THE COMPETITIVE MARKET EQUILIBRIUM RISK LOAD FORMULA FOR INCREASED LIMITS RATEMAKING

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DISCUSSION BY IRA ROBBIN

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AUTHOR'S REPLY TO DISCUSSION

Abstract

In his discussion of the author's paper, Ira Robbin takes issue with several aspects of the proposed risk load formula. In this response, the author seeks to clarify some of these differences and expand upon the role of reinsurance in the pricing of high limit policies. In particular, he shows how the risk load formula can be used to develop an efficient reinsurance program.

1. INTRODUCTION

Ira Robbin [2] has provided a thought-provoking article on the subject of risk loads. The subject has historically been a controversial one among actuaries since it attempts to describe one of the more subjective elements of insurance pricing with a mathematical formula.

Part of the problem has been a confusion in the terminology used to describe the pricing of insurance. Terms include expected losses, various insurer expenses, investment income, risk loads, and profit loads, all of which can be overridden by marketing considerations. I believe many of the differences between Robbin and myself can be attributed to differences in terminology. But when he combines these differences in terminology with an improper interpretation of the role of an advisory organization (which he calls a rating bureau), he draws conclusions about my paper which I neither *implicitly* (his word, my italics) nor explicitly intended.

2. THE ROLE OF AN ADVISORY ORGANIZATION

It should be kept in mind that the Competitive Market Equilibrium (CME) risk load formula was developed for use in ISO advisory increased limits filings. We (ISO) do not view our role as simply to provide increased limits factors on a "take it or leave it" basis. We recognize that our increased limits factors will not be appropriate for every situation, yet the development of these factors contains information of value to all insurers. We view our job as providing information to aid the insurer in deciding what its increased limits factors (or more generally, rates) should be. To do this job effectively, we must explicitly identify the various components that make up the increased limits factors so that insurers can more easily implement whatever changes they want to make.

This becomes particularly important when reinsurance is involved.

Robbin defines the risk load so that it contains a provision for reinsurance expenses, while in my definition there is no such provision. My risk load is for "pure" risk, and the reinsurance expenses are addressed separately. Since the purpose of reinsurance is to spread risk, Robbin's definition might be considered reasonable. However, it presents problems because there are many purposes of reinsurance, and a diverse population of reinsurance buyers. For this reason we decided to presume as little as possible about the nature of an insurer's reinsurance arrangements, and to provide information that will aid the insurer to account effectively for the use of reinsurance in increased limits pricing.

Thus, in our advisory increased limits filings we explicitly assume the insurer is retaining the entire risk. If an insurer wishes to obtain excess of loss reinsurance, it can use the filed factors to obtain its price up to the amount it retains, and then add on the price of reinsurance. In addition, we provide circulars and software that may be useful in planning for excess of loss reinsurance. The software handles reinsurance expense in the manner described in Section 8 of the paper.

My definition of risk load is motivated by institutional, rather than fundamental, reasons. While I have no fundamental objection to Robbin's definition of risk load, when he combines it with his interpretation of a "rating bureau," he draws inferences with which I strongly disagree. For example, he writes that I believe "that the bureau should file ILFs under the hypothesis that layering is not allowed," or that "implicitly, Meyers has prohibited insurers from entering into transactions that his theory says are beneficial."

Instead, the theory provides a tool to aid in the development of an efficient reinsurance program, and to incorporate reinsurance into the pricing of increased limits. However, we feel the responsibility for doing this lies with the insurer, and not with an advisory organization.

3. REINSURANCE PLANNING

In spite of our differences, I would like to recommend many of the ideas in Robbin's section on "Putting Reinsurance into the Model" for serious consideration in reinsurance planning. The exercise of finding the reinsurance program that results in the most competitive rate should be a regular activity for the insurer. He offered a solution for quota share reinsurance. Here I give an example which illustrates how an insurer might proceed when both excess of loss and quota share reinsurance are available. This example will be a continuation of the example started in Section 7 of the paper. Table 1 gives the ground up increased limits factors derived in this example.

