# PRICING TO OPTIMIZE AN INSURER'S RISK-RETURN RELATION

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*Abstract*

*It is appealing to estimate loss discount rates and risk loads for categories of an insurer's premium by using the categories' contributions to surplus variation. However, as will be explained, there has been a theoretical obstacle to this approach.*

*This paper presents a method that overcomes the obstacle. It produces a surprisingly simple result. The risk load (in dollars) of a category is proportional to the covariance of the yearly return on surplus with the category's yearly profit.*

*The paper analyses the use of the above result to optimize an insurer's risk-return relation. Some examples of computations of risk loads and risk-based discount rates for losses are presented. The relationship between the method of this paper, the Capital Asset Pricing Model, and several other models is discussed.*

## 1. INTRODUCTION

A few years ago, a Nobel Prize was awarded to Harry Markowitz [10] for developing a method of producing a diversified portfolio of stocks with the optimal relationship between expected rate of return and expected variability. In other words, Markowitz showed how to maximize the expected rate of return for a fixed amount of expected variability and, alternatively, how to minimize the variability at a fixed rate of return. Markowitz's method has been widely used by large investors because of their desire to lower the variability of their results.

Insurance company managers are also interested in reducing variability. Taking steps to reduce risk helps a company with its Best's rating and also increases the security of its employees and its policyholders. These actions help in attracting good business and retaining good employees, and produce increased profitability in the long run. Therefore, insurers generally require a greater profit margin on a risk with greater volatility.

Suppose that an insurer expects to write a certain volume and mix of business in the next year, and that the insurer has a certain target profit. The method presented in this paper produces a risk load for each risk such that the total expected profit equals the target and each risk is equally advantageous to the insurer in the following sense. If the insurer can charge more than the indicated risk load for any type of risk, then by increasing the proportion of that type of risk in the total book of business, the insurer can increase the expected return without increasing the surplus variability. Conversely, if the insurer charges less than the indicated price, increasing the proportion of that type of risk will decrease the expected return if variability is left constant.

The term "risk load" is sometimes given a different meaning than it is given above. Other meanings of the term include:

1. The risk load that a customer is willing to pay. This may be based on the market, or on the risk aversion of the customer.
2. The risk load that an underwriter desires, based on the possible effect that a contract may have on the total results of the contracts he or she has underwritten, or on the results of a profit center within the company.

The method presented here produces an indicated price for each risk by discounting losses and loss adjustment expenses at a risk-based rate and then adding a risk load as well as other expenses. As will be explained later, the risk loads and discount rates are produced by allocating surplus to categories of underwriting and loss reserves. This allocation is based on the contribution of these categories to surplus variability. The measure of surplus variability used in this paper is the "standard deviation of surplus," which is defined below.

Assume that, at each time $t$, there is an estimated value of surplus. Let the random variable $X$ represent the estimated value one year in the future. The "standard deviation of surplus" is defined as the standard deviation of $X$.

A problem with allocating surplus based on each category's contribution to surplus variability is that the effect of a category on the standard deviation of surplus *cannot* be estimated by simply estimating the standard deviation of surplus with and without the category, and then taking the difference. The explanation of this is as follows. (See Gogol [7].)

The standard deviation of surplus equals the standard deviation of the sum of the effects on surplus of all the categories of underwriting, loss reserves, other liabilities, assets, and other sources of income and expense. Suppose those categories are arranged in a list. Suppose the effect of each category on the total standard deviation is defined as the difference between the standard deviation of the sum of the categories up to and including that category on the list, and the standard deviation of the sum of the categories prior to it on the list. The sum of all these "effects" equals the total standard deviation, but the effect of a particular category depends on the order of the list. (Suppose, for example, that there is a list of two independent categories each with standard deviation $\sigma$. The standard deviation of the sum is $2^{.5}\sigma$. The effect of the first category in the list is $\sigma$, and the effect of the second is $2^{.5}\sigma - \sigma$.)

This dependence on the order in which the categories are listed has been considered a barrier to using contribution to surplus variability to estimate required risk loads. This paper will propose a solution. The following quotations from Venter [12] give an interesting description of the problem.

In 1953, Harry Markowitz developed a way of selecting optimal holdings for each available security if you were clear about your preferred mean-variance trade-off. This has been applied to optimal line mix strategies for insurers as well.

It's tempting for actuaries to invent (or re-invent) the Mean-Variance Pricing Model (MVPM).

Presumably the change in variance of your whole portfolio of risks or securities is more important than that of the new entrant by itself.

MVPM could be applied to the portfolio with and without the new entrant, whose price then becomes the difference. But then the order of entry will influence the price, which it should not. Or you could estimate in advance the make-up of the portfolio and then pro-rate to each unit a credit based on the reduction in variance achieved by the combination. The mind boggles. Besides needing a fair way to allocate credits, which this theory does not provide, any difference from the predicted result will give the wrong price overall. Because of covariance, MVPM does not seem usable for pricing individual risks in a portfolio.

## 2. ESTIMATING RISK-BASED PREMIUM

### A. *Return on Allocated Surplus*

The surplus considered in this paper is a type of adjusted surplus, using the market value of assets and a risk-based discounted value for loss reserves.[1] Statutory liabilities such as equity in the unearned premium reserve are included in the surplus. The value of the assets necessary to offset the discounted loss reserve lia-

[1] In this paper "loss reserves" will mean loss and loss adjustment expense reserves, net of ceded losses. "Earned premium" will refer to premium net of cessions.

bility is considered here to be greater than the discounted value of loss reserves at the "risk-free" interest rate (see Butsic [3]). This is because it would be necessary to pay an insurer more than this amount, as a reward for risk, in order for them to be willing to assume this liability. By using a lower discount rate to determine the loss reserve liability, the following is expected to occur. In the course of a year, the value of the offsetting assets is expected to grow at a greater rate of interest than was used to discount the liability, providing a profit for the risk of having the liability.

Suppose that each category of loss reserves is considered to be offset by an amount of assets that is equal to the risk-based discounted value of the reserves. The expected effect on surplus one year in the future of a category of discounted loss reserves and offsetting assets equals the accumulated value of the assets after one year of reserve payouts, minus the discounted value of the remaining reserves and the tax effects of the assets and liabilities.

