have unlimited entry ratios that exceed r_G. Also we expect the amount by which the limited entry ratios exceed $\hat{r}_G = r_G - k$ to be less on average than the amount by which the unlimited entry ratios exceed r_G.

In summary, we expect the range of the integral corresponding to $\phi^*_{\hat{G}} - k$ to have fewer risks than the range of the integral corresponding to ϕ_G, and we expect the integrand of $\phi^*_{\hat{G}} - k$ to be less than the integrand of ϕ_G. Thus we expect $\phi^*_{\hat{G}} - k$ to be less than ϕ_G, as stated above. Therefore, the portion of the error term involving the maximums is expected to be positive, representing an overcharge to the insured.

Since this portion of the error term is Area M – Area V, one can arrive at the same conclusion by observing that on the Lee diagram, Figure 3, Area M is greater than Area V. One can use a geometric argument to show that, in general, Area M + Area U is greater than Area V + Area U.

Area M + Area U and Area V + Area U each are approximately right triangles, except that rather than a straight line hypotenuse, one has a portion of the curve $F(x)$ or $F^*(x)$. Area M + Area U is larger because, as will be shown, it has both a larger height and larger width than Area V + Area U.

First, we note that on Figure 3, the curves $F(x)$ and $F^*(x)$ start off equal and get further apart vertically as we go to the right. The area between $F(x)$ and $F^*(x)$ is k, while the horizontal axis goes from zero to one. Therefore, the average vertical distance between $F(x)$ and $F^*(x)$ is k. Thus the vertical distance between $F(x)$ and $F^*(x)$ is greater than k near the right edge of Figure 3, while it is less than k near the left edge.

Now the left-hand vertex of Area M + Area U occurs where $F(x)$ crosses the horizontal line r_G, which in this case occurs at the point $(.745, 1.2)$. Since, in this portion of the diagram, the vertical distance between $F(x)$ and $F^*(x)$ is greater than k, the point where $F^*(x)$ attains the same level of probability .745 will be more than $k = .314$ lower. In this example, that point on $F^*(x)$ is at $(.745, .815)$.

The horizontal line corresponding to $\hat{r}_G$ at .886 is $k = .314$ below r_G at 1.200. Therefore, since $F^*(x)$ is increasing, it intersects the horizontal line $\hat{r}_G$ to the right of $(.745, .815)$. In this example, this intersection of $F^*(x)$ and $\hat{r}_G$ is at $(.829, .886)$ and represents the left-hand vertex of Area V + Area U.

In general, the left-hand vertex of Area V + Area U will be to the right of the left-hand vertex of Area M + Area U. Since both triangular shapes have Probability = 1 as their right-hand edge, Area V + Area U has a smaller width than Area M + Area U.

In addition, since $F^*(x)$ is more than k below $F(x)$ while $\hat{r}_G$ is k below r_G, Area V + Area U has a smaller height than Area M + Area U. Therefore, Area V + Area U with both a smaller height and width is smaller than Area M + Area U. Thus, it has been shown geometrically that the portion of the error term involving the maximums is expected to be positive, representing an overcharge to the insured.

9. ERROR TERMS INVOLVING THE MINIMUM

There are two areas in the error term that relate to the minimum, which are subtracted from the terms involving the maximum. The minimum terms are:

$$\begin{aligned}\text{Area } D - \text{Area } Q &= (\text{Area } D + \text{Area } E) - (\text{Area } E + \text{Area } Q)\\ &= \psi_H - \psi^*_{\hat{H}}.\end{aligned}$$

We expect $\psi_H \geq \psi^*_{\hat{H}}$ or, equivalently, $\psi_H - \psi^*_{\hat{H}} \geq 0$. This follows from writing this difference of savings in terms of charges:[13]

$$\psi_H = \phi_H + r_H - 1, \qquad \text{and}$$

$$\psi^*_{\hat{H}} = \phi^*_{\hat{H}} + \hat{r}_H - 1 = (\phi^*_{\hat{H}} - k) + r_H - 1,$$

$$\text{therefore,} \quad \psi_H - \psi^*_{\hat{H}} = \phi_H - (\phi^*_{\hat{H}} - k).$$

But $\phi_H - (\phi^*_{\hat{H}} - k) \geq 0$ by the same reasoning that led to the conclusion that $\phi_G - (\phi^*_{\hat{G}} - k) > 0$.