Let us assume that the reinsurer charges the risk load indicated by the CME formula and charges an additional charge, which is expressed as a percentage of the expected loss for the layer, to cover expenses.

TABLE 1

GROUND UP INCREASED LIMITS FACTORS

Policy Limit	Average Severity	Process Risk	Parameter Risk	ILF with Risk Load
$25,000	$ 8,202	$28	$253	1.000
500,000	18,484	659	575	2.324
1,000,000	20,579	1,262	641	2.650
2,000,000	22,543	2,391	703	3.022
5,000,000	24,943	5,513	779	3.682

Note that the total average severity and parameter risk will be the same for all possible primary insurer retentions. However, the total process risk and the reinsurer expense charge will depend upon the retention. Thus, the search for the best retention leads to the question: *what retention will minimize the sum of the process risk and the reinsurance charge?* In the case of a single reinsurer, trial and error will quickly provide the answer. Tables 2 and 3 provide results for our example. In this case we assume that the reinsurance charge for expenses is 10% of the expected loss for the layer.

In the following tables, the increased limits factor is given by:

$$\frac{\text{Average Severity} + \text{Process Risk} + \text{Parameter Risk} + \text{Reinsurance Charge} \quad \text{for the increased limit}}{\text{Average Severity} + \text{Process Risk} + \text{Parameter Risk} \quad \text{for the basic limit}}$$

Table 2 illustrates the kind of search that can be taken to find the most economical reinsurance program with a single reinsurer for a $5,000,000 policy limit.

TABLE 2

SEVERAL SINGLE EXCESS LAYER PROGRAMS FOR $5,000,000 POLICY LIMIT

Layers	Average Severity	Process Risk	Parameter Risk	Reinsurance Charge	Proc. Risk + Reins. Charge	ILF
0 – 1,000,000	$20,579	$1,262	$641			2.650
1,000,000 – 5,000,000	4,364	2,506	137	$436		0.877
Combined		3,768			$4,204	3.527
0 – 1,900,000	22,402	2,281	699			2.992
1,900,000 – 5,000,000	2,541	1,301	80	254		0.492
Combined		3,582			3,836	3.484
0 – 2,000,000	22,543	2,391	703			3.022
2,000,000 – 5,000,000	2,400	1,202	76	240		0.462
Combined		3,593			3,833	3.484
0 – 2,100,000	22,676	2,500	707			3.051
2,100,000 – 5,000,000	2,267	1,109	71	227		0.433
Combined		3,609			3,835	3.484
0 – 3,000,000	23,632	3,463	738			3.281
3,000,000 – 5,000,000	1,311	476	41	131		0.231
Combined		3,939			4,070	3.512

Table 3 shows the results of a similar search for a single reinsurer program with other limits.

TABLE 3

SELECTED SINGLE EXCESS LAYER PROGRAMS

Layers	Average Severity	Process Risk	Parameter Risk	Reinsurance Charge	Proc. Risk + Reins. Charge	ILF
0 > 350,000	$17,353	$469	$540			2.165
350,000 > 500,000	1,131	32	36	$113		0.155
Combined		501			$614	2.319
0 > 600,000	19,048	783	593			2.408
600,000 > 1,000,000	1,532	111	48	153		0.217
Combined		894			1,047	2.625
0 > 900,000	20,269	1,144	632			2.599
900,000 > 2,000,000	2,273	428	72	227		0.354
Combined		1,572			1,800	2.952
0 > 2,000,000	22,543	2,391	703			3.022
2,000,000 > 5,000,000	2,400	1,202	76	240		0.462
Combined		3,593			3,833	3.484

By examining Table 3 one can see that excess of loss reinsurance can be used to reduce increased limits factors. It is tempting to ask if one can further reduce increased limits factors by using more than one reinsurer. A problem is that more reinsurers mean more administrative and transaction expenses. In Tables 4 and 5, we assume that the reinsurance charge for two and three reinsurers is respectively 15% and 20% of the expected losses for each reinsurer.

Tables 4 and 5 were derived by a systematic search for the least expensive reinsurance program for two and three excess reinsurers.

