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UNBIASED LOSS DEVELOPMENT FACTORS

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DISCUSSION BY DANIEL F. GOGOL, PH.D.

Daniel Murphy presents some powerful and useful techniques for estimating biases and variances of loss reserve estimators. It is mentioned in the introduction of Murphy's paper that Casualty Actuarial Society literature is inconclusive regarding whether certain loss development techniques are biased or unbiased. It is also stated that the paper provides a model so that these questions, and others, can be answered.

Although the assumptions of Murphy's models enable him to show that the simple average development factor method and the weighted average development factor method are unbiased, actually they both are biased upwards. (The Bornhuetter–Ferguson and Stanard–Bühlmann methods also are biased upwards if they use these factors.) It is only because Murphy's models have unrealistic properties that it is possible to prove that the estimators are unbiased. Murphy is aware of this, as is shown by his discussion of claim count development in Appendix B. He states:

> Take the weighted average development method for example. Clearly there is a positive probability (albeit small) that $x = 0$, so the expected value of the weighted average development link ratio y/x is infinity.

Murphy also indicates that a general, heuristic argument that weighted average development yields biased estimates can be found in Stanard [3]. Stanard's argument for the bias of weighted average development factors [3, Appendix A], is actually only a derivation of an equation which must be satisfied in order for the factors to be unbiased. Stanard states without proof that the equation is not true in general. (It can be seen that the equation is untrue by considering Murphy's above point; i.e., the positive probability that $x = 0$.)

Murphy's statement that the expected value of the weighted average development link ratio is infinity is true; but of course, further analysis is necessary to give some idea of the amount of the bias.

In actual practice, a reserving actuary would obviously not use an infinite link ratio. In many situations the possibility of an infinite weighted average link ratio is remote and it may be judged that any weighted average link ratio greater than some R will be replaced by R in computing weighted average development factors. Suppose also that if the weighted average link ratio is $0/0$, the zeroes will be replaced judgmentally by some y/x.

However, even with this new system, weighted average development factors will be biased upwards if the probability that y/x is $0/0$ or $y/x > R$ is sufficiently small. The proof that will be given could be useful in estimating the amount of the bias.

The case of a single factor for a single accident year will be considered first although the same argument applies to policy years or report years. It will later be shown that the result demonstrated for a single factor applies to weighted or unweighted averages of factors.

Let X and Y be random variables which represent the reported losses for an accident year at evaluations x and y years after the start of the accident year. The factor which, when multiplied by X, produces an unbiased estimate of the mean of Y is $\mathrm{E}(Y)/\mathrm{E}(X)$, since

$$\mathrm{E}((\mathrm{E}(Y)/\mathrm{E}(X))X) = (\mathrm{E}(Y)/\mathrm{E}(X))\mathrm{E}(X) = \mathrm{E}(Y).$$

It is not true in a realistic model that, given a particular value x of X, $(\mathrm{E}(Y)/\mathrm{E}(X))x$ necessarily equals, or is even a good approximation of, $\mathrm{E}(Y \mid X = x)$. For example, it is not true that $\mathrm{E}(Y \mid X = 0) = (\mathrm{E}(Y)/\mathrm{E}(X))(0) = 0$.

The equality $(\mathrm{E}(Y)/\mathrm{E}(X))x = \mathrm{E}(Y \mid X = x)$ is, however, implicitly assumed in Murphy's Model II, which he described as follows: $y = bx + e$, $\mathrm{E}(e) = 0$, $\mathrm{Var}(e)$ is constant across accident years, and the e's are uncorrelated between accident years and are independent of x.

In a realistic model, a prior expectation E_p could be estimated for Y, based on data other than the loss experience of the accident year being considered. $\mathrm{E}(Y \mid X = x)$ can be approximated by a credibility weighting of E_p and the experience indication $(\mathrm{E}(Y)/\mathrm{E}(X))x$. (See [3].) The demonstration that will be given of the upward bias in development factors uses the assumption that, for $x \geq 0$, $\mathrm{E}(Y \mid X = x)$ is equal to a weighting of the form $(1 - Z)E_p + (Z)(\mathrm{E}(Y)/\mathrm{E}(X))x$. However, it can be seen that all the steps of the argument also hold true if the above weighting is a good approximation, as is generally true in actual practice.

