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ASSET/LIABILITY MATCHING (FIVE MOMENTS)

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Abstract

It is well known that re-investment risk can be greatly reduced if the assets which are assigned to support liabilities are "matched." In particular, matching two properties of the asset and liability cash flows, the dollar duration (DD1) *and dollar convexity* (DD2), *can provide a significant reduction in re-investment risk. This paper provides a rigorous mathematical treatment of the asset/liability matching problem.*

This paper initially shows that DD1 *and* DD2 *are the first two moments of a set of cash flows* (DDn). *By means of a Taylor expansion of the present value of a set of cash flows, the paper then shows why matching individual moments of an asset flow with the corresponding moments associated with a liability flow can reduce re-investment risk.*

Finally, for every cash flow and pair of interest rates, there exists a characteristic time T. *Even if the flow is originally priced to yield the first interest rate, and it is the* *second interest rate that prevails, the initial yield rate can be achieved by selling the flow at time* T. *The paper shows how this relates to asset/liability matching, and how* T *can be expressed in terms of the generalized moments,* DDn.

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1. INTRODUCTION

Whenever a liability takes the form of future cash outflows and assets earn interest, it is reasonable to discount the liability for interest before deciding whether or not assets are sufficient to "cover" the liability. In the discounting process, several assumptions are made. One assumption is that the size and timing of the cash outflows are known. A second assumption is that the interest rate used in the discounting process can be realized in asset yield. Both of these assumptions introduce an element of risk into the matching process. The latter risk has two distinct elements: Credit risk due to possible defaults as to principal and interest, and re-investment risk due to interest rate changes during the life of the asset.

The sources of re-investment risk and ways to reduce that risk have been the subject of several recent papers and articles (see [1]-[6]). It has been demonstrated that re-investment risk can be greatly reduced if two moments of the asset and liability cash flows are matched; namely dollar duration ($DD1$), and dollar convexity ($DD2$). Another simpler moment, weighted term duration (WTD), is mentioned, but usually not considered further.

Two moments in time are also discussed when considering the reduction of re-investment risk: the initial time (implicit in the re-investment rate) and one implied by Ferguson's Table C in which a characteristic time equal to 4.13 years is shown to have special significance for a five-year, 9% par bond [1]. With the exception of Appendix B in Ferguson's paper, relationships between the five moments listed above are usually demonstrated by means of examples, rather than by a more rigorous mathematical exposition. Following the spirit of Ferguson's appendix, this paper recasts the discussion into a mathematically rigorous format, and, in Appendix B, applies the results to reflect higher order terms in Ferguson's bond example. In the process we gain some insight into the nature of the relationships and see that all of them are approximations. It is not the objective to produce a better method for reducing re-investment risk but, rather, to place the current work into a unified theoretical framework. Credit risk is beyond the scope of this paper.

2. DEFINITIONS

Assume a set of discrete cash flows $\{CF_t\}$, where CF_t is the flow at time, t. These flows may represent either an income producing asset (in which case the CF represents inflows) or a liability (in which case the CF represents outflows).

The nominal value of the flow is given by the sum of the flows over time, as follows:

$$Nom = \sum_{t=0}^{\omega} CF_t , \tag{2.1}$$

where ω is the largest value of t for which CF_t is non-zero. The t need not be an integer, and some CF_t with $t < \omega$ can be zero.

The present value of the flow, under an assumed interest rate, i, is

$$PV = \sum_{t=0}^{\omega} v(i)^t \, CF_t , \tag{2.2}$$

where

$$v(i) = 1/(1+i) . \tag{2.3}$$

The weighted term duration is defined by

$$WTD = \sum_{t=0}^{\omega} t\, CF_t \,/ \sum_{t=0}^{\omega} CF_t\,. \tag{2.4}$$

The dollar duration is given by

$$DD1(i) = \sum_{t=0}^{\omega} t\, v(i)^t\, CF_t \,/ \sum_{t=0}^{\omega} v(i)^t\, CF_t\,. \tag{2.5}$$

The usual notation does not explicity draw attention to the fact that $DD1$ depends upon the assumed interest rate. For much of what follows, this dependence will be significant. Dollar convexity is defined as the second moment (in time) of the cash flow, as follows:

$$DD2(i) = \sum_{t=0}^{\omega} t^2 v(i)^t\, CF_t \,/ \sum_{t=0}^{\omega} v(i)^t\, CF_t\,. \tag{2.6}$$