Note that the increased limits factor decreased for only the top two limits. For the lower two limits, using multiple reinsurers in this example did not reduce the process risk by an amount sufficient to cover the extra expense involved. We press on and add another reinsurer for these top two limits.

TABLE 4

SELECTED DOUBLE EXCESS LAYER PROGRAMS

Layers	Average Severity	Process Risk	Parameter Risk	Reinsurance Charge	Proc. Risk + Reins. Charge	ILF
0–350,000	$17,353	$469	$540			2.165
350,000–425,000	620	9	20	$93		0.087
425,000–500,000	511	7	16	77		0.072
Combined		486		170	$655	2.324
0–500,000	18,483	659	575			2.324
500,000–700,000	1,034	39	33	155		0.149
700,000–1,000,000	1,062	60	33	159		0.155
Combined		758		314	1,072	2.628
0–800,000	19,919	1,025	621			2.542
800,000–1,300,000	1,419	129	45	213		0.213
1,300,000–2,000,000	1,205	155	38	181		0.186
Combined		1,310		394	1,704	2.941
0–1,400,000	21,549	1,722	672			2.823
1,400,000–2,900,000	1,993	519	63	299		0.339
2,900,000–5,000,000	1,401	530	44	210		0.258
Combined		2,772		509	3,281	3.419

TABLE 5

SELECTED TRIPLE EXCESS LAYER PROGRAMS

Layers	Average Severity	Process Risk	Parameter Risk	Reinsurance Charge	Proc. Risk + Reins. Charge	ILF
0–800,000	$19,919	$1,025	$621			2.542
800,000–1,100,000	938	53	30	$188		0.142
1,100,000–1,500,000	887	67	28	177		0.137
1,500,000–2,000,000	799	76	25	160		0.125
Combined		1,221		525	$1,746	2.946
0–1,000,000	20,579	1,262	641			2.650
1,000,000–2,000,000	1,963	343	62	393		0.326
2,000,000–3,300,000	1,339	316	42	268		0.232
3,300,000–5,000,000	1,062	334	33	212		0.193
Combined		2,255		873	3,128	3.401

Here we see a reduced increased limits factor for only the $5,000,000 policy limit. Table 6 summarizes the results we have obtained so far. The boldface numbers represent the lowest increased limit factor obtained for each policy limit.

TABLE 6

SUMMARY OF INCREASED LIMITS FACTORS

Policy Limit	Without Reinsurance	1 Excess Layer	2 Excess Layers	3 Excess Layers	Without Risk Load
$25,000	1.000	1.000	1.000	1.000	1.000
500,000	2.324	**2.319**	2.324	—	2.254
1,000,000	2.650	**2.625**	2.628	—	2.509
2,000,000	3.022	2.952	**2.941**	2.946	2.748
5,000,000	3.682	3.484	3.419	**3.401**	3.041

In this example we see that for the $500,000 and $1,000,000 policy limits the lowest increased limits factor comes as a result of using a single reinsurer. For the $2,000,000 and $5,000,000 policy limits, the lowest increased limits factor comes as a result of using, respectively, two and three reinsurers.

In examining various reinsurance agreements, one often finds a single excess layer shared by two or more reinsurers on a quota share basis. It is demonstrated in Appendix A that if r reinsurers share an excess layer equally on a quota share basis, the total process risk gets reduced by a factor of $1/r$, while the total parameter risk remains the same.

In a final example we examine the effect of quota share for the single excess layer. For the $2,000,000 policy limit, the retention of the primary insurer was $800,000, the reinsurance charge was 15% of the expected losses, and two reinsurers were involved. For the $5,000,000 policy limit, the retention of the primary insurer was $1,000,000, the reinsurance charge was 20% of the expected losses, and three reinsurers were involved. In this example we keep the same assumptions except that each reinsurer shares the excess loss equally on a quota share basis. The results are in Table 7.