The expected effect of a category of underwriting on the surplus one year in the future equals the effect of the premium minus the effect of the corresponding paid losses, discounted loss reserves, expenses, and taxes.

Suppose an amount of surplus is allocated to a category of underwriting, or to a category of loss reserves and offsetting assets. Then the expected return on the allocated amount during the year is the after-tax investment gain on it plus the expected effect of the category on surplus. The rate of return is the return divided by the amount of surplus.

### *B. Method of Allocation*

Just as there is a probability distribution of the amount of surplus one year in the future, there are probability distributions of the effects on surplus of each category of underwriting, or of each category of discounted loss reserves and offsetting assets. A basic part of the method of this paper is the idea that the appropriate amount of surplus to allocate to a category of underwriting, or to a category of discounted loss reserves and matching assets, is equal to total surplus times

$$\frac{\text{Cov(total surplus, effect of category on total surplus)}}{\text{Var(total surplus)}}.$$

It will be shown below, by Theorem 1, that in a certain sense the above covariance of a category with surplus is proportional to the category's effect on surplus variability. It is shown by Theorem 2 that if surplus is allocated to each category of underwriting according to the above formula, and the appropriate risk-based loss discounting rate is used, the following is true. Each category will improve the risk-return relation of the insurer if, and only if, its rate of return on allocated surplus is greater than the rate of return on the total amount of surplus allocated to underwriting.

It is a property of covariance that the covariance with surplus of a sum of categories equals the sum of the covariances. Therefore, the surplus allocated to a sum of categories is the same whether the surplus is allocated based on the covariance of the sum, or allocated to each individual category based on its covariance. This would not be true if surplus were allocated in proportion to the standard deviation or variance of a category's effect on surplus.

Thus, the amount of surplus allocated to a category is independent of how finely the categories are subdivided. For example, the amount of surplus allocated to private passenger auto does not depend on whether it is considered to be one category or whether it is split into private passenger auto liability and private passenger physical damage.

Surplus variability is caused not only by underwriting and by loss reserves and offsetting assets, but also by other assets. If surplus is allocated to all sources of surplus variability, and these sources are referred to as "categories" 1 through $n$, then

$$\sum_{i=1}^{n} \text{Cov(surplus, effect of category } i \text{ on surplus)}$$

$$= \text{Cov}\left(\text{surplus, effect of} \left(\sum_{i=1}^{n} \text{category } i\right) \text{ on surplus}\right)$$

$$= \text{Cov(surplus, surplus)} = \text{Var(surplus)}.$$

Therefore, the proportions of surplus allocated to the categories sum to unity.

### *C. Risk-Based Underwriting Margin and Discount Rate*

To understand how to apply the method of this paper, it is helpful to consider the following questions:

1. What risk-based discount rate should be used for loss reserves?
2. How much surplus should be allocated to loss reserves, and how much to underwriting?

Suppose the insurer's loss reserves are discounted, both at the beginning and end of the year, at a discount rate $d$. Suppose that, with this rate $d$, surplus is allocated by the above covariance formula to underwriting and to discounted loss reserves and offsetting assets. Lastly, suppose that the rate of return on allocated surplus from underwriting and from discounted loss reserves and offsetting assets is called rate $R$.

Call the amounts of surplus allocated to underwriting and to discounted loss reserves and offsetting assets $S_u$ and $S_r$, respectively. It was mentioned above that the surplus allocated to a sum of categories by the covariance method is equal to the sum of the amounts allocated to the individual categories. Suppose for the moment that, for each category of loss reserves and offsetting assets, the discount rate $d$ produces the same rate of return on allocated surplus. Since the sum of the amounts of surplus allocated to each category equals $S_r$, this rate of return equals $R$.

Suppose that, for some underwriting category $c$, the rate of return on the surplus allocated to the category, using the discount rate $d$, is $R$. Thus, the premium not only provides a rate of return on allocated surplus equal to the rate of return on $S_r$ and $S_u$, but also provides for the offsetting assets for its loss reserves at the end of the year. Assuming that the required discount rate remains the same, these reserves and offsetting assets are expected to produce a rate of return $R$ on allocated surplus in each following year. This is a key point, since it means that the expected effect on surplus of the loss reserve runoff from Category $c$ neither helps nor hurts the insurer's risk-return relation.

It will be shown by Theorem 2 that in a certain sense the covariance method allocates surplus in proportion to a category's effect on surplus, and it follows that the Category $c$ neither helps nor hurts the insurer's risk-return relation. This explains what conditions a category or contract must satisfy in order to help optimize that relation.

A discount rate $d$ with the above properties may be found by iteration, as outlined below. (See Example A in Section 4 for additional explanation.) Suppose the insurer expects to earn a given amount of premium in the coming year, with a given expected loss ratio and expense ratio. Certain estimates are made relating to loss payout rates, loss reserve variability, asset variability, underwriting variability, and various correlations, and an initial value of the discount rate is selected.

The value of the discount rate affects the estimated amount of surplus as well as:

1. the covariance with surplus of the total effect on surplus of discounted loss reserves and offsetting assets;
2. the covariance with surplus of the total effect on surplus of all underwriting categories;
3. the total amounts of surplus allocated by the above two covariances; and
4. the rates of return on the above two amounts of surplus.

Iteration is used to find a discount rate $d$ that makes the above two rates of return equal.

It isn't actually necessary to assume that a single discount rate $d$ produces the same rate of return on the amounts of surplus allocated to each category of loss reserves and offsetting assets. The indicated discount rate may vary for different categories, and thus it may be appropriate to use different discount rates in estimating the required risk-based premiums for different underwriting categories. This would require a more complicated iteration than the one described above. This may not be preferable from a practical point of view. The need for a great deal of judgment in estimating covariances with surplus will be discussed further in Section 3E.

The theoretical significance of the allocation method is indicated by the following two theorems. The proofs[2] are in the Appendix.