Again, this notation explicitly displays the dependence of the dollar convexity upon the assumed interest rate. Continuing on, higher moments of the cash flow distribution are defined by:

$$DDn(i) \equiv \sum_{t=0}^{\omega} t^n\, v(i)^t\, CF_t \,/ \sum_{t=0}^{\omega} v(i)^t\, CF_t\;. \tag{2.7}$$

As was previously mentioned, the time scale can be drawn as finely as the cash flow pattern dictates. For some flows, the payment pattern will be nearly continuous. For those flows, approximate the set of discrete flows, $\{CF_t\}$, with a flow rate $\sigma(t)$ such that $\sigma(t)dt$ represents the cash flow from time t to $t + dt$ (an infinitesimal time later). Further, define a normalized discounted flow density $\rho(i, t)$ as follows:

$$\rho(i, t) = v(i)^t\, \sigma(t) / \int_0^{\omega} v(i)^t\, \sigma(t)\, dt\,. \tag{2.8}$$

Using the definition of $\rho(i, t)$, Equations 2.1, 2.2, and 2.4-2.7 can be recast into continuous form:

$$Nom = \int_0^\omega \sigma(t)dt\,, \tag{2.9}$$

$$PV = \int_0^\omega v(i)^t\,\sigma(t)dt\,, \tag{2.10}$$

$$WTD = \int_0^\omega t\,\rho(0, t)\,dt = DD1\,(0) \tag{2.11}$$

$$DD1(i) = \int_0^\omega t\,\rho(i, t)\,dt\,, \tag{2.12}$$

$$DD2(i) = \int_0^\omega t^2\,\rho(i, t)\,dt\,, \text{ and} \tag{2.13}$$

$$DDn(i) = \int_0^\omega t^n\,\rho(i, t)\,dt. \tag{2.14}$$

In this form, the integrals for DDn ($n = 1, 2, 3....$) are clearly moments of the distribution (of cash flows) given by $\rho(i, t)$.

While the weekly payments of workers' compensation lifetime disability benefits may be reasonably approximated by a continuous cash flow, very few assets yield a nearly continuous cash flow.

A final definition allows the rigorous dealing with any discrete cash flow as if it were continuous—allowing us to work in the continuous case whenever the mathematical manipulations are easier. The device is called a Dirac delta, $\delta(x - x_0)$. Standing alone, the Dirac delta is undefined; but its action within an integral is well defined. Consider a function $f(x)$, then

$$\int_a^b f(x)\delta(x-x_0)dx = \begin{cases} f(x_0) \text{ if } a \le x_0 \le b \\ 0 \text{ if not} \end{cases} \tag{2.15}$$

If one writes, for the discrete set $\{CF_{t_0}, CF_{t_1}, CF_{t_2}, \ldots CF_{t_\omega}\}$,

$$\sigma(t) = \sum_{m=0}^{\omega} CF_{t_m}\, \delta(t - t_m)\,, \tag{2.16}$$

then, for example,

$$\int_0^{\omega} \sigma(t)dt = \sum_{m=0}^{\omega} CF_{t_m} \text{ and} \tag{2.17}$$

$$\int_0^{\omega} t^n \rho(i,t)dt = \sum_{m=0}^{\omega} t^n\, v(i)^{t_m}\, CF_{t_m} \Big/ \sum_{m=0}^{\omega} v(i)^{t_m}\, CF_{t_m}\,. \tag{2.18}$$

3. ASSET/LIABILITY MATCHING: CASE 1

The usual case considered is when a discounted liability cash flow,

$$PV_L\,(i) = \int_0^{\omega_L} v(i)^t\, \sigma_L\,(t)\, dt\,, \tag{3.1}$$

is matched with (set equal to) an asset with an identical present value (but not, necessarily, identical cash flows),

$$PV_A\,(i) = \int_0^{\omega_A} v(i)^t\, \sigma_A\,(t)\, dt \tag{3.2}$$

at time equals zero, the interest rate changes to j. The asset and liability continue to be matched if

$$PV_L\,(j) = PV_A\,(j)\,. \tag{3.3}$$

The trivial (in a mathematical sense) solution to Equation 3.3 involves selecting an asset for which:

$$\sigma_A\,(t) = \sigma_L\,(t)\,. \tag{3.4}$$

In this case, while both $PV_L\,(i)$ and $PV_A\,(i)$ are functions of the interest rate, their difference,

$$PV_L\,(i) - PV_A\,(i) = \int_0^{\max(\omega_L,\,\omega_A)} v(i)^t \cdot 0 \cdot dt = 0 \tag{3.5}$$

is independent of i.