TABLE 7

QUOTA SHARE FOR EXCESS LAYER

Layers	Average Severity	Process Risk	Parameter Risk	Reins. Charge	Proc. Risk + Reins. Charge	ILF
0 > 800,000	$19,919	$1,025	$621			2.542
800,000 > 2,000,000	2,624	175	83	$394		0.386
Combined		1,200			$1,594	2.928
0 > 1,000,000	20,579	1,262	641			2.650
1,000,000 > 5,000,000	4,364	627	137	873		0.707
Combined		1,889			2,762	3.357

Here we see that sharing the excess layer on a quota share basis produces even lower increased limits factors. The results of this example are summarized in Table 8.

TABLE 8

SUMMARY OF INCREASED LIMITS FACTORS

Policy Limit	Without Reins.	1 Excess Layer	2 Excess Layers	3 Excess Layers	Quota Share Excess	Without Risk Load
$25,000	1.000	1.000	1.000	1.000	1.000	1.000
500,000	2.324	**2.319**	2.324	—	—	2.254
1,000,000	2.650	**2.625**	2.628	—	—	2.509
2,000,000	3.022	2.952	2.941	2.946	**2.928**	2.748
5,000,000	3.682	3.484	3.419	3.401	**3.357**	3.041

These examples do not illustrate the entire story. While quota share reinsurance may exhibit superior risk load reduction, it also involves more administrative expense since all reinsurers must look at every claim. However, sound reinsurance underwriting may remove the need to examine every claim.

This certainly explains why quota share reinsurance is often used on excess layers. But at some level, the risk-sharing advantages of quota share reinsurance and the effect of reinsurance underwriting may overcome the additional administrative expense.

Another common feature of reinsurance contracts is that the primary insurer can take a pro-rata share of the excess layer. The possible reduction in the "morale hazard" may make the contract more attractive to reinsurers, but it comes at the expense of higher total risk load.

How to balance all these aspects of reinsurance contracts is not clear. What is clear is that there are many problems involved in making an advisory filing which attempts to build all this into its increased limits factors.

4. CONSISTENCY

Another "definition" problem between Robbin and myself involves the notion of consistency. Consistency means that the price of a layer of insurance of a given width does not increase as the initial attachment point increases. It is argued that since a loss covered by the insurance does not increase as the initial attachment increases, the price should not increase. Robbin believes that this definition should restrict the pricing formula to taking the difference between the ILFs of the layer limit and the initial attachment point. I believe one should use the method that is actually used in pricing the layer. Since the motivation for consistency refers to price, the definition should refer to price.

At this point I would like to confess to an error in my original paper. Robert Bear, a CAS member, recently pointed out that my proof of consistency for the CME formula (by my definition) contained an error in the part that involved parameter uncertainty for the severity distribution. Upon further investigation I discovered conditions when the CME formula can produce inconsistent layer prices. Conditions under which the CME formula will be consistent are given in Appendix B. Generally speaking, inconsistency can occur for low layers when most of the parameter uncertainty is in the severity distribution. Thus the status of consistency with respect to excess layers can be summarized as follows: (1) the expected loss is consistent; (2) process risk is consistent; (3) the part of parameter risk due to uncertainty in the claim count distribution is consistent; but (4) the part of parameter risk due to uncertainty in the severity distribution can be inconsistent. However, the consistency of the first three parts can overpower the inconsistency in the fourth part.

Table 9 gives an example of the CME formula producing inconsistent layer pricing. This example was produced by modifying the previous example by putting all the parameter uncertainty into the severity, and increasing the risk load multiplier, λ, by a factor of 100. This produces inconsistency for the parameter risk up to \$5,000 and for the total price up to \$2,000. It was necessary to increase the risk load multiplier drastically to produce the inconsistency in the total price.