It follows from the above assumption that

$$\mathrm{E}(Y \mid X = x) > (\mathrm{E}(Y)/\mathrm{E}(X))x$$

if x is sufficiently less than $\mathrm{E}(X)$. Similarly,

$$\mathrm{E}(Y \mid X = x) < (\mathrm{E}(Y)/\mathrm{E}(X))x$$

if x is sufficiently greater than $\mathrm{E}(X)$. Also, it is clear that $\mathrm{E}(Y \mid X = x)/x$ is a monotonically decreasing function of x.

Let X' and Y' be the random variables which represent the values of X and Y, respectively, after they have been judgmentally changed, as described previously, if $X = 0$. If the probability that $X = 0$ is sufficiently small, then $\mathrm{E}(Y')/\mathrm{E}(X')$ is very close to $\mathrm{E}(Y)/\mathrm{E}(X)$. It will be shown that

$$\mathrm{E}(Y' \mid X') > \mathrm{E}(Y')/\mathrm{E}(X'). \tag{1}$$

Let $f(x)$ be the probability density function of X'. Then

$$\int_0^{\infty} (\mathrm{E}(Y'/X') \mid X' = x) x f(x) dx = \mathrm{E}(Y').$$

Therefore,

$$\int_0^\infty (\mathrm{E}(Y'/X') \mid X' = x)(x/\mathrm{E}(X'))f(x)\,dx = \mathrm{E}(Y')/\mathrm{E}(X'). \tag{2}$$

However,

$$\begin{aligned}\mathrm{E}(Y'/X') &= \int_0^\infty (\mathrm{E}(Y'/X') \mid X' = x)f(x)\,dx \\ &> \int_0^\infty (\mathrm{E}(Y'/X') \mid X' = x)(x/\mathrm{E}(X'))f(x)\,dx, \end{aligned} \tag{3}$$

as will be shown below. It follows from (2) and (3) that (1) is true.

In the two integrals in the above inequality, the same function of x, i.e., $\mathrm{E}(Y'/X') \mid X' = x$, is multiplied by $f(x)$ and by $(x/\mathrm{E}(X'))f(x)$, respectively.

It was mentioned above that $f(x)$ is the probability density function of X'. The function $(x/\mathrm{E}(X'))f(x)$ is also a probability density function, since

$$\int_0^\infty (x/\mathrm{E}(X'))f(x)\,dx = (1/\mathrm{E}(X'))\int_0^\infty xf(x)\,dx = 1.$$

Note that:

$$(x/\mathrm{E}(X'))f(x) < f(x) \qquad \text{for} \quad x < \mathrm{E}(X'), \quad \text{and}$$

$$(x/\mathrm{E}(X'))f(x) > f(x) \qquad \text{for} \quad x > \mathrm{E}(X').$$

Thus, the density function $(x/\mathrm{E}(X'))f(x)$ gives less weight than $f(x)$ to values of $\mathrm{E}(Y'/X') \mid X' = x$ for which $x < \mathrm{E}(X')$, and more weight to values of $\mathrm{E}(Y'/X') \mid X' = x$ for which $x > \mathrm{E}(X')$. However, $\mathrm{E}(Y/X) \mid X = x$ is a monotonically decreasing function of x. If X' and Y' are not too different from X and Y, then $\mathrm{E}(Y'/X') \mid X = x$ is close enough to a monotonically decreasing function so that Equation 3 is true. This completes the proof of upward bias for a single factor.

Murphy uses the assumptions that the expected development pattern for each accident year is identical, and that development is independent for each accident year. It follows that the expected value of a simple average of development factors, at the same evaluations, equals the expected value of any individual factor. Therefore, using Murphy's assumptions and the proof above, the simple average of development factors is biased upwards.

It also follows from Murphy's assumptions that weighted average development factors are biased upwards. For a set of accident years, let X_i and Y_i represent the reported losses for accident year i at evaluations x and y years after the start of accident year i. Then the random variable Y/X, where $Y = \sum Y_i$ and $X = \sum X_i$, equals the weighted average development factor. It can easily be verified that the proof of Equation 1 is valid with the previous definitions of X and Y (and the corresponding X' and Y') replaced by these new definitions.

REFERENCES

[1] Gogol, Daniel, "Using Expected Loss Ratios in Reserving," *Insurance: Mathematics and Economics* 12, 1993, pp. 297–299.

[2] Murphy, Daniel, "Unbiased Loss Development Factors," *PCAS* LXXXI, 1994, pp. 154–222.

[3] Stanard, James N., "A Simulation Test of Prediction Errors of Loss Reserve Estimation Techniques," *PCAS* LXXII, 1985, pp. 124–148.