*Theorem 1*

Using any discount rates for each category of loss reserves and for each category of underwriting, suppose a pro rata share of $1/n$ of each category of one year underwriting results, loss reserves and offsetting assets, and other assets, liabilities, expenses, and sources of income affecting surplus is added to a list, and

[2]It will be assumed in the proofs that the covariance of a category with surplus is not zero. The case in which the covariance equals zero will be left to the reader.

this is done $n$ times. The limit as $n$ approaches infinity of the total of the $n$ effects of a category on the standard deviation of surplus,[3] divided by the total standard deviation of surplus, equals

$$\text{Cov(surplus, effect of category on surplus)}/\text{Var(surplus)}.$$

*Theorem 2*

Suppose that an insurer can charge more premium for a category of underwriting than the required risk-based premium described above. Then, by increasing the proportion of that category in the total book of business, the insurer can increase the expected return without increasing surplus variability. Specifically, there is some epsilon such that the expected return on surplus will increase if the following are assumed.

a. The premium for the category is increased by less than epsilon.
b. The expected underwriting return and the standard deviation of underwriting return for the category increase by the same proportion as the premium, and the correlation of its return with surplus is unchanged.
c. The rest of the insurer's premium is reduced by an amount such that total surplus variance remains the same.
d. The expected underwriting return and standard deviation of underwriting return for the rest of the premium decrease by the same proportion as the rest of the premium, and the correlation of its return with surplus is unchanged.

Conversely, a contract written at less than the required risk-based premium will decrease the expected return.

[3]The effect of each category on the standard deviation was defined in Section 1 as the difference between the standard deviation of the sum of the categories up to and including that category on the list, and the standard deviation of the sum of the categories prior to it.

## 3. DISCUSSION OF THE METHOD

### *A. Overall Premium Targets*

The method presented above estimates the required risk-based premium for a contract or category, given certain overall expectations or targets of the insurer. These expected values or targets include the overall loss ratio, expense ratio, payout rate, and mix of business for the coming year. Covariances of categories with surplus are estimated based on these expected values. The method applies to individual underwriting decisions concerning contracts or categories of business, but it does not indicate what the overall mix or amount of premium should be. It is assumed that there are practical constraints against making drastic shifts in the current mix of business. An insurer is not free to simply choose any portfolio of business in the way that a stockholder can choose a portfolio of stocks.

If an insurer increases or decreases its premium, or changes the mix of business, these changes have an immediate effect, as well as an additional long term effect, on the insurer's combined ratio, total return on surplus, and variability of surplus. In the long run, increased variability can make an insurer less attractive to its employees and its clients, and can adversely affect its combined ratio and return on surplus.

If certain estimates are made, it is possible to use the Capital Asset Pricing Model (CAPM) to help in selecting the volume of premium which maximizes the market value of the insurer. This model (Lintner [9] and Sharpe [11]) will be discussed further in the last section of the paper. In actual practice, insurer managements are more likely to use informed judgment than CAPM.

### *B. One Year Variability*

The one year time frame used for optimizing the risk-return relation is also intended to optimize this relation over the long term. Long term variability may be thought of as a sum of one year random variables.

Sometimes it may be more natural to estimate the long term variability for a category than to estimate the one year variability. Loss reserves for environmental and mass tort (E/MT) claims is an example of such a category. The estimate of the one year variability for E/MT reserves should be selected in a way that is consistent with estimated long term variability.

Let random variable $X_i$ represent the effect of this category on surplus in the year $i$. Let $Y_i$ equal $X_i - \mathrm{E}(X_i)$. $\mathrm{E}(X_{i+1})$ is based on the probability distribution of $X_{i+1}$ at the end of year $i$, so the fact that $X_i$ is greater or less than $\mathrm{E}(X_i)$ has no bearing on how $X_{i+1}$ will differ from the mean of its distribution. Therefore, $Y_{i+1}$ is independent of $Y_i$.

Similarly, for each integer $k > 1$, $Y_{i+k}$ is independent of $Y_i$. Therefore, the sequence of observations $Y_1, Y_2, \ldots$ is a stochastic process for which each value is independent of previous values.

### *C. Loss Reserve Variability and Discounting*

The estimates of loss reserves referred to in this paper are assumed to be unbiased, although annual statement estimates may be biased. Thus, the estimates do not necessarily equal the risk-based discounted values of annual statement estimates.

The reader may have noticed that the variability of loss reserves has been addressed in the paper, but not the variability of the unearned premium reserve. This is because the variability associated with this reserve is included in the underwriting variability for the coming year.

The definition of surplus in this paper uses a risk-based discounted value for the loss reserves. The corresponding value of surplus is not necessarily the market value of the insurer. For one thing, it excludes franchise value. However, it appears that optimizing the risk-return relation for this surplus, as discussed in this paper, should be a good approximation to optimizing the risk-return relation for market value.

### *D. Asset Variability*

An attempt can be made to minimize the effects of interest rate variability on surplus. A relatively simple method is to choose a mix of assets with a "duration" (see Ferguson [5]) such that interest rate changes have the same effect on the value of assets as on the value of liabilities. To apply this duration method, using the definition of surplus in this paper, it is necessary to estimate the effect of interest rate changes on the risk-based loss discounting rate. The correlation between interest rates and inflation, and the effect of inflation on estimated loss reserves, must also be estimated.

An insurer may find that duration matching of assets and liabilities requires an asset portfolio with a shorter duration than is desired. Shorter duration bonds have a lower interest rate.

Changing the mix of assets, including stocks, can be used as a tool in attempting to optimize an insurer's risk-return relation. The correlation of the insurer's return with "market return" (i.e., the average return for the market of all capital assets) should be taken into account in such an attempt. This is discussed briefly in Section 5, which contains a comparison of the method of this paper with the Capital Asset Pricing Model. However, the subject of optimizing an insurer's mix of assets is beyond the scope of this paper.

### *E. Estimation Problems*

The covariance between the effects on surplus of any two categories $a$ and $b$ will be denoted by $\mathrm{Cov}(a,b)$. The covariance of Category $c$ with all other sources of surplus variability will be denoted by $\mathrm{Cov}(c,s-c)$.

Let the variance of the effect on surplus of a Category $c$ be denoted by $(\sigma_c)^2$. Denote the correlation between the category and surplus by $\rho_{c,s}$. Note that

$$\mathrm{Cov}(c,s) = \mathrm{Cov}(c,c) + \mathrm{Cov}(c,s-c) = (\sigma_c)^2 + \sigma_c\sigma_{s-c}\rho_{c,s-c}.$$

Therefore, for a Category $c$ that is small, the estimate of $\mathrm{Cov}(c,s)$ is very sensitive to the estimate of $\rho_{c,s-c}$. This is a problem, due to the low credibility of the related data. From a practical point of view, it is best to implement the method of this paper by starting with estimates relating to the largest categories.