TABLE 9

Layer	Average Severity	Process Risk	Parameter Risk	ILF
0 > 25,000	$8,202	$2,811	$12,012	1.000
0 > 1,000	903	17	226	0.050
1,000 > 2,000	751	15	492	0.055
2,000 > 3,000	641	12	606	0.055
3,000 > 4,000	559	11	649	0.053
4,000 > 5,000	494	10	659	0.050
5,000 > 6,000	443	9	651	0.048
6,000 > 7,000	400	8	634	0.045
7,000 > 8,000	365	7	613	0.043

This example is extreme. But occasionally it is instructive to push a theory to its extreme cases to examine its theoretical foundations. Here, we examine it from the viewpoint of utility theory.[1]

Let X_1 and X_2 be losses for layer a_1 to $a_1 + h$ and a_2 to $a_2 + h$, respectively, where $a_1 < a_2$. Let P_1 and P_2 be the premium obtained for insuring against X_1 and X_2. If $P_1 = P_2 = P$, an insurer, I, with utility function u_I, will prefer to sell a policy for X_2 since:

$$\mathrm{E}\,[u_I(P - X_1)\,] < \mathrm{E}\,[u_I(P - X_2)\,]\,.$$

Each insurer, with its own utility function, will prefer to sell a policy for X_2. Thus you should expect P_2 to be less than P_1. Here we have a case where the CME risk load formula and utility theory disagree.

Suppose we have an insured, G, with utility function u_G. If insurance is being bought for X_2, we must have:

$$u_G(-P_2) \geq \mathrm{E}\,[u_G(-X_2)\,]\,.$$

[1] The utility of an insurance policy depends upon many variables, such as initial wealth. Here I will not write down variables which are the same for all situations.

Now let's suppose we have inconsistency, i.e. $P_1 < P_2$. Then one of the following three cases must happen.

$$\text{Case 1: } \mathrm{E}\,[u_G(-X_2)] \geq u_G(-P_1) \;>\; u_G(-P_2)$$

$$\text{Case 2: } u_G(-P_1) > \mathrm{E}\,[u_G(-X_2)] \;\geq\; u_G(-P_2)$$

$$\text{Case 3: } u_G(-P_1) > u_G(-P_2) \;\geq\; \mathrm{E}\,[u_G(-X_2)]\,.$$

In Case 1, no insurance will be bought for either layer. In Case 2, no insurance will be bought for the second layer. In Case 3, insurance will be bought for the second layer in spite of the inconsistency. Note that the derivation of the CME risk load formula assumes that the demand for insurance is fixed; i.e., it only considers Case 3 where inconsistency can be tolerated.

5. WHERE DO WE GO FROM HERE?

The CME is derived using a variance constraint on insurance portfolios. I consider utility theory to be a better measure of risk than variance. However, risk loads derived from variance principles often provide good practical approximations to the results that are obtained using utility theory. I regard the results on inconsistency discussed above as evidence that the approximation is not perfect.

For now anyway, the inconsistency appears to be a theoretical rather than a practical problem. The lengths to which one has to go to produce inconsistent results seem far removed from real pricing decisions. Should real life cases where this becomes a problem arise in the future, I offer the following avenues of research to deal with these and other problems.

1. Replace the insurer's maximum variance constraint in the CME derivation with a minimum utility constraint.

2. Allow for flexibility in the demand for insurance. The assumption of constant demand has problems at both the very high and the very low layers.

3. The parameter uncertainty in the current CME formulation is very restricted. It allows only for uncertainty in the scale of the severity distribution. A potentially bigger problem is uncertainty in the shape of the severity distribution. Work on this needs to be done. Also, it is conceivable that incorporating other kinds of parameter uncertainty may make the consistency issue more pressing.

The problem of determining risk loads is perhaps one of the most difficult in all of actuarial science. Its solution will not come about with any single brilliant insight, but will evolve slowly after much trial and error. I would like to think that my work, as well as the work of Dr. Robbin and Mr. Bear, makes a positive contribution to this effort.

REFERENCES

[1] Meyers, Glenn, "The Competitive Market Equilibrium Formula for Increased Limits Ratemaking," *PCAS* LXXVIII, 1991

[2] Robbin, Ira, Discussion of Meyers: "The Competitive Market Equilibrium Formula for Increased Limits Ratemaking," *PCAS* LXXIX, 1992.