One could always transfer the liability to a third party in exchange for a single payment equal to the selling price of the asset (remember, we are not considering timing risk or default risks, so the price should equal the present value). The purchase of zero coupon bonds, which mature as the liabilities become due, produces just such a solution to the re-investment risk problem.

When the two $\sigma(t)$ are not identical, approximate solutions to Equation 3.3 may be found via a Taylor expansion of the present value as a function of the interest rate, i. In particular, for $j = i + \Delta i$,

$$PV_L\,(j) = \sum_{n=0}^{\infty} (i/n!)\,[d^n PV_L\,(k)/dk^n]\mid_{k=i} (\Delta i)^n, \tag{3.6}$$

$$PV_A\,(j) = \sum_{n=0}^{\infty} (i/n!)\,[d^n PV_A\,(k)/dk^n]\mid_{k=i} (\Delta i)^n, \tag{3.7}$$

or

$$PV_L(j) - PV_A(j) =$$

$$\sum_{n=0}^{\infty} (i/n!)\,[d^n PV_L\ (k)/dk^n - d^n PV_A\ (k)/dk^n]\,|_{k=i}\,(\Delta i)^n. \quad (3.8)$$

The set $\{(\Delta i)^n\}$ for Δi not equal to zero and for $n = 0, 1, 2, 3....$ forms an independent basis for a vector space. As such, a null vector, implying $PV_L(j) = PV_A(j)$, can only be obtained if each component,

$$a_n = 1/n!\,[d^n PV_L(k)/dk^n - d^n PV_A(K)/dk^n]\,|_{k=i} \quad (3.9)$$

is zero. We therefore conclude that the solution for Equation 3.3 obtained by setting $\sigma_L(t)$ equal to $\sigma_A(t)$ is not only the trivial solution, but it is the only exact solution (since satisfying Equation 3.9 to all orders would cause the two functions to be identical). For small i, the higher order terms in the Taylor series can be expected to decrease rapidly, allowing for an acceptable degree of error to remain if only one or two terms are matched (i.e., Equation 3.9 is satisfied).

The zero order terms are initially equal if the asset and liability have equal present values before the (time zero) interest rate change. The first order term requires a matching of (from Equation 3.9 with $n = 1$),

$$dPV_L(k)/dk\,|_{k=i} = dPV_A(k)/dk\,|_{k=i}\,. \quad (3.10)$$

From Equation 3.1,

$$dPV_L(k)/dk\,|_{k=i} = \int_0^{\omega} \sigma_L(t)\, dv(k)^t/dk\,|_{k=i}\, dt \quad (3.11)$$

$$= -v(i) \int_0^{\omega} tv(i)^t\, \sigma_L(t)\, dt$$

$$= -v(i)\, DD1_L(i)\, PV_L(i)\,.$$

Likewise, for the asset,

$$dPV_A(k)/dk\,|_{k=i} = -v(i)\, DD1_A(i)\, PV_A(i)\,. \tag{3.12}$$

Equation 3.10 will be satisfied, in view of Equations 3.11 and 3.12, if

$$DD1_L(i) = DD1_A(i)\,, \tag{3.13}$$

which is the usual condition that dollar durations be matched. (Note that $PV_L = PV_A$ when the asset was originally selected.)

The next term introduces convexity. Setting

$$d^2PV_L(k)/dk^2\,|_{k=i} = d^2PV_A(k)/dk^2\,|_{k=i}$$

produces matching to second order in Δi,

$$d^2PV(k)/dk^2\,|_{k=i} = \int_0^{\omega} \sigma(t)\, d^2v(k)/dk^2\,|_{k=i_{dt}} \tag{3.14}$$

$$= v(i)^2 \int_0^{\omega} (t^2 + t)v(i)^t\sigma(t)dt$$

$$= v(i)^2[DD2(i) + DD1(i)]\, PV(i)\,.$$

As long as PV and $DD1$ have been matched, Equation 3.14 adds the convexity matching requirement, or

$$DD2_L(i) = DD2_A(i) \tag{3.15}$$

for second order agreement.