For example, a practical first step would be to allocate surplus to the category of all loss reserves and offsetting assets and to the category of all underwriting. This determines the risk-based discount rate for the category of all loss reserves, and the risk-based profit margin on discounted underwriting results.

A reasonable second step would be to allocate surplus to the sum of all property underwriting categories and to the sum of all casualty underwriting categories. (Note that the sum of these two amounts of surplus equals the amount of surplus allocated in the first step to the category of all underwriting.) These allocations determine risk-based profit margins for property and casualty as a whole.

The problem of implementing the method is a vast one, and the examples in the next section are intended only as illustrations. In practice, it is necessary to use a considerable amount of judgment, in addition to making a study of relevant historical data.

## 4. EXAMPLES OF APPLICATIONS

### A. *Overall Underwriting Risk Load and Overall Discount Rate*

Suppose the following for some insurer:

1. Risk-free interest rate on assets = 6%.
2. Loss reserves at start of year discounted at 3% = \$500,000,000.
3. Discounted value of amount of loss reserves expected to be paid during year = \$100,000,000.

4. Present discounted value of loss reserves not expected to be paid during year = \$400,000,000.
5. Expected earned premium for coming year = \$150,000,000.
6. Expected underwriting expenses to be incurred during year = \$40,000,000.
7. Expected current accident year losses to be paid during year = \$45,000,000.
8. Expected value of loss reserves at end of year for current accident year discounted at 3% = \$50,000,000.
9. The pre-tax contributions to surplus of loss reserves and offsetting assets, and of underwriting, are in the same proportion as the corresponding after-tax effects.

Assume that the expected expense and loss ratios equal the targets that were discussed in Section 3A. "Risk load" will be taken to mean "risk-based underwriting margin," which was discussed in Section 2C. The after-tax effect on surplus of loss reserves and offsetting assets will be called the return from loss reserves. The after-tax effect on surplus of underwriting will be called underwriting return. These returns do not include investment income on allocated surplus.

Using the 3% discount rate, the expected one year pre-tax return from loss reserves and offsetting assets, assuming loss reserves paid during the year are paid on average in the middle of the year, is (as explained below):

$$(\$500{,}000{,}000)(1.06) - (\$100{,}000{,}000)(1.03)^{.5}(1.06)^{.5}$$

$$- (\$400{,}000{,}000)(1.03) = \$13{,}511{,}000.$$

By the end of the year, the \$400 million in loss reserves that are not expected to be paid during the year grows to \$400,000,000 × (1.03) due to one year's unwinding of discounting. The \$400 million in offsetting assets grows, from investment income, to \$424 million, producing a pre-tax return of $\$400{,}000{,}000 \times (1.06 - 1.03)$. A loss reserve payment of $\$100{,}000{,}000 \times (1.03)^{.5}$ is made in the middle of the year (on average), reducing the assets that were offsetting those reserves to $\$100{,}000{,}000 \times ((1.06)^{.5} - (1.03)^{.5})$. By the end of the year, these assets grow by a factor of $(1.06)^{.5}$ to $\$100{,}000{,}000 \times ((1.06) - (1.03)^{.5}(1.06)^{.5})$. This expression plus the above $\$400{,}000{,}000 \times (1.06 - 1.03)$ is equal to the left side of the above equation.

If it is assumed, for the sake of simplicity, that the earned premium is received in the middle of the year, and that the underwriting expenses and accident year losses are paid in the middle of the year, then the expected pre-tax return on underwriting is

$$(1.06)^{.5}(\$150{,}000{,}000 - \$40{,}000{,}000 - \$45{,}000{,}000)$$

$$- \$50{,}000{,}000 = \$16{,}922{,}000.$$

Approaches to estimating the covariances of loss reserve return with surplus, and of underwriting return with surplus, will be discussed after the following brief description of the iterative process.

Suppose that, using the 3% discount rate, the above two covariances, respectively, are in the proportion $A : 1$. The corresponding rates of return on allocated surplus are then in the proportion $13{,}511/A : 16{,}922$. Call this proportion $B : 1$. Suppose that using a 4% discount rate changes the proportion of rates of return from $B : 1$ to $C : 1$. Since the goal is to make the rates of return equal, a reasonable next step in the iteration would be

$$4\% + (3\% - 4\%)((1 - C)/(B - C)).$$

Suppose for the sake of illustration that the 3% rate is the solution to the iteration. It then follows from the formula for pre-tax return on underwriting that

$$\begin{aligned}\$150{,}000{,}000 &= \$40{,}000{,}000 + \$45{,}000{,}000\\ &\quad + (1.06)^{-.5}(\$50{,}000{,}000)\\ &\quad + (1.06)^{-.5}(\$16{,}922{,}000).\end{aligned}$$

In other words, the premium equals expected expenses (i.e., \$40,000,000) + expected discounted losses (i.e., \$45,000,000 + $(1.06)^{-.5}$(\$50,000,000)) + risk load (i.e., $(1.06)^{-.5}$(\$16,922,000)).

The covariance of the loss reserve return, and of the underwriting return, with surplus can be estimated based on the insurer's historical data. The insurer's loss reserve runoff variability, its loss ratio and expense ratio variability, the duration of its loss reserves, the duration of its assets, and the historical variability of interest rates are all relevant.

Variability in the loss reserve return is caused by differences between the estimated loss reserve and the one year runoff, changes in market values of offsetting assets, changes in estimated risk-based discount rates, and changes in estimated payout rates for loss reserves. To some extent, changes in asset values caused by interest rate changes are offset by corresponding changes in discount rates. Variability in the underwriting return results from variability in asset values, loss ratios, expense ratios, payout rates, and discount rates.

One way of estimating the covariances is as follows. For some period of years, estimates are made of what the expected increases in surplus, and the expected returns from loss reserves and underwriting, would have been at the beginning of each year. (Note that surplus is increased by the return on other assets as well as those offsetting reserves.) These estimates are then compared with what would have been estimated for each of those returns at the end of the same year.

For each year, all the above estimates can be brought to the level of the current year. The estimated loss reserves return for each year can be multiplied by a factor equal to the reserves at the beginning of the current year divided by the beginning reserves for the year. A similar on-level adjustment can be made for estimated underwriting return, based on the premium for the years. For the on-level factor for return on assets other than those offsetting reserves, the amount of those assets can be used. As mentioned above, the estimated increase in surplus is the sum of the above three estimated returns, so the on-level estimate is the sum of the three on-level estimates.