While the entry ratios greater than r_H are overall greater than average, they are much closer to average than those greater than r_G. The extent to which these entry ratios are greater than average was the central thread of the reasoning that led to the conclusion that $\phi_G - (\phi^*_{\hat{G}} - k) > 0$. Therefore, we expect the difference $\phi_H - (\phi^*_{\hat{H}} - k)$ to be smaller than the difference $\phi_G - (\phi^*_{\hat{G}} - k)$. In the particular example, $\phi_H - (\phi^*_{\hat{H}} - k) =$ Area D – Area Q = .0439 – .0012 = .0427, while $\phi_G - (\phi^*_{\hat{G}} - k)$ = .0780.

We note that Area D is analogous to Area V and enters into the error terms for the same reason. Area D relates to the fact that the minimum premium is achieved at $\hat{r}_H = r_H - k$ rather than r_H, when an accident limit charge is included in the retro premium. Thus, with an accident limit, there are fewer times where the insured pays more due to the imposition of a minimum. Therefore, we are crediting the insured with too much savings. Area $D = \psi_H - \psi_{\hat{H}}$ represents the resulting undercharge of the insured.

Area Q is analogous to Area M and enters the error term for the same reason. Area Q relates to the interaction of the minimum and the accident limit. Some of the benefit of the accident limit is lost to the insured, because reducing the losses that enter

[13]Both of these equations can be derived in the Lee diagram. The second follows from the fact that the area under the line $\hat{r}_H$ is Area E + (Area Q + Area Z), so that $\hat{r}_H = \psi^*_{\hat{H}} + (1 - \phi^*_{\hat{H}})$.

the retro has no effect if one is already below the point at which the minimum premium will be charged.

For example, assume the minimum entry ratio $\hat{r}_H$ corresponds to \$400,000 in losses, there is an accident limit of \$100,000, and an insured had a single \$250,000 accident. The insured will pay the minimum premium whether the full \$250,000 enters the retro calculation or the accident limited to \$100,000 enters the retro calculation. In this case the insured gained no benefit from the accident limit. Yet the ELF is based on the loss elimination ratio k, which includes as part of its average this \$150,000 reduction. Thus the insured is being charged for something which provides no benefit. Area $Q = \psi^*_{\hat{H}} - \psi_{\hat{H}}$ quantifies this overcharge to the insured.

10. ERROR TERM, SUMMARY

The error due to the separate use of Table M and ELFs has four terms:

$$\begin{aligned}\text{Error} &= (\text{Area } M - \text{Area } V) - (\text{Area } D - \text{Area } Q)\\ &= \{\phi_G - (\phi^*_{\hat{G}} - k)\} - \{\phi_H - (\phi^*_{\hat{H}} - k)\}\\ &= \{\phi_G - (\phi^*_{\hat{G}} - k)\} - (\psi_H - \psi^*_{\hat{H}}).\end{aligned}$$

The first two terms are related to the maximum premium and the second two terms are related to the minimum premium. In actual applications, we expect to find generally that the error is a difference of two positive terms with the first one being larger, resulting in a positive error. In general, we expect an overcharge to the insured.

In the particular example here, the error = .0780 – .0427 = 3.53% of expected losses.

It may also be useful to rewrite the error as

$$\text{Error} = (\phi_G - \psi_H) - (\phi^*_{\hat{G}} - k - \psi^*_{\hat{H}}).$$

In order to correct for this error one would remove the net Table M insurance charge $\phi_G - \psi_H$ and substitute the net Table L insurance charge $\phi^*_{\hat{G}} - \psi^*_{\hat{H}}$, excluding the charge for the loss limitation k, at a lower set of entry ratios $\hat{r}_G$ and $\hat{r}_H$.

11. CONCLUSION

The graphical methods in Lee have been used to demonstrate how to quantify the error that would result from a separate use of Table M and Excess Loss Factors. This error usually represents a net overcharge to the insured. There are two main concepts that are responsible for this error. First, the effect of the maximum or minimum premiums each interact with the effect of an accident limit; one must be careful to not count the same effect twice. Secondly, the addition of an accident limit charge into the retrospective rating formula lowers the entry ratios corresponding to the maximum and minimum premiums.

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