While higher order terms can be matched, a small Δi raised to a large power makes the terms less significant. Nonetheless, we observe that each additional order introduces an additional moment along with the previously matched moments. Again, if all of the moments are equal, the two distributions must be equal. Practically speaking, it may be extremely difficult to match $DD2$, let alone to find assets for which higher orders of DDn are matched.

It is interesting to note that expressions for the change in price frequently omit terms and factors from the Taylor series. (Ferguson draws attention to the missing factor of $v(i)$ in the first order term.) In particular, both Babbel and Stricker [5], and Diembiec, et al [2] omit the $DD1$ contribution to the second order term, and the $v(i)$ factor at all orders. The correct expression is

$$\begin{aligned} \Delta\text{Price}/(\text{Original Price}) &= [PV(j) - PV(i)]/PV(i) \qquad (3.16)\\ &= -v(i)\, DD1(i)\Delta i \\ &\quad + \tfrac{1}{2}\, v(i)^2[DD2(i) + DD1(i)](\Delta i)^2 \\ &\quad + R(\Delta i^3) \end{aligned}$$

where R is a residual term of order Δi^3 and higher. The previously published residual term contains contributions of the same order as those that are explicity displayed. The expressions also appear to confuse price with Δprice/original price. Of course, the missing terms and factors are common to both the asset and the liability, so their absence in the price expansion does not introduce any errors into the matching process, or the conclusion that convexity matching is a significant improvement over dollar duration matching.

4. ASSET/LIABILITY MATCHING: CASE 2

Ferguson alludes to a second method of re-investment risk management. Given an initially matched asset and liability and an initial change of interest rate, there is some time, T, (not equal to zero) at which the asset and liability could be exchanged (assuming no intervening interest rate changes). He implies that T is equal to the duration (which is true only to first order in Δi).

Before demonstrating the degree of approximation in this assertion, it will be shown that this second time of price equality can be determined exactly in closed form. As in the previous case, assume that the asset and liability are price matched under the initial interest rate assumption,

$$PV_L(i) = PV_A(i), \tag{4.1}$$

and that at time $t = 0$, interest rates abruptly change from i to j. We have already seen that, if the change is small and $DD1(i)$ and $DD2(i)$ are equal, then $PV_L(j)$ will be approximately equal to $PV_A(j)$.

After some time has elapsed, there is a time, T, at which the asset can be sold such that the accumulated value of prior payments at the new rate, j, plus the sale price (determined at the new rate, j, for the remaining flows) yields the original rate, i. If the corresponding liability has the same characteristic T, an exchange could be made at time T without suffering the consequences of re-investment risk.

At the original yield rate and price, $PV_A(i)$ would have accumulated to $PV_A(i) \cdot (1+i)^T$ by time T. Instead, the prior payments will have accumulated to

$$\sum_{t=0}^{T} (1+j)^{T-t} CF_t = \sum_{t=0}^{T} v(j)^{t-T} CF_t, \tag{4.2}$$

using the discrete notation for simplicity. The present value of the future payments at time T are given by

$$\text{selling price} = \sum_{t=T+1}^{\omega} v(j)^{t-T} CF_t. \tag{4.3}$$

Combining Equations 4.2 and 4.3 to obtain the total wealth after selling the asset at time T and comparing it to the original asset price,

$$PV_A(i) * (1+i)^T = \sum_{t=0}^{T} v(j)^{t-T} CF_t + \sum_{t=T+1}^{\omega} v(j)^{t-T} CF_t \qquad (4.4)$$

$$= (1+j)^T \sum_{t=0}^{\omega} v(j)^t CF_t$$

$$= (1+j)^T PV_A(j),$$

where $PV_A(j)$ is the *original* price of the asset under an assumed interest rate, j. Solving for T gives the exact solution,

$$T(i,j) = \ln[PV_A(j)/PV_A(i)]/\ln[(1+i)/(1+j)]. \qquad (4.5)$$

While any logarithm base could be used, we have selected the natural base. T depends upon both interest rates, so it is not a function of the original bond price alone (as one might believe after reading Ferguson's example).