The covariances of the loss reserves and underwriting returns with surplus can then be estimated as shown in the example below. The example is intended to illustrate a method of computation, but in actual practice many more years of data would be used. For each year listed, each of three types of return for the year are estimated at 1/1 and then at 12/31. It is assumed that the 1/1 estimates equal the means of the probability distributions of possible 12/31 estimates.

TABLE 1

| | Estimated Loss Reserve Return (000's) | | Estimated Underwriting Return (000's) | | Estimated Increase in Surplus (000's) | |
|---|---|---|---|---|---|---|
| Year | 1/1 | 12/31 | 1/1 | 12/31 | 1/1 | 12/31 |
| 1990 | $13,600 | $12,800 | $33,000 | $28,600 | $81,600 | $75,600 |
| 1991 | $13,200 | $14,200 | $31,400 | $25,600 | $80,800 | $86,000 |
| 1992 | $19,400 | $18,600 | $28,400 | $39,600 | $77,400 | $81,900 |
| 1993 | $17,000 | $15,000 | $21,400 | $18,200 | $62,200 | $57,200 |
| 1994 | $18,900 | $14,400 | $22,700 | $24,200 | $63,100 | $59,500 |

Based on the data in Table 1, the estimated covariances with surplus are as follows (000,000's):

*Loss Reserve Return:*

$$(1/5)((12{,}800 - 13{,}600)(75{,}600 - 81{,}600)$$
$$+ (14{,}200 - 13{,}200)(86{,}000 - 80{,}800)$$
$$+ (18{,}600 - 19{,}400)(81{,}900 - 77{,}400)$$
$$+ (15{,}000 - 17{,}000)(57{,}200 - 62{,}200)$$
$$+ (14{,}400 - 18{,}900)(59{,}500 - 63{,}100)) = 6{,}520{,}000.$$

*Underwriting Return:*

$$(1/5)((28{,}600 - 33{,}000)(75{,}600 - 81{,}600)$$
$$+ (25{,}600 - 31{,}400)(86{,}000 - 80{,}800)$$
$$+ (39{,}600 - 28{,}400)(81{,}900 - 77{,}400)$$
$$+ (18{,}200 - 21{,}400)(57{,}200 - 62{,}200)$$
$$+ (24{,}200 - 22{,}700)(59{,}500 - 63{,}100)) = 11{,}448{,}000.$$

Another method of estimating the covariances of loss reserve return and underwriting return with surplus is to analyze the covariance structure and estimate the component parts.

Let $\sigma_l$, $\sigma_u$, and $\sigma_a$ denote the standard deviations of the following random variables:

$L$: return from loss reserves;

$U$: return from underwriting; and

$A$: return on assets other than those offsetting loss reserves.

Let the correlations between the above returns be denoted by $\rho_{l,u}$, $\rho_{l,a}$, and $\rho_{u,a}$. Let $\mathrm{Cov}(L,S)$ and $\mathrm{Cov}(U,S)$ denote the covariances of the indicated returns with surplus. Then,

$$\begin{aligned}
\mathrm{Cov}(L,S) &= \mathrm{Cov}(L, L+U+A)\\
&= \mathrm{Cov}(L,L) + \mathrm{Cov}(L,U) + \mathrm{Cov}(L,A)\\
&= (\sigma_l)^2 + \sigma_l\sigma_u\rho_{l,u} + \sigma_l\sigma_a\rho_{l,a}, \qquad \text{and}\\
\mathrm{Cov}(U,S) &= \mathrm{Cov}(U, L+U+A)\\
&= \mathrm{Cov}(U,L) + \mathrm{Cov}(U,U) + \mathrm{Cov}(U,A)\\
&= \sigma_u\sigma_l\rho_{l,u} + (\sigma_u)^2 + \sigma_u\sigma_a\rho_{u,a}.
\end{aligned}$$

*B. Risk Loads for Property and Casualty*

Since 1980, the variation in industry casualty loss ratios has been much greater than the variation in property loss ratios. Also, casualty loss ratio variation has been significantly correlated with variation in loss reserve estimates. Both loss ratios and reserve estimates were affected by trends in loss severity.

Suppose that, for some insurer:

1. All premiums are either casualty or property.
2. The overall underwriting risk load (discussed in the previous example) is 8% of premium.
3. The covariances with casualty return and with property return of the return on assets other than those offsetting loss reserves are zero.
4. Expected property and casualty earned premiums are \$100,000,000 and \$150,000,000, respectively, and total risk-based discounted loss reserves are \$400,000,000.
5. The expected pre-tax returns from property and casualty premiums are in the same proportion as the corresponding after-tax returns.

6. The estimated covariances of property return, casualty return, and loss reserves return with each other are based on Table 2.

TABLE 2

| Year | Change from 1/1 to 12/31 in Estimated Property Return (000's) | Change from 1/1 to 12/31 in Estimated Casualty Return (000's) | Change from 1/1 to 12/31 in Estimated Loss Reserves Return (000's) |
|---|---|---|---|
| 1983 | −\$2,500 | −\$20,800 | −\$14,600 |
| 1984 | −\$6,100 | −\$29,700 | −\$16,400 |
| 1985 | −\$400 | \$6,100 | \$1,300 |
| 1986 | \$8,700 | \$16,500 | \$4,600 |
| 1987 | \$4,100 | \$28,800 | \$8,900 |
| 1988 | −\$600 | \$6,200 | \$1,400 |
| 1989 | −\$500 | \$1,500 | \$4,800 |
| 1990 | −\$6,000 | −\$1,700 | \$2,100 |
| 1991 | −\$3,600 | −\$1,400 | \$5,700 |
| 1992 | \$2,100 | −\$2,500 | \$5,900 |
| 1993 | \$4,800 | −\$3,800 | \$1,200 |
| 1994 | −\$1,500 | \$900 | −\$1,100 |