APPENDIX A

QUOTA SHARE REINSURANCE

The CME risk load is given by $\bar{\lambda}\,(\boldsymbol{U} + 2\boldsymbol{V}\,\bar{\boldsymbol{n}})$ with:

$$u_i = \mathrm{E}_\alpha[\mathrm{E}[Z_i^2 \mid \alpha]\,] + \mathrm{E}_\alpha[\mathrm{E}[Z_i \mid \alpha]^2\,] \bullet d$$

$$v_{ij} = (1 + c) \bullet \mathrm{E}_\alpha[\mathrm{E}[Z_i \mid \alpha] \bullet \mathrm{E}[Z_j \mid \alpha]\,] - \mathrm{E}_\alpha[\mathrm{E}[Z_i \mid \alpha]\,] \bullet \mathrm{E}_\alpha[\mathrm{E}[Z_j \mid \alpha]\,]\,.$$

If we multiply the loss in the i^{th} line of insurance by $1/r$ we get:

$$u_i' = \mathrm{E}_\alpha[\mathrm{E}[(Z_i/r\,)^2 \mid \alpha]\,] + \mathrm{E}_\alpha[\mathrm{E}[Z_i/r \mid \alpha\,]^2] \bullet d$$

$$= u_i/r^2.$$

$$v_{ij}' = (1{+}c) \bullet \mathrm{E}_\alpha\,[\mathrm{E}[Z_i/r \mid \alpha] \bullet \mathrm{E}[Z_j \mid \alpha]\,] - \mathrm{E}_\alpha[\mathrm{E}[Z_i/r \mid \alpha]\,] \bullet \mathrm{E}_\alpha[\mathrm{E}[Z_j \mid \alpha]\,]$$

$$= v_{ij}/r\,.$$

The total risk load contributed by the r reinsurers in the i^{th} line of insurance is:

$$r\bar{\lambda}(\boldsymbol{U}' + 2\boldsymbol{V}'\bar{\boldsymbol{n}})_i = r\bar{\lambda}(\boldsymbol{U}/r^2 + 2\boldsymbol{V}\,\bar{\boldsymbol{n}}/r) = \bar{\lambda}(\boldsymbol{U}/r + 2\boldsymbol{V}\,\bar{\boldsymbol{n}}).$$

Thus the total process risk is reduced by a factor of $1/r$ and the total parameter risk is unchanged.

APPENDIX B

CONSISTENCY

In this appendix, it is assumed that the reader is familiar with the results in Appendices C and E of the original paper.

Let the claim severity distribution be given by $S(z)$ and the expected claim cost for the layer from a to $a + h$ be given by $M_1(a,h)$.

Recall from Lemma E.1 of the original paper that:

$$M_1(a,h) = \int_a^{a+h} (1 - S(z)) \bullet dz .$$

Now:

$$M_1(a,h \mid \alpha) = \int_a^{a+h} \left(1 - S\left(\frac{z}{\alpha}\right)\right) \bullet dz = \alpha \bullet \int_{a/\alpha}^{(a+h)/\alpha} (1 - S(z)) \bullet dz ,$$

with the second equality being derived by substituting z for z/α.

Lemma B.1:

$\frac{\partial}{\partial\alpha} M_1(a,h \mid \alpha)$ is positive.

Using the product rule and the fundamental theorem of calculus we get:

$$\frac{\partial}{\partial\alpha}\left(\alpha \bullet \int_{a/\alpha}^{(a+h)/\alpha} (1-S(z)) \bullet dz\right)$$

$$= \left[\int_{a/\alpha}^{(a+h)/\alpha} (1-S(z)) \bullet dz - \frac{h}{\alpha}\left(1-S\left(\frac{a+h}{\alpha}\right)\right)\right] + \frac{a}{\alpha}\left(S\left(\frac{a+h}{\alpha}\right) - S\left(\frac{a}{\alpha}\right)\right). \qquad \text{(B.1)}$$

The bracketed expression is positive since $S(z) < S\left(\frac{a+h}{\alpha}\right)$ for all z in the interval from a/α to $(a+h)/\alpha$. The interval is of length h/α. The remainder of the expression is also positive since S is an increasing function.