To see how T is related to $DD1$ and $DD2$, expand T in a Taylor series to first order in (Δi). Here, however, the derivatives are not quite as simple as they were for the PV expansion. The Taylor series in powers of $\Delta i = j - i$ is given by

$$T(i,j) = T(i,k)\,|_{k=i} + dT(i,k)/dk\,|_{k=i}\,\Delta i + R(\Delta i^2). \qquad (4.6)$$

Due to the presence of $\ln[PV(k)/PV(i)]$ in the numerator of $T(i,k)$ and $\ln[(1+i)/(1+k)]$ in the denominator, each of these terms involves the indeterminate form $0/0$ when k is set equal to i. One or more applications of l'Hopital's rule (see Appendix A) allows us to evaluate each term giving

$$T(i,j) = DD1(i) - \tfrac{1}{2}\, v(i)[DD2(i) - DD1(i)^2]\Delta i + R(\Delta i^2). \qquad (4.7)$$

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APPENDIX A

EVALUATION OF THE INDETERMINATE FORMS, $T(i, k) \mid_{k=i}$

Zero Order Term, $T(i, k) \mid_{k=i}$

Using l'Hopital's rule for the form 0/0, we replace

$$\lim_{k \to i} T(i, k) = \lim_{k \to i} \ln[PV(k)/PV(i)]/\ln[(1+i)/(1+k)] \quad (A.1)$$

with the equivalent

$$\lim_{k \to i} T(i, k) = \lim_{k \to i} d/dk \ln[PV(k)/PV(i)]/\lim_{k \to i} d/dk \ln[(1+i)/(1+k)]\,, \quad (A.2)$$

and evaluate the derivatives,

$$\lim_{k \to i} T(i, k) = \lim_{k \to i} [PV(k)^{-1}\, dPV(k)/dk]/\lim_{k \to i} [(1+k)\, d(1+k)^{-1}/dk]\,. \quad (A.3)$$

This expression can be evaluated further if the discrete form expression for $PV(k)$ is substituted, as follows:

$$\lim_{k \to i} T(i, k) = \lim_{k \to i} [(d/dk \sum_{t=0}^{\omega} v(k)^t CF_t)/ \sum_{t=0}^{\omega} v(k)^t CF_t]/\lim_{k \to i} (-v(k))$$

$$= \sum_{t=0}^{\omega} tv(i)\, CF_t / \sum_{t=0}^{\omega} v(i)^t CF_t\,, \quad (A.4)$$

which is quickly identified as the discrete form of $DD1(i)$. Therefore,

$$T(i, k) \mid_{k=i} = DD1(i)\,. \quad (A.5)$$

First Order Term $dT(i, k)/dk \mid_{k=i}$

The first order term involves taking the first derivative of $T(i, k)$ with respect to k, or more specifically,

$$dT(i,k)/dk = \{\ln[(1+i)/(1+k)] \cdot d/dk \ln[PV(k)/PV(i)] \qquad \text{(A.6)}$$

$$-\ln[PV(k)/PV(i)] \cdot d/dk \ln[(1+i)/(1+k)]\}$$

$$\div \{\ln[(1+i)/(1+k)]\}^2,$$

an expression which is rich in indeterminate forms when $k = i$.

The derivative in the first term is identical to the numerator in Equation A.2, $-v(k)DD1(k)$, and the derivative in the second term is identical to the one taken in Equation A.2, or $-v(k)$. Making these substitutions into A.6 gives

$$dT(i,k)/dk \qquad \text{(A.7)}$$

$$= \{-\ln[(1+i)/(1+k)] \cdot v(k) \cdot DD1(k) + v(k) \cdot \ln[PV(k)/PV(i)]\}$$

$$\div \{\ln[(1+i)/(1+k)]\}^2,$$

which is clearly of the form 0/0 when $k = i$ because $v(k)$ and $DD1(k)$ are finite positive numbers for all non-negative interest rates.