The covariance between any two of the returns in Table 2 is estimated by taking the average of the products of the numbers in each row of the two columns of returns. It is assumed that the 1/1 estimates equal the means of the probability distributions of possible 12/31 estimates. Let $P$, $C$, $R$, and $A$ denote random variables which equal the returns from property, casualty, reserves, and other assets, and let $S$ denote a random variable which equals the change in surplus. Then, since $\mathrm{Cov}(P,A)$ and $\mathrm{Cov}(C,A)$ are zero by Assumption 3 above,

$$\begin{aligned}\mathrm{Cov}(P,S) &= \mathrm{Cov}(P,P) + \mathrm{Cov}(P,C) + \mathrm{Cov}(P,R) + \mathrm{Cov}(P,A)\\ &= \mathrm{Var}(P) + \mathrm{Cov}(P,C) + \mathrm{Cov}(P,R) = 74.14 \text{ million},\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(C,S) &= \mathrm{Cov}(C,P)+\mathrm{Cov}(C,C)+\mathrm{Cov}(C,R)+\mathrm{Cov}(C,A)\\ &= \mathrm{Cov}(C,P)+\mathrm{Var}(C)+\mathrm{Cov}(C,R) = 342.83 \text{ million.}\end{aligned}$$

The ratio of the risk load, in dollars, for property to that of casualty is $74.14:342.83$; i.e., $.216:1$. It was assumed above that overall underwriting risk load is 8% of premium, so if $x$ represents the casualty risk load in dollars,

$$\begin{aligned}x + .216x &= .08 \times (\$250{,}000{,}000)\\ x &= \$16.447 \text{ million.}\end{aligned}$$

Therefore, the risk loads for casualty and property, as percentages of premium, are, respectively, $16.447/150 = 11.0\%$, and $(.216(16.447))/100 = 3.6\%$.

Suppose that expenses are 30% of premium for both casualty and property, and that the respective risk-based present value factors for the losses are .800 and .970. It then follows, using the above risk loads of 11.0% and 3.6%, that the target combined ratio for casualty is given by

$$30 + (100 - 30 - 11)/.800 = 103.8,$$

and the target for property is given by

$$30 + (100 - 30 - 3.6)/.970 = 98.5.$$

### C. *Catastrophe Cover Risk Load*

In this example, in order to estimate the value of a catastrophe cover to a ceding company, we will suppose that the ceding company re-assumes the cover, and we will estimate the required risk load.

Assume that:

1. The probability of zero losses to the catastrophe cover is .96, and the probability that the losses will be \$25 million

is .04. Therefore, the variance $(\sigma_c)^2$ of the losses is 24 trillion, and the expected losses are \$1 million.

2. Property premium earned for the year is \$100 million, and there is no casualty premium.
3. The standard deviation of pre-tax underwriting return is 15 million.
4. The expected pre-tax return from underwriting is \$8 million.
5. Taxes have the same proportional effect on the expected pre-tax returns on total premium and on the catastrophe cover, and on the standard deviations of the returns.
6. The covariance between the catastrophe cover's losses and losses net of the cover is equal to .50 times the variance of the cover's losses.
7. The discount rate for losses is zero.
8. Total underwriting return, and the return on the catastrophe cover, are statistically independent of non-underwriting sources of surplus variability.

It follows from 1 and 6 above that the covariance with surplus of the pre-tax return on the catastrophe cover is 24 trillion + .50(24 trillion); i.e., 36 trillion. It follows from 3 that the corresponding covariance for total underwriting is $(15 \text{ million})^2$; i.e., .225 trillion. Therefore, it follows from assumption 4 that the risk load for the catastrophe cover should be such that the pre-tax return from the catastrophe cover is given by (36/225)(8 million) = \$1.28 million. This is greater than the cover's expected losses.

The insurer may be able to cede the catastrophe cover for a price that is mutually beneficial to it and a reinsurer. For example, if a reinsurer is much larger and more diversified than the ceding company, and it pools its assumed catastrophe covers with other reinsurers, it may not require as great a risk load for the cover as would the ceding company.

### D. *Risk Load by Layer*

Suppose that for some insurer:

1. All premium is property premium.
2. The accident year expected property losses for the \$500,000 excess of \$500,000 layer, and the 0 – \$500,000 layer, respectively, are \$10 million and \$90 million. Expected losses excess of \$1 million are zero.
3. The accident year property losses for each of the above layers are independent of all non-underwriting sources of surplus variation.
4. The discount rate is zero.
5. The coefficients of variation (ratios of standard deviations to means) of the higher and lower layers are .30 and .15, respectively.
6. The correlation between the two layers is .5.
7. Taxes have the same proportional effect on the returns of both layers.

Let $\sigma_1$ and $\sigma_2$ denote the standard deviations of the losses to the higher and lower layers, respectively. Let $\rho$ denote the correlation. With the above assumptions, the pre-tax covariances with surplus for the higher and lower layers, respectively, are given by:

$$\begin{aligned}\sigma_1^2 + \rho\sigma_1\sigma_2 &= ((10\text{ million})(.30))^2 \\ &\quad + (.5)(10\text{ million})(.30)(90\text{ million})(.15) \\ &= 29.25\text{ trillion}, \qquad \text{and} \\ \sigma_2^2 + \rho\sigma_1\sigma_2 &= ((90\text{ million})(.15))^2 \\ &\quad + (.5)(10\text{ million})(.30)(90\text{ million})(.15) \\ &= 202.5\text{ trillion}.\end{aligned}$$

The allocated surplus for the 0 – \$500,000 layer is 202.5/29.25 (i.e., 6.9) times as great as the allocated surplus for the \$500,000 excess of \$500,000 layer. The expected losses are nine times as great for the lower layer. Therefore, the required risk load, as a percentage of expected losses, is 1.3 (i.e., ((9)(29.25))/202.5) times as great for the higher layer as it is for the lower layer. This is expected due to the higher layer's larger coefficient of variation.

Note the contrast of the use of covariances to the use of variances or standard deviations. The covariances for the lower and higher layers are 202.5 trillion and 29.25 trillion, respectively. The corresponding variances are 182.25 trillion and 9 trillion, and the corresponding standard deviations are 13.5 million and 3 million. Thus the ratio of total risk loads, in dollars, for the lower and higher layers is about 7 for the covariance method, about 20 for the variance method, and exactly 4.5 for the standard deviation method.

## 5. SOME RELATED METHODS

It will be shown that the Capital Asset Pricing Model (CAPM) can be useful in selecting the overall premium and combined ratio targets that are used in this paper to set targets for individual categories. Also, the significance of the method of this paper from a CAPM perspective will be discussed.