Lemma B.2:

If $zS'(z)$ is decreasing for $z > a_0$, then $\frac{\partial}{\partial\alpha}M_1(a,h \mid \alpha)$ is a decreasing function of a for $a > \alpha a_0$.

$$\frac{\partial}{\partial a}\frac{\partial}{\partial\alpha}M_1(a,h \mid \alpha) = \frac{\partial}{\partial a}\left[\int_{a/\alpha}^{(a+h)/\alpha}(1-S(z)) \bullet dz - \frac{h}{\alpha}\left(1-S\left(\frac{a+h}{\alpha}\right)\right) + \frac{a}{\alpha}\left(S\left(\frac{a+h}{\alpha}\right) - S\left(\frac{a}{\alpha}\right)\right)\right]$$

$$= \frac{1}{\alpha}\left[\frac{a+h}{\alpha}S'\left(\frac{a+h}{\alpha}\right) - \frac{a}{\alpha}S'\left(\frac{a}{\alpha}\right)\right] < 0 \text{ for } a > \alpha a_0.$$

Thus $\frac{\partial}{\partial\alpha}M_1(a,h \mid \alpha)$ is a decreasing function of a for $a > \alpha a_0$.

The condition that $zS'(z)$ be decreasing for $z > a_0$ is a common property of severity models. Consider the Pareto distribution:

$$S(z) = 1 - \left(\frac{b}{z+b}\right)^q \quad \text{and} \quad zS'(z) = \frac{zqb^q}{(z+b)^{q+1}}.$$

Now

$$\frac{d}{dz}(zS'(z)) = qb^q\left(\frac{1}{(z+b)^{q+1}} - \frac{z(q+1)}{(z+b)^{q+2}}\right)$$

is negative if and only if $z > b/q$. The reader can verify that this property holds for many other distributions such as the Weibull and the lognormal.

Theorem B.3:

If the severity distribution satisfies the property that $zS'(z)$ is decreasing for $z > a_0$, then there exists a limit D so that increased limits factors are consistent for retentions $a > D$.

It is instructive to consider the (incorrect) "proof" given in my paper. Let $a_2 > a_1$.

$$v_{1j} = (1+c) \bullet \mathrm{E}_\alpha[M_1(a_1,h \mid \alpha) \bullet \mathrm{E}[Z_j \mid \alpha]] - \mathrm{E}_\alpha[M_1(a_1,h \mid \alpha)] \bullet \mathrm{E}_\alpha[\mathrm{E}[Z_j \mid \alpha]]$$

$$> (1+c) \bullet \mathrm{E}_\alpha[M_1(a_2,h \mid \alpha) \bullet \mathrm{E}[Z_j \mid \alpha]] - \mathrm{E}_\alpha[M_1(a_2,h \mid \alpha)] \bullet \mathrm{E}_\alpha[\mathrm{E}[Z_j \mid \alpha]]$$

$$= v_{2j} .$$

It then follows that:

$$(\boldsymbol{V} \bullet \overline{\boldsymbol{n}})_1 > (\boldsymbol{V} \bullet \overline{\boldsymbol{n}})_2 .$$

Robert Bear's contribution was to point out that while

$$c\mathrm{E}_\alpha[M_1(a_1,h \mid \alpha) \bullet \mathrm{E}[Z_j \mid \alpha]]$$

is greater than

$$c\mathrm{E}_\alpha[M_1(a_2,h \mid \alpha) \bullet \mathrm{E}[Z_j \mid \alpha]] ,$$

it does not follow that

$$\mathrm{E}_\alpha[M_1(a_1,h \mid \alpha) \bullet \mathrm{E}[Z_j \mid \alpha]] - \mathrm{E}_\alpha[M_1(a_1,h \mid \alpha)] \bullet \mathrm{E}_\alpha[\mathrm{E}[Z_j \mid \alpha]]$$

is greater than (B.2)

$$\mathrm{E}_\alpha[M_1(a_2,h \mid \alpha) \bullet \mathrm{E}[Z_j \mid \alpha]] - \mathrm{E}_\alpha[M_1(a_2,h \mid \alpha)] \bullet \mathrm{E}_\alpha[\mathrm{E}[Z_j \mid \alpha]].$$

It is this last inequality, Equation B.2, that we must demonstrate in order to make the claim of consistency.