L'Hopital's rule, therefore, can be applied to the right side of Equation A.7 in order to determine $dT(i, k)/dk$ as k approaches i. The application of l'Hopital's rule to Equation A.7 involves the algebraic manipulation of some rather lengthy expressions. To simplify the process we define A, B, and C as follows:

$$A(k) = v(k) \ln[PV(k)/PV(i)], \qquad \text{(A.8a)}$$

$$B(k) = \ln[(1+i)/(1+k)]\, v(k)\, DD1(k), \qquad \text{(A.8b)}$$

$$C(k) = \{\ln[(1+i)/(1+k)]\}^2. \qquad \text{(A.8c)}$$

In terms of A, B, and C, Equation A.7 becomes

$$dT(i,k)/dk = [A(k) - B(k)]/C(k)\ , \tag{A.9}$$

and l'Hopital's rule leads to

$$dT(i,k)/dk\,|_{k=i}$$
$$= [\lim_{k\to i} dA(k)/dk - \lim_{k\to i} dB(k)/dk]/\lim_{k\to i} dC(k)/dk\ , \tag{A.10}$$

from which each term may be evaluated separately.

$$\begin{aligned} dA(k)/dk &= \ln[PV(k)/PV(i)]\ dv(k)/dk \\ &\quad + v(k)\ d/dk \ln[PV(k)/PV(i)] \\ &= -v(k)^2 \ln[PV(k)/PV(i)] - v(k)^2\ DD1(k). \end{aligned} \tag{A.11}$$

$$\begin{aligned} dB(k)/dk &= v(k)\ DD1(k)\ d/dk \ln[(1+i)/(1+k)] \\ &\quad + \ln[(1+i)/(1+k)]\ v(k)\ d \\ &\quad \div dk \left\{ \sum_{t=0}^{\omega} tv(k)^t\ CF_t / \sum_{t=0}^{\omega} v(k)^t\ CF_t \right\} \\ &= -v(k)^2\ DD1(k) - v(k)^2\ DD1(k) \ln[(1+i)/(1+k)] \\ &\quad + \ln[(1+i)/(1+k)]\ v(k)^2\ DD1(k)^2 \\ &\quad - v(k)^2\ DD2(k) \ln[(1+i)/(1+k)]\ . \end{aligned} \tag{A.12}$$

$$\begin{aligned} dC(k)/dk &= d/dk \left\{ \ln[(1+i)/(1+k)] \right\}^2 \\ &= 2 \ln[(1+i)/(1+k)]\ v(k)\ . \end{aligned} \tag{A.13}$$

The full expression becomes

$$dT(i, k)/dk \mid_{k=i} \tag{A.14}$$

$$= {}^{-}\tfrac{1}{2} \left\{ \underset{k \to i}{\text{Limit}}\ v(k) \ln[PV(k)/PV(i)/\underset{k \to i}{\text{Limit}} \ln[(1+i)/(1+k)] \right\}$$

$${}^{-}\tfrac{1}{2}\, v(i)\, DD1(i) - \tfrac{1}{2}\, v(i)\, [DD2(i) - DD1(i)^2]\ ,$$

where the first term is *still* indeterminate!

A reapplication of l'Hopital's rule to the first term quickly discloses (in view of the evaluation of the zero order term) that

$$dT(i, k)/dk \mid_{k=i} = {}^{-}\tfrac{1}{2}\, v(i)\, [DD2(i) - DD1(i)^2]\ . \tag{A.15}$$

APPENDIX B

A Numerical Example

Consider a five-year, par \$1,000 bond with 9% semi-annual coupons, redeemed at par. Table B.1 displays the moments necessary to price the bond to yield 9% and to determine $DD1(0.09)$ and $DD2(0.09)$. Column 2 displays the set of cash flows, with CF_5 consisting of both the final coupon and the redemption of the bond. Columns 4-6 are the components of the zero, first, and second moments of the discounted cash flow in time.

An example of the type of re-investment risk to be managed would be an abrupt change in yield rates from the 9% assumed when the bond was priced to 6.5%. Assume that the change in yield takes place at time equals zero.

Table B.2 repeats the first four columns of Table B.1, but under a 6.5% yield assumption. Had the actual re-investment rate been known when Bond 1 was priced, it would have cost \$1,109.87 rather than the \$1,007.70 purchase price.