According to CAPM, the price of a capital asset depends on its expected rate of return and the covariance of this rate with the overall rate of return on the market of all capital assets. (See Brealey and Myers [1], Lintner [9], and Sharpe [11].) There is some similarity between CAPM and the method presented here, since CAPM estimates prices based on the covariance of an asset with the market, and the method presented here estimates prices based on the covariance of a contract with surplus. The similarity is limited, however. The derivation of the CAPM formula for a capital asset uses the fact that holders of capital assets are able to use Markowitz diversification. The method presented here requires that the mix of business of an insurer be approximated in advance. The method applies to a risk-return optimization problem, but for an insurer with a stable, or almost stable, book of business.

According to CAPM, each asset $j$ in the market of all capital assets will have a market price such that

$$E_j = R_f + (E_m - R_f)((\mathrm{Cov}(R_j, R_m))/(\sigma_m)^2),$$

where

$E_j$ = the expected rate of return on asset $j$,

$E_m$ = the expected rate of return on the market portfolio,

$\sigma_m$ = the standard deviation of the rate of return on the market portfolio,

$R_f$ = the risk-free rate of return,

$R_m$ = the market rate of return, and

$R_j$ = the rate of return on asset$j$.

The market value of an insurer's assets, not including franchise value, minus its liabilities will be called the market value of its surplus. Suppose for the sake of illustration that for some insurer, the market value of surplus equals the market value of the insurer. In other words, the franchise value is zero. Suppose also that the expected market value of surplus one year in the future equals the expected market value of the insurer one year in the future. It then follows that the expected change in this value of surplus in the coming year, divided by the present surplus, is equal to $E_j$ in the above formula if $R_j$ represents the rate of return on the market value of the insurer.

This expected rate of return, which makes the market value of the insurer equal the runoff value (market value) of the assets and liabilities, could be considered to be the minimum acceptable expected return on surplus for the insurer.

Suppose that, due to a change in management, the expected change in surplus in the coming year increases, and there is no change in the expression $R_f$ or

$$(E_m - R_f)(\mathrm{Cov}(R_j, R_m)/\sigma_m^2).$$

Since $E_j$ does not change, the market value of the insurer theoretically increases and becomes greater than the market value of surplus. This creates what is known as franchise value.

The amount of premium that is required for a category in order to neither improve nor worsen the insurer's risk-return relation is not necessarily the same as the amount that neither increases nor decreases the market value of the insurer according to CAPM.

Suppose that surplus is allocated according to the method of this paper, and the estimated rate of return on the surplus allocated to a Category $a$ is less than the rate of return of the insurer. Suppose also that, according to the application of CAPM to Category $a$ and its allocated surplus, this rate of return is above the acceptable minimum for the insurer discussed above. Also, suppose that according to CAPM the rate of return of the insurer is equal to the acceptable minimum.

In the above example, Category $a$ would be estimated by CAPM to increase the market value of the insurer if certain intangible effects of worsening the risk-return relation are ignored.

Advantages that the insurer gains by improving the risk-return relation were described in the second paragraph of the introduction to this paper. (The risk-return relation has an influence on policyholders, employees, and rating organizations.) In the long run, these advantages can translate into lower expected combined ratios. In the case of the above example, the long-term effects of worsening the risk-return relation should be weighed against a CAPM estimate that ignores them.

An insurer can also use CAPM to evaluate the effects on its market value of changes in its amount of written premium or the composition of its asset portfolio. Here again, the effects on the risk-return relation are important, as well as the effects on the CAPM estimate of market value. The intangible effects of variability on rating organizations, customers, and employees should be considered.

Kreps [8] presented a method of determining risk load by marginal surplus requirements. A problem with Kreps's method was discussed in the introduction. The sum of the effects of all categories on the standard deviation of surplus, as measured by Kreps, does not equal the total standard deviation. Kreps does not address the variability of loss reserves or the discounting of losses.

Feldblum [4] suggested a modified version of CAPM for determining risk loads for insurers:

> The market return $R_m$ in the CAPM model should be replaced by the return on a fully diversified insurance portfolio.

Feldblum's method could be used to estimate required return on allocated surplus for an insurance contract. The subscript $m$ for market is replaced in three places in the CAPM formula by $i$ for insurance industry. Feldblum's method does not address the problem of discounting, but it could be expanded to do so.

Feldblum's method is somewhat similar to the method in this paper in that it addresses the problem, for an insurer, of optimizing the risk-return relation. The key difference between Feldblum's method and the method in this paper is the following: Feldblum's method evaluates insurance contracts for an insurer that is free to use an insurance analogue of Markowitz diversification to produce a portfolio of insurance contracts. (In actual practice, there are constraints on an insurer.) The method in this paper estimates the effect of a contract on surplus variance given an approximated mix of earned premium for the coming year.

Brubaker [2] and Ferrari [6] discuss methods of maximizing an insurer's profit, given a constraint on variance, by selecting an insurance portfolio. They don't address the problems of variability of loss reserves or discounting of losses. Underwriting profit margins by category are estimated prior to selecting the portfolio.

## 6. CONCLUSION

The method in this paper is an attempt to address the problem of risk-based pricing for an insurer in a way that is useful and also meaningful in the context of financial theory. Although there is considerable judgment and effort involved in applying the method, it provides a new theoretical framework for dealing with the challenge of improving an insurer's risk-return relation.

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## APPENDIX

### *Proof of Theorem 1*

Let the random variable $X$ equal the effect of a Category $x$ on surplus in a one year period. Let the random variable $Y$ equal the combined effect of all other sources of surplus variation in the one year period.

Suppose a $1/n$ pro rata share of each category, including $x$, which contributes to surplus variation is added in any order. Suppose the process is repeated until Category $x$ is about to be added for the $(k+1)^{\text{st}}$ time, where $k+1 \leq n$. Let $V_1$ denote the variance of the effect on surplus of the set of pro rata shares before $x$ is added, and let $V_2$ denote the variance afterwards.