It will help if we introduce some shorthand notation. Let:

$$m_1(\alpha) = M_1(a_1, h \mid \alpha) \text{ and } \overline{m}_1 = \mathrm{E}_\alpha[m_1(\alpha)],$$

$$m_2(\alpha) = M_1(a_2, h \mid \alpha) \text{ and } \overline{m}_2 = \mathrm{E}_\alpha[m_2(\alpha)], \text{ and}$$

$$e(\alpha) = \mathrm{E}[Z_j \mid \alpha] \text{ and } \overline{e} = \mathrm{E}_\alpha[\mathrm{E}[Z_j \mid \alpha]].$$

Equation B.2 can then be written as:

$$\int_0^\infty (m_1(\alpha)-\overline{m}_1)(e(\alpha)-\overline{e})f(\alpha)d\alpha > \int_0^\infty (m_2(\alpha)-\overline{m}_2)(e(\alpha)-\overline{e})f(\alpha)d\alpha .$$

The left hand side of this expression can be evaluated using integration by parts:

$$u = m_1(\alpha)-\overline{m}_1 \qquad dv = (e(\alpha)-\overline{e})f(\alpha)d\alpha$$

$$du = m_1'(\alpha)d\alpha \qquad v = \int_0^\alpha (e(t)-\overline{e})f(t)dt$$

$$\text{LHS} = \underbrace{\lim_{r\to\infty} (m_1(r)-\overline{m}_1)\int_0^r (e(t)-\overline{e})f(t)dt}_{=0} - \lim_{r\to\infty}\int_0^r \underbrace{m_1'(\alpha)}_{>0} \underbrace{\int_0^\alpha (e(t)-\overline{e})f(t)dt}_{<0}\, d\alpha$$

$$= -\lim_{r\to\infty}\int_0^r m_1'(\alpha)\int_0^\alpha (e(t)-\overline{e})f(t)dt\, d\alpha \qquad \text{(which is positive).}$$

Similarly:

$$\text{RHS} = -\lim_{r\to\infty}\int_0^r m_2'(\alpha)\int_0^\alpha (e(t)-\overline{e})f(t)dt\, d\alpha \qquad \text{(which is positive).}$$

If we evaluate the outer integrals by a numerical integration formula (as I do in my paper):

$$\text{LHS} = \sum_{i=1}^{L} m_1'(\alpha_i) \int_0^{\alpha_i} (e(t)-\bar{e})f(t)dt \text{ and}$$

$$\text{RHS} = \sum_{i=1}^{L} m_2'(\alpha_i) \int_0^{\alpha_i} (e(t)-\bar{e})f(t)dt.$$

If we choose $D = \max\{\alpha_i\}a_0$, then by Lemma B.2, $m_1'(\alpha_i) > m_2'(\alpha_i)$ for all i. Thus LHS is greater than RHS. This proves Equation B.2 and consequently, Theorem B.3.

A close examination of the above proof reveals that if $zS'(z)$ is increasing for $z < a_0$, as it does for the Pareto distribution with $a_0 = b/q$, then it is possible to have inconsistent increased limits factors for $a_2+h < \min\{\alpha_i\}a_0$. Our example can be modified to produce inconsistent increased limits factors as follows. Change c from .02 to 0, a from .001 to .02, and $\bar{\lambda}$ from 2×10^{-7} to 2×10^{-5}. This yields min $\{\alpha_i\}a_0 = 3{,}432$ and max $\{\alpha_i\}a_0 = 5{,}659$. The results are in Table 9. Note that the parameter risk shifts from inconsistent to consistent in the interval from 3,432 to 5,659. The shift from inconsistent to consistent for increased limit factors occurs at a lower level because of the consistency of the average severity and the process risk.