Using the two prices and yield rates together with the exact Equation 4.5 for $T(i, j)$, we find that Bond 2 can be sold to yield the original 9% rate at $T(0.09, 0.065) = 4.1621$ years (approximately two months into the fifth year).

Solving for $T(i, j)$ to four decimal places, by means of the Taylor expansion, gives $T(i, j) =$

4.1383 years, using zero order term $DD1(i)$	(-0.57% error at zero order in $j - i$)
+ 0.0238 years, (first order correction term)	
= 4.1621 years, to first order in $j - i$	(0.00% error at first order in $j - i$)

Given the rather straightforward nature of the exact solution, there would be little reason to use the Taylor series in lieu of Equation 4.5. Assuming that $T(i, j) = DD1(i)$ would introduce an unnecessary error into the calculation. An advantage of using Equation 4.5 over the approximate $DD1(i)$ is that the sensitivity to the magnitude of change from i to j can be tested, because Equation 4.5 explicitly contains the new interest rate, j.

If Equation 3.16 is solved for the new price, one obtains

$$PV(j) = \tag{B.1}$$

$$PV(i) - v(i) \cdot DD1(i) \cdot PV(i) \cdot (j - i) + \tfrac{1}{2} \cdot v(i)^2$$

$$\cdot\ [DD2(i) + DD1(i)] \cdot PV(i) \cdot (j - 1)^2 + R(\Delta i^3)\ .$$

From Table B.2, $PV(0.065)$ should be \$1,109.87. The Taylor series produces the following approximations.

$PV(j) =$

\$1,007.70 to zero order in $(j - i)$ (-9.21% error at zero order in $(j - i)$)

+ \$95.65 (first order correction)

= \$1,103.35 to first order in $(j - i)$ (-0.59% error at first order in $(j - i)$)

+ 6.19 (second order correction)

=\$1,109.54 to second order in $(j - i)$ (–0.03% error at second order in $(j - i)$)

which verifies that, at least for this example, matching dollar convexity significantly improved the matching process.

TABLE B.1

BOND 1

Years to maturity: five years
Coupon rate: 9.00% paid semi-annually
Par value: $1,000
Redemption value: $1,000
Priced to yield i: 9.00% annually

(1) t (in years)	(2) CF_t	(3) $v(i)^t$	(4) $t^0 * v(i)^t * CF_t$	(5) $t^1 * v(i)^t * CF_t$	(6) $t^2 * v(i)^t * CF_t$
0.0	0.00	1.0000000	0.00	0.00	0.00
0.5	45.00	0.9578263	43.10	21.55	10.78
1.0	45.00	0.9174312	41.28	41.28	41.28
1.5	45.00	0.8787397	39.54	59.31	88.97
2.0	45.00	0.8416800	37.88	75.75	151.50
2.5	45.00	0.8061832	36.28	90.70	226.74
3.0	45.00	0.7721835	34.75	104.24	312.73
3.5	45.00	0.7396176	33.28	116.49	407.71
4.0	45.00	0.7084252	31.88	127.52	510.07
4.5	45.00	0.6785483	30.53	137.41	618.33
5.0	1,045.00	0.6499314	679.18	3,395.89	16,979.46
Total			1,007.70	4,170.14	19,347.57

$PV(i)$ = $1,007.70 = total (4)
$DD1(i)$ = 4.1383 = total (5) / total (4)
$DD2(i)$ = 19.1997 = total (6) / total (4)

TABLE B.2
Bond 2

Years to maturity: five years

Coupon rate: 9.00% paid semi-annually

Par value: $1,000

Redemption value: $1,000

Priced to yield j: 6.50% annually

(1) t (in years)	(2) CF_t	(3) $v(i)^t$	(4) $t^0 * v(i)^t * CF_t$
0.0	0.00	1.0000000	0.00
0.5	45.00	0.9690032	43.61
1.0	45.00	0.9389671	42.25
1.5	45.00	0.9098621	40.94
2.0	45.00	0.8816593	39.67
2.5	45.00	0.8543306	38.44
3.0	45.00	0.8278491	37.25
3.5	45.00	0.8021884	36.10
4.0	45.00	0.7773231	34.98
4.5	45.00	0.7532285	33.90
5.0	1,045.00	0.7298808	762.73
Total			1,109.87

$PV(j) =$ \$1,109.87 $=$ total (4)