In the following argument, the expression $\approx$ will be used to indicate that the ratio of the expression on the left to the one on the right approaches 1 as $k$ and $n$ approach infinity. It can be seen that

$$V_1 \approx 2(k/n)^2\rho_{x,y}\sigma_x\sigma_y + (k/n)^2\sigma_x^2 + (k/n)^2\sigma_y^2, \qquad \text{and}$$

$$V_2 \approx 2((k+1)/n)(k/n)\rho_{x,y}\sigma_x\sigma_y + ((k+1)/n)^2\sigma_x^2 + (k/n)^2\sigma_y^2.$$

The change in standard deviation, $\Delta$ Std. Dev., is $V_2^{.5} - V_1^{.5}$.

It follows from $(V_1^{.5} + \Delta \text{ Std. Dev.})^2 = V_2$ that

$$\begin{aligned}
\Delta \text{ Std. Dev.} &\approx .5((V_2 - V_1)/V_1^{.5}) \\
&\approx .5((2k/n^2)\rho_{x,y}\sigma_x\sigma_y + ((2k+1)/n^2)\sigma_x^2)/ \\
&\quad (2(k/n)^2\rho_{x,y}\sigma_x\sigma_y + (k/n)^2\sigma_x^2 + (k/n)^2\sigma_y^2)^{.5} \\
&\approx ((1/n)(\rho_{x,y}\sigma_x\sigma_y + \sigma_x^2))/(2\rho_{x,y}\sigma_x\sigma_y + \sigma_x^2 + \sigma_y^2)^{.5}.
\end{aligned}$$

Therefore, it can be seen that

$$\lim_{n\to\infty}\sum_{1}^{n}\Delta\text{ Std Dev.} = (\rho_{x,y}\sigma_x\sigma_y + \sigma_x^2)/(2\rho_{x,y}\sigma_x\sigma_y + \sigma_x^2 + \sigma_y^2)^{.5}.$$

$$= \text{Cov}(X+Y,X)/\text{Std. Dev. }(X+Y)$$

$$= \text{Cov}(\text{surplus},X)/\text{Std. Dev. (surplus)},$$

and

$$\left(\lim_{n\to\infty}\sum_{1}^{n}\Delta\text{ Std. Dev.}\right)/\text{Std. Dev. (surplus)}$$

$$= \text{Cov}(\text{surplus},X)/\text{Var(surplus)}.$$

*Proof of Theorem 2*

Let the random variable $X$ equal the effect of the Category $x$ on surplus in a one year period. Let the random variable $S$ equal the change in surplus in the one year period.

It is assumed that the insurer gets more than the required risk-based premium for Category $x$. Therefore,

$$\text{E}(X) > \text{E}(S)(\text{Cov}(X,S)/\text{Var}(S)). \tag{A.1}$$

It follows that,

$$\text{E}(S-X) = (\text{E}(S)-\text{E}(X)) < \text{E}(S)(1-(\text{Cov}(X,S)/\text{Var}(S)))$$

$$= \text{E}(S)(\text{Cov}(S-X,S)/\text{Var}(S)).$$

Therefore,

$$\text{E}(S-X) < \text{E}(S)(\text{Cov}(S-X,S)/\text{Var}(S)). \tag{A.2}$$

Suppose the premium for Category $x$ is multiplied by some number $1+a$, where $a>0$, and that the total premium for the rest of the book is multiplied by some number $1-b$, where $b>0$. Suppose also that the insurer's total surplus variance is unchanged. Therefore,

$$\begin{aligned}\mathrm{Var}(S) &= \sigma_x^2(1+a)^2 + (\sigma_{s-x})^2(1-b)^2 \\ &\quad + 2(1+a)(1-b)\rho_{x,s-x}\sigma_x\sigma_{s-x} \\ &= \sigma_x^2 + (\sigma_{s-x})^2 + 2\rho_{x,s-x}\sigma_x\sigma_{s-x}.\end{aligned}$$

Let $\Delta\mathrm{Var}(S)$ represent the first of the above two expressions minus the second. There is an expression $f(a,b)$ such that

$$0 = \Delta\mathrm{Var}(S)$$

$$0 = \sigma_x^2(2a) + (\sigma_{s-x})^2(-2b) + (2a-2b)\rho_{x,s-x}\sigma_x\sigma_{s-x} + f(a,b)$$

$$0 = 2a\sigma_x(\sigma_x + \rho_{x,s-x}\sigma_{s-x}) - 2b\sigma_{s-x}(\sigma_{s-x} + \rho_{x,s-x}\sigma_x) + f(a,b)$$

$$0 = 2a(\mathrm{Cov}(X,S)) - 2b(\mathrm{Cov}(S-X,S)) + f(a,b)$$

and the limit as $a$ and $b$ approach zero of $f(a,b)/a$, and of $f(a,b)/b$, is zero.

It follows from the above that

$$\begin{aligned}&a\mathrm{E}(S)(\mathrm{Cov}(X,S)/\mathrm{Var}(S)) \\ &\quad = b\mathrm{E}(S)(\mathrm{Cov}(S-X,S)/\mathrm{Var}(S)) + g(a,b),\end{aligned} \tag{A.3}$$

where $g(a,b)/a$ and $g(a,b)/b$ approach zero as $a$ and $b$ approach zero.

Now,

$$\begin{aligned}&\mathrm{E}((1+a)X + (1-b)(S-X)) \\ &\quad = \mathrm{E}(X + (S-X) + aX - b(S-X)) \\ &\quad = \mathrm{E}(S) + a\mathrm{E}(X) - b\mathrm{E}(S-X).\end{aligned} \tag{A.4}$$

It follows from Inequalities A.1 and A.2 that the formula above equals

$$\begin{aligned} &\mathrm{E}(S) + a(\mathrm{E}(S)(\mathrm{Cov}(X,S)/\mathrm{Var}(S)) \\ &\quad - b(\mathrm{E}(S)(\mathrm{Cov}(S-X,S)/\mathrm{Var}(S)) + ad + be, \end{aligned} \tag{A.5}$$

where $d > 0$ and $e > 0$.

It was mentioned above that $a > 0$ and $b > 0$. As $a$ and $b$ approach zero, $d$ and $e$ above remain constant and, by Equations A.3, A.4 and A.5,

$$\begin{aligned} &\mathrm{E}((1+a)X + (1-b)(S-X)) \\ &\quad = \mathrm{E}(S) + g(a,b) + ad + be > \mathrm{E}(S). \end{aligned}$$

This completes the proof of Theorem 2 for the case in which Category $x$ is written at more than the required risk-based premium. The proof of the converse is similar.